On a Distance between Directions in an Aleksandrov Space of Curvature $\leq K$

I. D. Berg & I. G. Nikolaev

1. Introduction

In this note we consider an analog of the notion of angle between two directions, possibly based at different points, for a space of curvature bounded above.

The set of all unit tangent vectors of an n-dimensional Riemannian manifold $\mathcal{M}^n$ constitutes a *sphere-bundle* $T_1(\mathcal{M}^n)$ over $\mathcal{M}^n$. In [13; 14], Sasaki introduced a natural Riemannian metric on a sphere-bundle of a Riemannian space $\langle \mathcal{M}^n, ds^2 \rangle$, which now is known as the *Sasaki metric*. Let $\boldsymbol{\xi}$ and $\tilde{\boldsymbol{\xi}}$ be a pair of vectors in $T_1(\mathcal{M}^n)$ at the nearby points $\mathbf{x}$ and $\tilde{\mathbf{x}}$. Translate the vector $\tilde{\boldsymbol{\xi}}$ in a parallel way to the point $\mathbf{x}$ by Levi–Civita parallelism along a minimizing geodesic joining $\mathbf{x}$ and $\tilde{\mathbf{x}}$, and denote the angle between the tangent vector thus obtained and the tangent vector $\boldsymbol{\xi}$ by $\Delta\theta$. Then, by a standard limiting process,

$$d\sigma^2 = ds^2 + d\theta^2$$

specifies the Sasaki metric tensor in $T_1(\mathcal{M}^n)$. Thus the restriction of the Sasaki metric to a sphere fiber is the canonical round metric; that is, a sphere fiber is isometric to a unit sphere in a Euclidean space and, for a smooth field of unit vectors along a smooth curve $\gamma : [a, b] \to \mathcal{M}^n$, the length of the corresponding curve $\Xi(t) = (\gamma(t), \boldsymbol{\xi}(t))$ in $T_1(\mathcal{M}^n)$ is given by

$$\int_a^b \sqrt{\|\dot\gamma(t)\|^2 + \|\nabla_{\dot\gamma(t)}\boldsymbol{\xi}(t)\|^2}\, dt,$$

where $\dot\gamma = \gamma_*\left(\frac{d}{dt}\right)$, ∇ is the Levi–Civita connection, and $\|\cdot\|$ denotes the norm of a vector relative to ds^2.

It is natural that an angle measurement should produce the canonical Sasaki metric defined via the Levi–Civita parallel transport for Riemannian spaces. In a metric space it should be sensitive enough to distinguish admittedly nonparallel directions; in particular, it should agree in a very strong sense with the well-established notion of the (upper) angle for two directions based at the same point in an Aleksandrov space. Finally, as a working hypothesis, directions on a geodesic should be, up to reversal, at angle zero to each other. We present such a construction and discuss some natural modifications.

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It is not clear what an ideal theory could be. The simultaneous achievement of sensitivity and stability may simply be impractical because of the instability of the exponential map. Indeed, for two fixed directions based at nearby points it appears unreasonable for such an angle measurement between the two to be a continuous function of the directions and base points. As an elementary example consider a graph and a triple of directions $\boldsymbol{\xi}$, $\boldsymbol{\zeta}$, and $\boldsymbol{\eta}$ that are tangent to the graph (see Figure 1). Since the polygonal lines XOT and YOT are minimizing geodesics in the graph, according to the principles of the angle measurement, $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$ should be "parallel" to $\boldsymbol{\eta}$ (i.e., in both cases the angle should be zero), while $\boldsymbol{\xi}$ is not parallel to $\boldsymbol{\zeta}$ since the angle between them equals π.

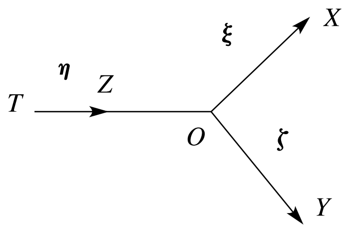

Figure 1

Our major technical apparatus is that of a development of estimates on the metric constructions called quadrilateral cosine and quadrilateral sine. These provide machinery with which to construct a Sasaki metric and appear to be of interest in the general context of Aleksandrov spaces. For example, we show that equidistant cross-bars forming nested isosceles triangles with a fixed vertex are approximately parallel (Lemma 23).

Recent works [5; 6; 7; 8] show the growing interest of analysts in spaces of curvature $\leq K$ in the sense of Aleksandrov. These spaces give a natural generalization of Riemannian manifolds. However, Aleksandrov's spaces are of much more general nature.

In the present paper we propose a construction that in a certain sense extends the notion of the sine of the angle θ to the spaces of curvature bounded above in the sense of Aleksandrov. We call this generalized sine the *quadrilateral sine*. In a metric space a concept of *direction* replaces the notion of a vector in a sphere-bundle. The quadrilateral sine of a pair of directions makes sense in any metric space, as does Aleksandrov's upper angle. However, it can be equal to infinity in the general case. In our paper we prove that, in an $\Re_K$ domain of a space of curvature $\leq K$, the quadrilateral sine possesses a stability property. In other words, it is independent of the choice of the curves specifying a given pair of directions; the quadrilateral sine is always bounded and, for a pair of directions $\boldsymbol{\xi}, \boldsymbol{\zeta}$ emanating from one point, the quadrilateral sine coincides with $2\sin(\angle(\boldsymbol{\xi}, \boldsymbol{\zeta})/2)$. In addition, [11, Prop. 10] ensures that the quadrilateral sine induces the Sasaki metric in the sphere-bundle of a Riemannian manifold. The quadrilateral sine arises in a natural way from the notion of the quadrilateral cosine.

The basis of the definition of the quadrilateral cosine (notation: cosq) is the following generalization of the cosine formula in $\mathbb{E}_3$:

$$\cos\angle\big(\overrightarrow{AX},\overrightarrow{BY}\big)=\frac{\big|\overrightarrow{AY}\big|^2+\big|\overrightarrow{BX}\big|^2-\big|\overrightarrow{AB}\big|^2-\big|\overrightarrow{XY}\big|^2}{2\big|\overrightarrow{AX}\big|\cdot\big|\overrightarrow{BY}\big|},$$

which is a substitute for the Levi–Civita parallel transport in metric spaces. We show that the quadrilateral cosine possesses certain additivity properties (Section 3.2), which are essential in the proofs of our results.

Though cosq can be infinite in a general metric space, the quadrilateral cosine possesses a boundedness property in Aleksandrov spaces that depends surprisingly sharply on Aleksandrov's curvature (Corollaries 11 and 12), and these boundedness estimates are equally vital and interesting in their own right. Indeed, in a space of curvature not greater than zero in the sense of Aleksandrov, 1 is a local bound for the absolute value of the quadrilateral cosine. Conversely, under reasonably general hypotheses that we shall discuss later, a local lower positive bound of Aleksandrov's curvature implies that the quadrilateral cosine will exceed 1 for certain configurations (Proposition 13).

In particular, we establish that a Riemannian space has nonpositive sectional curvature if and only if 1 is a local bound for the absolute value of the quadrilateral cosine (Proposition 14).

Along this line we establish an extremal property of the quadrilateral cosine (Theorem 15): In a geodesic metric space where the absolute value of the quadrilateral cosine is bounded by 1, we show that two pairs of distinct points for which cosq achieves this bound have a convex (geodesic) hull that is either isometric to a trapezoid in $\mathbb{E}_2$ or to a segment of straight line.

In a Riemannian space we can see $\cos\Delta\theta-\operatorname{cosq}(\boldsymbol{\xi},\tilde{\boldsymbol{\xi}})=O(|\mathbf{x}-\tilde{\mathbf{x}}|^2)$. Thus, one cannot use the quadrilateral cosine directly to construct the Sasaki metric. However, in $\mathbb{E}_3$ we see

$$2\sin\angle\frac{\big(\overrightarrow{AX},\overrightarrow{BY}\big)}{2}=\sup_{\overrightarrow{CZ}\in\mathbb{E}_3}\cos\big(\overrightarrow{AX},\overrightarrow{CZ}\big)-\cos\big(\overrightarrow{BY},\overrightarrow{CZ}\big)$$

and, it turns out,

$$2\sin\angle\frac{(\boldsymbol{\xi},\tilde{\boldsymbol{\xi}})}{2}=\sup_{\boldsymbol{\eta}\in T_1(\mathcal{M}^n)}\{\operatorname{cosq}(\boldsymbol{\eta},\boldsymbol{\xi})-\operatorname{cosq}(\boldsymbol{\eta},\tilde{\boldsymbol{\xi}})\},$$

which is a basis of the definition of the generalized sine, gives the correct result in Riemannian spaces.

The quadrilateral cosine is geometrically accessible but, as we see, is inadequate even in the Riemannian case for analytic purposes. The quadrilateral sine, defined by a family of quadrilateral cosines, is harder to compute but it is more satisfactory analytically; for example, it is adequate to define parallel transport in the Riemannian case. For Aleksandrov spaces of curvature bounded from above it is not difficult to show that directions on a geodesic (up to reversal) are "parallel" in the sense that the quadrilateral sine between them equals zero (Remark 8). One of our basic results (see Theorem 21 and Corollary 22) states that, in a space of curvature $\le K$ in the sense of Aleksandrov, the quadrilateral sine locally produces the same distance between a pair of directions stemming from one point as the Aleksandrov upper angle does. This result is not quite simple since, in contrast to the upper angle, computation of the quadrilateral sine involves pairs of points not lying on the geodesics. Thus we show that the angle measurement specified by the quadrilateral sine obeys the principles stated at the beginning of this section.

The quadrilateral cosine (under another name) was introduced in [10]. The construction of a generalized sine was developed in [3; 11] and was applied to the problem of synthetic description of Riemannian geometry. In the present paper we use a modification of that generalized sine because, in a space of curvature $\le K$, the old definition does not give the correct answer for a pair of directions emanating from one point.

2. Basic Definitions

In what follows, for a pair of points A, B in a metric space $(\mathcal{M}, \rho)$ we will denote by $AB = \rho(A, B)$ the distance between A and B.

2.1. Upper Angle between Curves

Let $(\mathcal{M}, \rho)$ be a metric space and let $\mathcal{L}, \mathcal{N}$ be a pair of curves in $(\mathcal{M}, \rho)$ emanating from a point $P \in \mathcal{M}$. Consider $X \in \mathcal{L}$ and $Y \in \mathcal{N}$ $(X, Y \neq P)$. Define $\alpha_0(X, Y)$ by means of the equation

$$\cos \alpha_0(X, Y) = \frac{PX^2 + PY^2 - XY^2}{2PX \cdot PY}.$$

Aleksandrov's upper angle between $\mathcal{L}$ and $\mathcal{N}$ is defined as follows:

$$\cos \bar{\alpha}(\mathcal{L}, \mathcal{N}) = \underline{\lim}_{X, Y \to P} \cos \alpha_0(X, Y).$$

2.2. Space of Directions at a Point

A curve γ starting at a point P has a *direction* if $\bar{\alpha}(\gamma, \gamma) = 0$. Consider the set $\Lambda_P(\mathcal{M})$ of all curves emanating from the point P and having a direction at this point. Let $\gamma_1, \gamma_2 \in \Lambda_P(\mathcal{M})$. We introduce an equivalence relation: $\gamma_1 \sim \gamma_2$ if $\bar{\alpha}(\gamma_1, \gamma_2) = 0$. Then $\Omega_P(\mathcal{M}) = \Lambda_P(\mathcal{M})/(\sim)$ is called a *space of directions*. We denote by $\Pi\colon \Lambda_P(\mathcal{M}) \to \Omega_P(\mathcal{M})$ the canonical projection.

The *distance* $\bar{\alpha}$ between two directions is the upper angle between any of their representatives.

2.3. Quadrilateral Cosine

We will keep the notation $\vec{u} = \overrightarrow{PQ}$ for an ordered pair of points (P, Q) in a metric space $(\mathcal{M}, \rho)$ and $|\vec{u}|$ for the distance between the points P and Q, that is, $|\overrightarrow{PQ}| =$

PQ. Let $\{P, X, Q, Y\}$ be a quadruple of points in a metric space $(\mathcal{M}, \rho)$ such that $P \neq X$ and $Q \neq Y$. The *quadrilateral cosine between* $\overrightarrow{PX}$ *and* $\overrightarrow{QY}$ is defined as

$$\operatorname{cosq}\bigl(\overrightarrow{PX}, \overrightarrow{QY}\bigr) = \frac{PY^2 + QX^2 - XY^2 - PQ^2}{2 \cdot PX \cdot QY}.$$

Remark 1. It follows immediately from the definition that

$$\operatorname{cosq}\bigl(\overrightarrow{PX}, \overrightarrow{QY}\bigr) = -\operatorname{cosq}\bigl(\overrightarrow{XP}, \overrightarrow{QY}\bigr).$$

Remark 2. Let (P, X) and (Q, Y) be two pairs of distinct points in $\mathbb{R}^n$. Observe that

$$\begin{aligned} &PY^2 + QX^2 - XY^2 - PQ^2 \\ &\qquad = \bigl|\overrightarrow{PY}\bigr|^2 + \bigl|\overrightarrow{PX} - \overrightarrow{PQ}\bigr|^2 - \bigl|\overrightarrow{PY} - \overrightarrow{PX}\bigr|^2 - \bigl|\overrightarrow{PQ}\bigr|^2 = 2\overrightarrow{PX} \cdot \overrightarrow{QY}, \end{aligned}$$

whence

$$\operatorname{cosq}\bigl(\overrightarrow{PX}, \overrightarrow{QY}\bigr) = \cos \angle\bigl(\overrightarrow{PX}, \overrightarrow{QY}\bigr).$$

Thus, in a Euclidean space, $\bigl|\operatorname{cosq}\bigl(\overrightarrow{PX}, \overrightarrow{QY}\bigr)\bigr| \le 1$. In a general metric space the quadrilateral cosine can be as large as one wishes, but we will show that a curvature bound affords a bound on the quadrilateral cosine.

Example 1 [3]. On the set $\mathbb{R}^2$ we specify the norm $\|(x, y)\|_1 = |x| + |y|$. In the resulting normed space we consider the rays $\mathbf{r}_1(t) = (0, 1 + t)$ and $\mathbf{r}_2(t) = (1, 1+t)$, $t \ge 0$. Let $P = (0, 1)$, $X = (0, 1+s)$, $Q = (1, 1)$, and $Y = (1, 1+t)$. Then

$$\operatorname{cosq}\bigl(\overrightarrow{PX}, \overrightarrow{QY}\bigr) = \frac{2t + 2s - 2|t - s| + 2st}{2st} = \begin{cases} 1 + 2/t & \text{if } t \ge s, \\ 1 + 2/s & \text{if } s \ge t. \end{cases}$$

Thus, $\operatorname{cosq}\bigl(\overrightarrow{PX}, \overrightarrow{QY}\bigr)$ can be arbitrarily large. By Remark 1, the quadrilateral cosine of $\overrightarrow{PX}$, $\overrightarrow{QY}$ can be an arbitrarily small negative number.

Remark 3. $\operatorname{cosq}\bigl(\overrightarrow{PX}, \overrightarrow{PY}\bigr) = \cos \alpha_0(X, Y)$.

Remark 4. Let $\mathcal{L}, \mathcal{N}$ be a pair of curves emanating from the point P in a metric space $\mathcal{M}$. Consider $X \in \mathcal{L}$ and $Y \in \mathcal{N}$ $(X, Y \neq P)$. Then

$$\cos \bar{\alpha}(\mathcal{L}, \mathcal{N}) = \underline{\lim}_{X,Y \to P} \operatorname{cosq}\bigl(\overrightarrow{PX}, \overrightarrow{PY}\bigr).$$

2.4. Quadrilateral Sine

Let $\vec{u}, \vec{v}, \vec{w} \in \mathcal{M} \times \mathcal{M}$ be three ordered pairs of distinct points. We define the *quadrilateral sine* of the triple $\vec{u}, \vec{v}, \vec{w}$ as follows:

$$\operatorname{sinq}(\vec{u}, \vec{v}; \vec{w}) = |\operatorname{cosq}(\vec{u}, \vec{w}) - \operatorname{cosq}(\vec{v}, \vec{w})|.$$

In a Euclidean space, $\sup_{\vec{w} \neq 0} \operatorname{sinq}(\vec{u}, \vec{v}; \vec{w}) = 2\sin(\angle(\vec{u}, \vec{v})/2)$. This motivates our definition of the quadrilateral sine in what follows.

Although the notion of the quadrilateral cosine of two ordered pairs is clearly very useful and the quadrilateral sine of a triple of ordered pairs is a sensible idea, we should point out that there are several reasonable notions for quadrilateral sine of two directions $\boldsymbol{\xi}, \boldsymbol{\zeta}$, all of which present drawbacks. The question is: What is the appropriate form in which to choose "test vector" $\vec{w}$? We will establish that all sensible choices would coincide for the particular important case of two directions parting at a point.

Two reasonable choices of $\vec{w}$ are:

(1) all $\vec{w} = \overrightarrow{AB}$ with A, B close to a geodesic segment joining points P and Q, where P is the tail of $\vec{u}$ and Q is the tail of $\vec{v}$;

(2) all $\vec{w}$ with A, B in the convex hull $\{X, Y, P, Q\}$ (as defined below), where X, Y are points on a pair of curves tangent to directions $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$ that are arbitrarily close to P and Q.

We remark that even in Aleksandrov spaces of curvature bounded above, the first case fails to yield that the quadrilateral sine of a pair of directions tangent to a geodesic equals zero. That is why we choose case (2) to define the quadrilateral sine of a pair of directions (see Remark 8).

2.5. *Quadrilateral Sine of Directions*

Let P, Q be a pair of points in a metric space $(\mathcal{M}, \rho)$. Assume that any pair P, Q of points in $\mathcal{M}$ can be joined by a geodesic segment. Denote by $\mathcal{G}[P, Q]$ the set of points each of which belongs to a geodesic segment joining the points P and Q. Let $\mathcal{A} \subset \mathcal{M}$. We define $\mathcal{G}[\mathcal{A}]$ as follows:

$$\mathcal{G}[\mathcal{A}] = \bigcup_{P, Q \in \mathcal{A}} \mathcal{G}[P, Q].$$

Denote $\mathcal{A}$ by $\mathcal{G}^0[\mathcal{A}]$ and $\underbrace{\mathcal{G}[\mathcal{G}[\ldots\mathcal{G}[\mathcal{A}]]]}_{n \text{ times}}$ by $\mathcal{G}^n[\mathcal{A}]$.

The *convex hull* of $\mathcal{A}$ is defined as

$$\mathcal{GC}[\mathcal{A}] = \bigcup_{n=0}^{\infty} \mathcal{G}^n[\mathcal{A}].$$

In what follows we assume that any pair of sufficiently close points in $\mathcal{M}$ can be joined by a geodesic segment.

Let $\mathcal{L}$ and $\mathcal{N}$ be a pair of curves in $\mathcal{M}$ emanating from points P and Q, respectively. Consider $X \in \mathcal{L}$ $(X \neq P)$ and $Y \in \mathcal{N}$ $(Y \neq Q)$ such that $PX = QY = h$. Note that if the points P and Q are sufficiently close to each other then the convex hull of the set $\{P, Q, X, Y\}$ is well-defined.

The *quadrilateral sine of* $\mathcal{L}$ *and* $\mathcal{N}$ is defined as

$$\operatorname{sinq}(\mathcal{L}, \mathcal{N}) = \lim_{a \to 0+} \sup_{n=0,1,2\ldots} \varlimsup_{h \to 0} \sup_{A, B \in \mathcal{G}^n[P, Q, X, Y];\, AB = ah} \left|\operatorname{sinq}\left(\overrightarrow{PX}, \overrightarrow{QY}; \overrightarrow{AB}\right)\right|. \quad (1)$$

Let $\boldsymbol{\xi} \in \Omega_P(\mathcal{M})$ and $\boldsymbol{\zeta} \in \Omega_Q(\mathcal{M})$. The *quadrilateral sine of the pair* $(\boldsymbol{\xi}, \boldsymbol{\zeta})$ is defined as

$$\operatorname{sinq}(\boldsymbol{\xi}, \boldsymbol{\zeta}) = \inf_{\mathcal{L} \in \Pi^{-1}(\boldsymbol{\xi}),\, \mathcal{N} \in \Pi^{-1}(\boldsymbol{\zeta})} \operatorname{sinq}(\mathcal{L}, \mathcal{N}).$$

3. Quadrilateral Cosine and Sine in a Space of Curvature $\leq K$

3.1. Domain $\Re_K$ [1]

For more details, see [1], [2], and [3].

By S_K, we denote a complete simply connected 2-dimensional space of curvature K (i.e., a sphere, a plane, or a Lobachevskii plane of curvature K). For a triangle $T = ABC$ in a metric space $(\mathcal{M}, \rho)$, denote by $T^K = A^K B^K C^K$ a triangle on S_K that has sides of the same length as T (for $K > 0$ one must assume that the perimeter of T is no greater than $2\pi/\sqrt{K}$).

An $\Re_K$ domain, abbreviated by $\Re_K$, is a metric space with the following properties.

(i) Any two points in $\Re_K$ can be joined by a geodesic segment.
(ii) Each triangle in $\Re_K$ has nonpositive K-excess, that is, for the angles α, β, γ of the triangle
$$\alpha + \beta + \gamma - (\alpha_K + \beta_K + \gamma_K) \leq 0,$$
where α_K, β_K, and γ_K are the corresponding angles in the triangle T^K.
(iii) If $K > 0$ then the perimeter of each triangle in $\Re_K$ is less than $2\pi/\sqrt{K}$.

By a space of curvature $\leq K$ in the sense of Aleksandrov we understand a metric space, each point of which is contained in some neighborhood of the original space, which is an $\Re_K$ domain.

REMARK 5. Another term for an $\Re_K$ domain is a CAT(K) space. However, we will use Aleksandrov's original notation [1].

We will need the following basic properties of $\Re_K$, established by Aleksandrov [1].

(a) In an $\Re_K$ domain the geodesic segment joining a pair of points is unique.
(b) Between any two geodesic segments in $\Re_K$ starting from one point there exists an angle, that is, the limit $\lim_{X,Y\to P} \alpha_0(X, Y)$.
(c) (Angle comparison theorem): The angles α, β, γ of an arbitrary triangle T in $\Re_K$ are not greater than the corresponding angles $\alpha_K, \beta_K, \gamma_K$ of the triangle T^K on S_K.
(d) (K-concavity): Let X and Y be points on the sides AB and AC of the triangle $T = ABC$ in $\Re_K$, and let X' and Y' be the points on the corresponding sides of the triangle $T^K = A^K B^K C^K$ such that $A^K X' = AX$ and $A^K Y' = AY$. Then $XY \leq X'Y'$.

For a pair of points $A, B \in \Re_K$ we will denote by $\mathcal{AB}$ the (unique) geodesic segment joining A and B.

REMARK 6. Another name for property (d) is CAT(K)-inequality. In fact, in [1] Aleksandrov called K-concavity a property equivalent to (d).

Let $\vec{u} = \overrightarrow{PQ}$ and $0 \leq t \leq 1$. Denote by $t \cdot \vec{u}$ the ordered pair $\overrightarrow{PP_t}$, where P_t is a point on the (unique) geodesic segment $\overrightarrow{PQ}$ such that $PP_t = t \cdot PQ$. Define

$(-1) \cdot \vec{u}$ to be the ordered pair $\overrightarrow{QP}$. If $-1 \le t \le 0$ then $t \cdot \vec{u} = |t|[(-1)\vec{u}]$. Let $\vec{v} = \overrightarrow{QR}$. We define $\vec{u} + \vec{v}$ to be the ordered pair $\overrightarrow{PR}$. If $\vec{z} = \overrightarrow{PR}$ we define $\vec{z} - \vec{u}$ to be equal to $\overrightarrow{QR}$. Observe that, by the definition, $t(\vec{u} \pm \vec{v}) = t\vec{u} \pm t\vec{v}$. For a vector $\vec{u}$ we denote by $|\vec{u}|$ the length of $\vec{u}$, that is, the distance between the points P and Q.

3.2. Averaging Property

We now establish an averaging property for cosq, which—together with an a priori upper bound depending only on curvature—allows us to establish control of differences of cosqs and hence of sinqs.

LEMMA 2. *Let $\vec{u} = \overrightarrow{PQ}$, $\vec{v} = \overrightarrow{QR}$, and $\vec{w} = \overrightarrow{LM}$. Then*

$$\operatorname{cosq}(\vec{w}, \vec{u} + \vec{v}) = \frac{|\vec{u}|}{|\vec{u} + \vec{v}|} \operatorname{cosq}(\vec{w}, \vec{u}) + \frac{|\vec{v}|}{|\vec{u} + \vec{v}|} \operatorname{cosq}(\vec{w}, \vec{v}).$$

Proof. The foregoing equation is equivalent to

$$\frac{LR^2 + PM^2 - MR^2 - PL^2}{2 \cdot ML \cdot PR} = \frac{PQ}{PR} \cdot \frac{LQ^2 + PM^2 - PL^2 - QM^2}{2 \cdot PQ \cdot ML} + \frac{QR}{PR} \cdot \frac{LR^2 + QM^2 - LQ^2 - MR^2}{2 \cdot QR \cdot ML},$$

which is obvious. □

COROLLARY 3. *Let $\vec{u} = \overrightarrow{PQ}$, $\vec{w} = \overrightarrow{LM}$, $\vec{z} = \overrightarrow{PR}$, and $R \neq Q$. Then*

$$\operatorname{cosq}(\vec{w}, \vec{u} - \vec{z}) = \frac{|\vec{u}|}{|\vec{u} - \vec{z}|} \operatorname{cosq}(\vec{w}, \vec{u}) - \frac{|\vec{z}|}{|\vec{u} - \vec{z}|} \operatorname{cosq}(\vec{w}, \vec{z}).$$

We now express cosq of two segments in terms of averages of cosq of subsegments.

Let $\vec{u} = \overrightarrow{PQ}$ and $\vec{w} = \overrightarrow{LM}$. Split the geodesic segment $\mathcal{PQ}$ evenly by the points

$$P = P_0 < P_1 < P_2 < \cdots < P_{m-1} < P_m = Q$$

and the geodesic segment $\mathcal{LM}$ evenly by the points

$$L = L_0 < L_1 < L_2 < \cdots < L_{n-1} < L_n = M.$$

Denote $\overrightarrow{P_{i-1}P_i}$ by $\vec{u}_i$ $(i = 1, \ldots, m)$ and $\overrightarrow{L_{j-1}L_j}$ by $\vec{w}_j$ $(j = 1, \ldots, n)$. For $m = 1$ and $n = 2$, see Figure 2.

COROLLARY 4.

$$\operatorname{cosq}(\vec{w}, \vec{u}) = \frac{1}{mn} \sum_{i=1,\ldots,m;\ j=1,\ldots,n} \operatorname{cosq}(\vec{w}_j, \vec{u}_i).$$

In particular,

$$\operatorname{cosq}(\vec{w}, \vec{u}) = \frac{1}{m} \sum_{i=1}^{m} \operatorname{cosq}(\vec{w}, \vec{u}_i).$$

The following form of averaging is an immediate generalization of Remark 1.

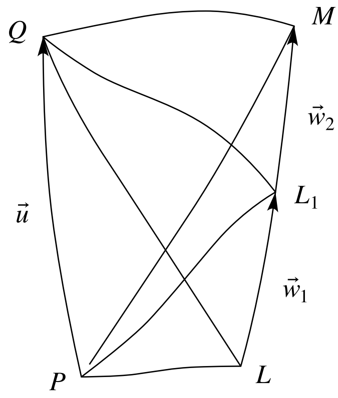

Figure 2

COROLLARY 5. *Let $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n$ form a closed polygon and let $\vec{u}$ be arbitrary. Then*

$$\sum_{i=1}^{n} |\vec{w}_i| \operatorname{cosq}(\vec{u}, \vec{w}_i) = 0.$$

3.3. A Priori Bound for the Quadrilateral Cosine

In contrast to the general case, $\operatorname{cosq}(\vec{u}, \vec{v})$ is uniformly bounded in an $\Re_K$ domain, with a bound depending on the curvature. These bounds are sharp, thus establishing that our results are involved with curvature.

We first mention an elementary formula for a limiting value of cosq in a smooth $\Re_K$ domain.

LEMMA 6. *Let distinct P, Q, X, Y be given. Let P_x be the unique point on the geodesic segment $\mathcal{PX}$ at distance x from P. Let Q_y be the unique point on geodesic segment $\mathcal{QY}$ at distance y from Q. Let $z(x, y) = P_x Q_y$ so that*

$$h(x, y) = \operatorname{cosq}\big(\overrightarrow{PP_x}, \overrightarrow{QQ_y}\big) = \frac{z(x,0)^2 + z(0,y)^2 - z(x,y)^2 - z(0,0)^2}{2xy}.$$

Then, if $z(x, y) \in C^2$ in a neighborhood of $(0, 0)$, *we have*

$$\lim_{x,y \to 0+} \operatorname{cosq}\big(\overrightarrow{PP_x}, \overrightarrow{QQ_y}\big) = -(z_y \cdot z_x + z \cdot z_{xy})|_{(0,0)}.$$

Proof. Let $w(x, y) = z^2(x, y)$. Then

$$\frac{[w(x,y) - w(x,0)] - [w(0,y) - w(0,0)]}{xy}$$
$$= \frac{[w_y(x,0)y + w_{yy}(x,\tilde{y})(y^2/2)] - [w_y(0,0)y + w_{yy}(0,\hat{y})(y^2/2)]}{xy}$$

$$= \frac{w_y(x,0) - w_y(0,0)}{x} + y\frac{w_{yy}(x,\tilde{y}) - w_{yy}(0,\hat{y})}{2x}$$
$$= w_{xy}(0,0) + (w_{xy}(\bar{x},0) - w_{xy}(0,0)) + \frac{y}{2x}[w_{yy}(x,\tilde{y}) - w_{yy}(0,\hat{y})].$$

In a similar way one obtains:

$$\frac{[w(x,y) - w(x,0)] - [w(0,y) - w(0,0)]}{xy}$$
$$= w_{xy}(0,0) + (w_{xy}(0,\bar{y}) - w_{xy}(0,0)) + \frac{x}{2y}[w_{yy}(\tilde{x},y) - w_{yy}(\hat{x},0)].$$

For the bracketed terms we choose the more convenient estimate. Thus, where $x \geq y$ we choose the first bracket, where $y/2x \leq 1/2$ and $w_{yy}(x,\tilde{y}) - w_{yy}(0,\hat{y}) \to 0$ uniformly in x and y by the continuity of w_{yy} at $(0,0)$; similarly, $w_{xy}(0,\bar{y}) - w_{xy}(0,0) \to 0$ uniformly. Recall that $0 \leq \tilde{y} \leq y$, and so on. Of course, where $y > x$, we choose the second estimate. Thus,

$$|h(x,y) + (z_y \cdot z_x + z \cdot z_{xy})|_{(0,0)}|$$
$$\leq \frac{3}{2} \max_{0 \leq s,s' \leq x, 0 \leq t,t' \leq y, i,j=1,2} |w_{ij}(s',t') - w_{ij}(s,t)|, \quad (2)$$

where w_{11} denotes $\partial^2 w/\partial x^2$ and so on. Therefore,

$$\lim_{x,y\to 0} h(x,y) = -\frac{1}{2}\frac{\partial^2(z^2(x,y))}{\partial x\,\partial y} = -(z_y \cdot z_x + z \cdot z_{xy})|_{(0,0)}. \qquad \square$$

We observe that, for this C^2 situation, by identifying cosq as a derivative we have freed ourselves from considering any relationship between x and y as they approach zero. In this note we are concerned with situations much worse than C^2, but we need some C^2 comparisons.

3.3.1. A Priori Bound on a Sphere. In what follows, $K = k^2 > 0$ and

$$S_K^+ = \{\,(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1/K = 1/k^2 \text{ and } z > 0\,\}.$$

We consider a domain in the upper hemisphere of radius properly less than $\pi/2k$ so that all distances encountered are less than π/k and so all geodesic segments are unique.

Let distinct P, Q, X, Y be given. We first calculate

$$\lim_{x,y\to 0+} \operatorname{cosq}\big(\overrightarrow{PP_x}, \overrightarrow{QQ_y}\big) = -(z_y z_x + z z_{xy})|_{(0,0)}$$

as described in Lemma 6. We include a sketch (see Figure 3).

LEMMA 7.

$$\lim_{x,y\to 0+} \operatorname{cosq}\big(\overrightarrow{PP_x}, \overrightarrow{QQ_y}\big) = \frac{kz(0,0)}{\sin kz(0,0)} \sin\xi_x \sin\xi_y + \cos\xi_x \cos\xi_y.$$

Here ξ_x denotes $\angle QPX$ and ξ_y is $\pi - \angle YQP$ as sketched.

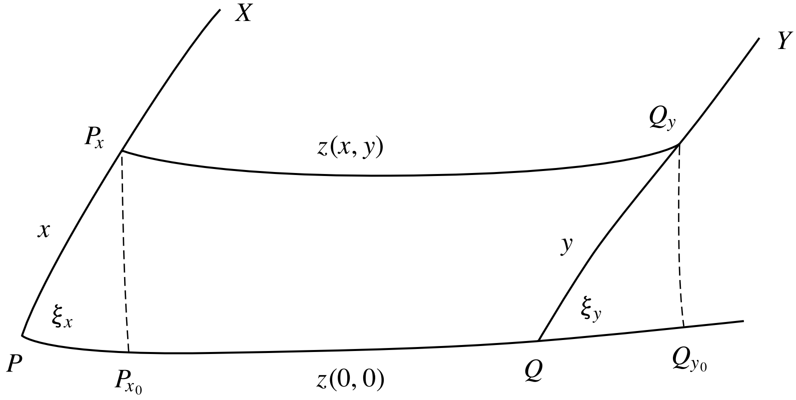

Figure 3

Proof. We prove Lemma 7 for the unit sphere. The adjustment for k is trivial. The second variation formula will yield a calculation; however, direct computation is particularly neat here.

It is clear that, on a sphere, we may break up the arbitrary geodesic segment $\mathcal{P}\mathcal{P}_x$ intersecting $\mathcal{P}\mathcal{Q}$ into a perpendicular meridian $\mathcal{P}_{x_0}\mathcal{P}_x$ and a transversal equatorial segment $\mathcal{P}\mathcal{P}_{x_0}$, and similarly with $\mathcal{Q}\mathcal{Q}_y$ (see Figure 3). Referring to the definition of the sum of ordered pairs (Section 3.1), we see that

$$\overrightarrow{PP_x} = \overrightarrow{PP_{x_0}} + \overrightarrow{P_{x_0}P_x} \quad \text{and} \quad \overrightarrow{QQ_y} = \overrightarrow{QQ_{y_0}} + \overrightarrow{Q_{y_0}Q_y}.$$

By applying twice the averaging property (Lemma 2), we have

$$\begin{aligned}
&\left|\overrightarrow{PP_x}\right| \cdot \left|\overrightarrow{QQ_y}\right| \operatorname{cosq}\left(\overrightarrow{PP_x}, \overrightarrow{QQ_y}\right) \\
&\quad = \left|\overrightarrow{P_{x_0}P_x}\right| \cdot \left|\overrightarrow{Q_{y_0}Q_y}\right| \cdot \operatorname{cosq}\left(\overrightarrow{P_{x_0}P_x}, \overrightarrow{Q_{y_0}Q_y}\right) \\
&\qquad + \left|\overrightarrow{PP_{x_0}}\right| \cdot \left|\overrightarrow{Q_{y_0}Q_y}\right| \cdot \operatorname{cosq}\left(\overrightarrow{PP_{x_0}}, \overrightarrow{Q_{y_0}Q_y}\right) \\
&\qquad + \left|\overrightarrow{P_{x_0}P_x}\right| \cdot \left|\overrightarrow{QQ_{y_0}}\right| \cdot \operatorname{cosq}\left(\overrightarrow{P_{x_0}P_x}, \overrightarrow{QQ_{y_0}}\right) \\
&\qquad + \left|\overrightarrow{PP_{x_0}}\right| \cdot \left|\overrightarrow{QQ_{y_0}}\right| \cdot \operatorname{cosq}\left(\overrightarrow{PP_{x_0}}, \overrightarrow{QQ_{y_0}}\right).
\end{aligned} \tag{3}$$

In order to compute the limits of these four terms, we will first establish the following special cases of Lemma 7.

Case (I) $\mathcal{P}\mathcal{P}_x$ and $\mathcal{Q}\mathcal{Q}_y$ are both meridians, and $\xi_x = \xi_y = \pi/2$. We have

$$\cos z(x, y) = \sin x \sin y + \cos x \cos y \cos z(0, 0),$$

where x and y are measured from the equator along the meridians. Hence, because $z(x, y)$ is bounded away from 0 and $\pi/2$, differentiation yields

$$z_x(0, 0) = z_y(0, 0) = 0 \quad \text{and} \quad z_{xy}(0, 0) = -\frac{1}{\sin z(0, 0)}.$$

Hence $\lim_{x,y\to 0} \operatorname{cosq}\left(\overrightarrow{PP_x}, \overrightarrow{QQ_y}\right) = \frac{z(0,0)}{\sin z(0,0)}$ as claimed.

Case (II) If $\mathcal{P}\mathcal{P}_x$ is a meridian and $\mathcal{Q}\mathcal{Q}_y$ is equatorial ($\xi_x = \pi/2$, $\xi_y = 0$), then $\cos z(x, y) = \cos x \cos(z(0,0) + y)$ and both $z_{xy}(0,0)$ and $z_x(0,0)$ equal zero, whence

$$\lim_{x,y\to 0} \operatorname{cosq}\big(\overrightarrow{PP_x}, \overrightarrow{QQ_y}\big) = 0.$$

Case (III) $\mathcal{P}\mathcal{P}_x$ and $\mathcal{Q}\mathcal{Q}_y$ are both equatorial, and $\xi_x = \xi_y = 0$. Then

$$\lim_{x,y\to 0} \operatorname{cosq}\big(\overrightarrow{PP_x}, \overrightarrow{QQ_y}\big) = 1,$$

since $\operatorname{cosq} \equiv 1$ there.

Let $\tilde{P}_s$, $\tilde{Q}_t$ be unique points on geodesic segments $\mathcal{P}_{x_0}\mathcal{P}_x$ and $\mathcal{Q}_{y_0}\mathcal{Q}_y$ at distances s and t respectively. By (I),

$$\lim_{s,t\to 0} \operatorname{cosq}\big(\overrightarrow{P_{x_0}\tilde{P}_s}, \overrightarrow{Q_{y_0}\tilde{Q}_t}\big) = \lim_{x,y\to 0} \frac{P_{x_0}Q_{y_0}}{\sin P_{x_0}Q_{y_0}}.$$

By (2) of Lemma 6, the function $\operatorname{cosq}\big(\overrightarrow{P_{x_0}\tilde{P}_s}, \overrightarrow{Q_{y_0}\tilde{Q}_t}\big)$ converges uniformly (relative to x and y) to its limit as $s, t \to 0$. Therefore

$$\lim_{x,y\to 0} \operatorname{cosq}\big(\overrightarrow{P_{x_0}P_x}, \overrightarrow{Q_{y_0}Q_y}\big) = \lim_{x,y\to 0} \frac{P_{x_0}Q_{y_0}}{\sin P_{x_0}Q_{y_0}} = \frac{PQ}{\sin PQ} = \frac{z(0,0)}{\sin z(0,0)}.$$

In a similar way, by invoking (II) and (2), one can see that

$$\lim_{x,y\to 0} \operatorname{cosq}\big(\overrightarrow{P_{x_0}P_x}, \overrightarrow{QQ_{y_0}}\big) = 0.$$

Finally, by (III),

$$\operatorname{cosq}\big(\overrightarrow{PP_{x_0}}, \overrightarrow{QQ_{y_0}}\big) = 1.$$

Observe that

$$\lim_{x\to 0} \frac{P_{x_0}P_x}{PP_x} = \sin \angle XPQ = \sin \xi_x,$$
$$\lim_{x\to 0} \frac{PP_{x_0}}{PP_x} = \cos \angle XPQ = \cos \xi_x,$$

and similarly for ξ_y, by the boundedness of curvature. Now we take the limit on the four terms that summed to $\operatorname{cosq}\big(\overrightarrow{PP_x}, \overrightarrow{QQ_y}\big)$. This yields

$$\lim_{x,y\to 0} \operatorname{cosq}\big(\overrightarrow{PP_x}, \overrightarrow{QQ_y}\big) = \sin \xi_x \sin \xi_y \frac{z(0,0)}{\sin z(0,0)} + \cos \xi_x \cos \xi_y,$$

as claimed in Lemma 7. □

Finally, in the case of two geodesics joining PQ at right angles from the same side, note that $\overrightarrow{PP_x}$ and $\overrightarrow{QQ_y}$ are parallel under parallel transport but that

$$\lim_{x,y\to 0+} \operatorname{cosq}\big(\overrightarrow{PP_x}, \overrightarrow{QQ_y}\big) = \frac{k(PQ)}{\sin k(PQ)},$$

which goes to infinity as $k(PQ) \to \pi$; hence our curvature dependence for the bound on cosq is sharp.

Because $z/(\sin z)$ is nondecreasing, we have the following corollary.

COROLLARY 8.

$$\lim_{X\to P,Y\to Q} \operatorname{cosq}\left(\overrightarrow{PX}, \overrightarrow{QY}\right) \leq \frac{kz(0,0)}{\sin kz(0,0)}.$$

We will keep the following useful notation:

$$\hat{z} = \hat{z}\left(\overrightarrow{PX}, \overrightarrow{QY}\right) = \sup\{\, z(x,y) \mid P_x \in \mathcal{PX},\ Q_y \in \mathcal{QY}\}.$$

Recall that $z(0,0) = PQ = \left|\overrightarrow{PQ}\right|$ and $z(PX, QY) = XY = \left|\overrightarrow{XY}\right|$.

LEMMA 9. *Let* $z \in (0, \pi/k)$ *and* $x, y \in (0, \pi/2k]$. *Then*

$$\left|\operatorname{cosq}\left(\overrightarrow{PX}, \overrightarrow{QY}\right)\right| \leq \frac{\kappa\hat{z}}{\sin\kappa\hat{z}}.$$

Proof. For $N > 0$, split the geodesic segments $\mathcal{PX}$ and $\mathcal{QY}$ by points

$$P = X_0 < X_1 < X_2 < \cdots < X_{n-1} < X_n = X,$$
$$Q = Y_0 < Y_1 < Y_2 < \cdots < Y_{n-1} < Y_n = Y,$$

so that

$$x_n = X_{i-1}X_i = \frac{PX}{n} \quad \text{and} \quad y_n = Y_{j-1}Y_j = \frac{QY}{n}, \quad i,j = 1,2,\ldots,n.$$

By Theorem 6 and Corollary 8, for an $\varepsilon > 0$ there is an n_ε such that

$$\operatorname{cosq}\left(\overrightarrow{X_{i-1}X_i}, \overrightarrow{Y_{j-1}Y_j}\right) \leq \frac{k\hat{z}}{\sin k\hat{z}} + \varepsilon, \quad i,j = 1,2,\ldots,n_\varepsilon - 1.$$

By Corollary 4,

$$\begin{aligned}\operatorname{cosq}\left(\overrightarrow{PX}, \overrightarrow{QY}\right) &= \frac{1}{n_\varepsilon^2}\sum_{i,j=1}^{n_\varepsilon} \operatorname{cosq}\left(\overrightarrow{X_{i-1}X_i}, \overrightarrow{Y_{j-1}Y_j}\right)\\ &\leq \frac{1}{n_\varepsilon^2}\sum_{i,j=1}^{n_\varepsilon}\left(\frac{k\hat{z}}{\sin k\hat{z}} + \varepsilon\right) = \frac{k\hat{z}}{\sin k\hat{z}} + \varepsilon.\end{aligned}$$

As ε is an arbitrary positive number, we have established Lemma 9. □

3.3.2. General Case. We remark that $\operatorname{diam}(\Re_K) < \pi/\sqrt{K}$ if $K > 0$, since otherwise there is a triangle in $\Re_K$ of perimeter no less than $2\pi/\sqrt{K}$.

LEMMA 10. *Let* P, X, Q, Y *be a quadruple of points in an* $\Re_K$ *domain, and let* $x = PX$ *and* $y = QY$. *Assume that* $PQ > 0$ *and that both*

$$0 < x, y < \pi/2\sqrt{K} \quad \text{and} \quad x + y + PQ < \pi/\sqrt{K}$$

if $K = k^2 > 0$. *Then there is a convex quadrangle* $P'X'Y'Q'$ *on* S_K^+ *if* $K > 0$, *and on a Euclidean plane if* $K \leq 0$, *such that*

$$\operatorname{cosq}\left(\overrightarrow{PX}, \overrightarrow{QY}\right) \leq \operatorname{cosq}\left(\overrightarrow{P'X'}, \overrightarrow{Q'Y'}\right).$$

Proof. Lemma 10 is an immediate consequence of a theorem by Reshetnyak [12]. Indeed, consider a closed broken line $PXYQP$ in the domain $\Re_K$. By the hypothesis of the lemma, the length of the line $PXYQP$ is less than $2\pi/k$ if $K > 0$. This allows us to apply the results of [12]. By [12, Thm.] there is a convex domain D on S_K^+ if $K > 0$, and on a Euclidean plane if $K \le 0$, as well as a nonexpanding map $\phi\colon D \to \Re_K$ that is length-preserving on the boundary ∂D of the domain D. Properties of the map ϕ ensure that ∂D is a geodesic quadrangle $P'X'Y'Q'$, where $PX = P'X'$, $XY = X'Y'$, $QY = Q'Y'$ and $PQ = P'Q'$. Because ϕ is a nonexpanding map,

$$PY \le P'Y' \quad \text{and} \quad QX \le Q'X'.$$

Thus,

$$\begin{aligned}\operatorname{cosq}\big(\overrightarrow{PX}, \overrightarrow{QY}\big) &= \frac{PY^2 + QX^2 - XY^2 - PQ^2}{2 \cdot PX \cdot QY} \\ &\le \frac{P'Y'^2 + Q'X'^2 - X'Y'^2 - P'Q'^2}{2 \cdot P'X' \cdot Q'Y'} = \operatorname{cosq}\big(\overrightarrow{P'X'}, \overrightarrow{Q'Y'}\big). \quad \square\end{aligned}$$

In what follows, by $\hat{z} = \hat{z}\big(\overrightarrow{PX}, \overrightarrow{QY}\big)$ we understand $\hat{z}\big(\overrightarrow{P'X'}, \overrightarrow{Q'Y'}\big)$ (see Section 3.3.1). We will refer to $\hat{z}$ as the *maximal distance of the configuration* $\{P, X, Q, Y\}$.

Invoking Corollary 4, by Remark 1 and Lemmas 9 and 10 we have the following.

COROLLARY 11. *For* $K = k^2 > 0$,

$$\big|\operatorname{cosq}\big(\overrightarrow{PX}, \overrightarrow{QY}\big)\big| \le \frac{k\hat{z}}{\sin k\hat{z}}$$

provided that $0 < x, y < \pi/2k$ *and* $PQ > 0$, *where* $\hat{z} = \hat{z}\big(\overrightarrow{PX}, \overrightarrow{QY}\big)$.

Our next corollary follows by virtue of Lemma 10 and Remarks 1 and 2.

COROLLARY 12. *For* $K \le 0$,

$$\big|\operatorname{cosq}\big(\overrightarrow{PX}, \overrightarrow{QY}\big)\big| \le 1.$$

3.4. Sign of the Curvature and Bounds of the Quadrilateral Cosine

At this point we note that bounds below on the curvature greater than $\kappa > 0$ in the sense of Aleksandrov guarantee a failure of the bound $\big|\operatorname{cosq}\big(\overrightarrow{PX}, \overrightarrow{QY}\big)\big| \le 1$ with very moderate side conditions. We observe that the proof technique does not require the full hypotheses, but we are not pursuing this line in the present paper and remark that Riemannian spaces are an easy special case.

Let $\mathcal{L}, \mathcal{N}$ be a pair of geodesic segments in a metric space $(\mathcal{M}, \rho)$ with a common starting point $P \in \mathcal{M}$. On $\mathcal{L}$ and $\mathcal{N}$ (respectively) we choose arbitrary points X and Y that are different from P. Let $x = PX$, $y = PY$, and $z = XY$. Let T^κ be a triangle $P^\kappa X^\kappa Y^\kappa$ in $\mathcal{S}_\kappa$ such that $P^\kappa X^\kappa = x$, $P^\kappa Y^\kappa = y$, and $X^\kappa Y^\kappa = z$. Set $\gamma^\kappa_{\mathcal{L}N}(x, y) = \angle X^\kappa P^\kappa Y^\kappa$.

An $\Re_\kappa^+$ domain, abbreviated by $\Re_\kappa^+$, is a metric space with the following properties.

(i) Any two points in $\Re_\kappa^+$ can be joined by a geodesic segment.

(ii) (κ-convexity): For any two geodesic segments $\mathcal{L}$ and $\mathcal{N}$ in $\Re_\kappa^+$ emanating from a common point P, the angle $\gamma_{\mathcal{LN}}^\kappa(x, y)$ is a nonincreasing function of x and y. That is,

$$\gamma_{\mathcal{LN}}^\kappa(x', y') \le \gamma_{\mathcal{LN}}^\kappa(x, y)$$

when $x \le x'$, $y \le y'$.

The property of κ-convexity implies:

(A) between any two geodesic segments in $\Re_\kappa^+$ starting from one point there is an angle—that is, the limit $\alpha(\mathcal{L}, \mathcal{N}) = \lim_{x,y\to 0} \gamma_{\mathcal{LN}}^\kappa(x, y)$ exists and is independent of κ;

(B) the angles α, β, γ of an arbitrary triangle T in $\Re_\kappa^+$ of perimeter less than $2\pi/\sqrt{\kappa}$ are not less than the corresponding angles $\alpha_\kappa, \beta_\kappa, \gamma_\kappa$ of the triangle T^κ in $\mathcal{S}_\kappa$ with the same lengths of sides as T

(see [3]).

Let $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ be geodesic segments emanating from the common point, and let $\mathcal{L}_1$ and $\mathcal{L}_3$ be branches of the same geodesic segment. Then the angles $\angle(\mathcal{L}_1, \mathcal{L}_2)$ and $\angle(\mathcal{L}_3, \mathcal{L}_2)$ are called the *adjacent* angles.

(C) The sum of adjacent angles is equal to π.

A space of *curvature bounded from below* is a metric space with intrinsic metric each point of which is contained in some neighborhood of the original space, which is an $\Re_\kappa^+$ domain for some κ. The notion of a space of curvature bounded from below is due to Aleksandrov [1].

It is useful to note that an analog (K-convexity) of the K-concavity described in (d) of Section 3.1 for $\Re_K$ domains holds for $\Re_\kappa^+$ domains.

Let $(\mathcal{M}, \rho)$ be a metric space and let $P \in \mathcal{M}$. We say that geodesics are *locally extendable at the point* P if there is an $r > 0$ such that each geodesic segment $\mathcal{PX}$ of length less than r can be extended to a geodesic segment $\mathcal{PX}'$ of length r in $(\mathcal{M}, \rho)$ for which X is an internal point.

PROPOSITION 13. *Let* $(\mathcal{M}, \rho)$ *be an* $\Re_\kappa^+$ *domain and let* $P \in \mathcal{M}$. *Let geodesics be locally extendable at the point* P *and suppose that there exist triangles in* $(\mathcal{M}, \rho)$ *with* P *as a vertex with sides* $\lambda, \lambda, \sqrt{2}\lambda$ *for sufficiently small* $\lambda > 0$. *Then, in a neighborhood of the point* P, *there exists a triple* $\{B, C, E\}$ *of distinct points such that* $PB > 0$ *and*

$$\operatorname{cosq}\left(\overrightarrow{PB}, \overrightarrow{CE}\right) > 1.$$

Proof. (I) In a neighborhood of the point P, construct a triangle $T = PBC$ with sides $\lambda, \lambda, \sqrt{2}\lambda$. Let D be the midpoint of the geodesic segment $\mathcal{BC}$. Assume that $PD < \sqrt{2}\lambda$. Extend the geodesic segment $\mathcal{PD}$ through the point D to a geodesic segment $\mathcal{PE}$ of length $\sqrt{2}\lambda$; since geodesics are locally extendable at P, this is possible if λ is sufficiently small. Note that, if $PD \ge \sqrt{2}\lambda$, our extension to $\mathcal{PE}$ is not necessary because then

$$\operatorname{cosq}\big(\overrightarrow{PB}, \overrightarrow{CD}\big) = \frac{PD^2 + (\sqrt{2}\lambda)^2 - \lambda^2 - \left(\dfrac{\lambda}{\sqrt{2}}\right)^2}{2\lambda\left(\dfrac{\lambda}{\sqrt{2}}\right)} = \frac{\left(\dfrac{PD}{\lambda}\right)^2 + \dfrac{1}{2}}{\sqrt{2}}$$
$$\geq \frac{5}{2\sqrt{2}} > 1.$$

We claim that, for sufficiently small λ,

$$0 < BE < \lambda$$

(see Figure 4). Let $B = E$. Then $PD + DB = PD + DE = PB$, since $\mathcal{PE}$ is a geodesic segment. However, $PD + DE = \sqrt{2}\lambda > \lambda = PB$ and so we have reached a contradiction. Hence $B \neq E$, whence $BE > 0$.

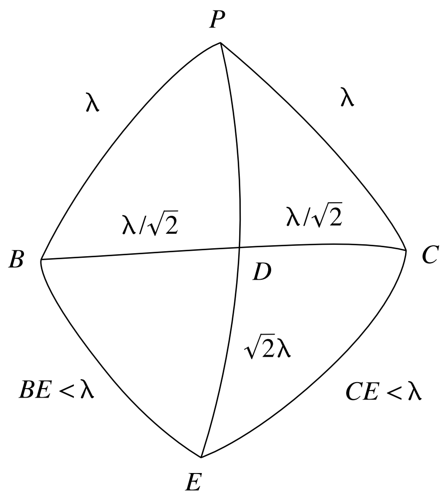

Figure 4

Now consider a Euclidean triangle $T' = P'D'B'$ having the same lengths of sides as the triangle $T = PDB$. Extend the segment $P'D'$ through the point D' to the segment $P'\tilde{E}$ of length $\sqrt{2}\lambda$, and denote by $\tilde{T}$ the Euclidean triangle $P'B'\tilde{E}$. We claim that

$$B'\tilde{E} \leq \lambda.$$

Indeed, $\angle B'P'\tilde{E} < \pi/2$ since $\angle B'P'\tilde{E} = \angle B'P'D'$ and $\lambda^2 + PD^2 - \frac{1}{2}\lambda^2 > 0$. Now observe that $B'\tilde{E}$ is maximized by the choice $P'D' = (\sqrt{2}/2)\lambda$ and so $B'\tilde{E} \leq \lambda$.

Next consider a Euclidean triangle $T'' = B'D'E'$ having the same lengths of sides as the triangle BDE. Let $\alpha = \angle BDP$ and $\beta = \angle BDE$. By (B),

$$\alpha \geq \alpha_\kappa \quad \text{and} \quad \beta \geq \beta_\kappa,$$

where α_κ is the corresponding angle in a triangle in $\mathcal{S}_\kappa$ having the same lengths of sides as the triangle PDB. Since the triangle $B'D'P'$ is nondegenerate,

$$\alpha_\kappa > \alpha_0 = \angle B'D'P'$$

when λ is sufficiently small. Since

$$\beta_\kappa \geq \beta_0 = \angle B'D'E',$$

we arrive at the inequality

$$\alpha + \beta > \alpha_0 + \beta_0. \tag{4}$$

By (C), $\alpha + \beta = \pi$ and we obtain that

$$\alpha_0 + \beta_0 < \pi.$$

Thus, the angle

$$\angle P'D'E' = \alpha_0 + \beta_0$$

in the Euclidean quadrangle $P'B'E'D'$ composed of the triangles T' and T'' is less than π. By rectifying the polygonal line $P'D'E'$ (preserving all four side lengths), we obtain a Euclidean triangle $P''E''B''$ with

$$P''E'' = PE, \quad B''E'' = BE, \quad P''B'' = PB, \quad \text{and} \quad \angle B''P''E'' < \angle B'P'\tilde{E}.$$

The last inequality implies that

$$BE = B''E'' < B'\tilde{E} \leq \lambda.$$

In a similar way, we obtain the inequality

$$0 < CE < \lambda.$$

(II) We see that

$$\operatorname{cosq}\left(\overrightarrow{PB}, \overrightarrow{CE}\right) = \frac{PE^2 + BC^2 - PC^2 - BE^2}{2PB \cdot CE} = \frac{2\lambda^2 + 2\lambda^2 - \lambda^2 - BE^2}{2\lambda \cdot CE}.$$

By (I),

$$\frac{2\lambda^2 + 2\lambda^2 - \lambda^2 - BE^2}{2\lambda \cdot CE} > \frac{2\lambda^2 + 2\lambda^2 - \lambda^2 - \lambda^2}{2\lambda^2} = 1.$$

Thus,

$$\operatorname{cosq}\left(\overrightarrow{PB}, \overrightarrow{CE}\right) > 1.$$

This completes the proof. $\square$

REMARK 7. Note that $\mathcal{CE}$ somewhat resembles the result of a half-step in Cartan's ladder construction of a parallel translate of $\mathcal{PB}$. We note also that, whenever geodesics are not bi-point unique, it is easy to realize $|\operatorname{cosq}| > 1$. Indeed, suppose that a pair of distinct points P and Q can be joined by a pair of distinct geodesic segments $\mathcal{L}$ and $\mathcal{L}'$. We let C and C' denote the midpoints of the geodesic segments $\mathcal{L}$ and $\mathcal{L}'$, respectively. Without loss of generality, one can assume that $C \neq C'$. Then

$$\operatorname{cosq}\left(\overrightarrow{PC}, \overrightarrow{C'Q}\right) = \frac{CC'^2 + PQ^2 - PQ^2/4 - PQ^2/4}{2(PQ^2/4)} = 1 + 2\frac{CC'^2}{PQ^2} > 1.$$

We do not bother with this common pathology, which occurs even locally in Aleksandrov spaces of positive curvature.

PROPOSITION 14. *A Riemannian space $\langle \mathbf{M}, g\rangle$ is of nonpositive sectional curvature if and only if each point $Q \in \mathbf{M}$ has a neighborhood such that, for each quadruple $\{A, B, C, D\}$ of distinct points in this neighborhood, the absolute value of the quadrilateral cosine is bounded by* 1; *that is,*

$$\left|\frac{AC^2 + BD^2 - AC^2 - BD^2}{2AB \cdot CD}\right| \le 1.$$

Proof. (I) It is known [1] that a Riemannian space $\langle \mathbf{M}, g\rangle$ of nonpositive curvature is an Aleksandrov space of curvature ≤ 0 (see also [3, Cor. 7.1]). Thus, each point $Q \in \mathbf{M}$ has a neighborhood that is an $\Re_0$ domain. Then Corollary 12 implies that locally the absolute value of the quadrilateral cosine is bounded by 1.

(II) Assume that there exists a point $P \in \mathbf{M}$ and a 2-dimensional section $\sigma \subset \mathbf{M}_P$ such that the sectional curvature $K_\sigma(P) > 0$. We cannot apply Proposition 13 directly because we do not assume that the sectional curvature of $\mathbf{M}$ is strictly positive at the point P for each 2-dimensional section in $\mathbf{M}_P$. However, we will show that a minor modification of arguments of Proposition 13 yields the desired contradiction. It is obvious that there are unit vectors $X, Y \in \mathbf{M}_P$ such that, for sufficiently small $\lambda > 0$,

$$\operatorname{Span}\{X, Y\} = \sigma, \qquad \operatorname{dist}(\exp_P \lambda X, \exp_P \lambda Y) = \sqrt{2}\lambda.$$

Let $B = \exp_P \lambda X$ and $C = \exp_P \lambda Y$. Then, as in the proof of Proposition 13, we define the points D and E.

Let $Z = \exp_P^{-1}(D)$. It is well known that the angle between bivectors $X \wedge Y$ and $X \wedge Z$ converges to 0 as $\lambda \to 0$. Thus, without loss of generality we can assume that $K_{X\wedge Z} \ge c > 0$ for all sufficiently small λ.

Let

$$X' = \exp_D^{-1}(C), \qquad Z' = \exp_D^{-1}(E),$$

and let X'' and Z'' be vectors in $\mathbf{M}_p$ that are results of the parallel translation of the vectors X' and Z' (respectively) along the geodesic segment $\mathcal{DP}$. Consider bivectors $X \wedge Z$, $X' \wedge Z'$, and $X'' \wedge Z''$. Define the angle between bivectors $X \wedge Z$ and $X' \wedge Z'$ (notation: $\alpha(X \wedge Z, X' \wedge Z')$) to be the angle between bivectors $X \wedge Z$ and $X'' \wedge Z''$. We now have (see [9, Lemma 9.8])

$$\lim_{\lambda\to 0} \alpha(X \wedge Z, X' \wedge Z') = 0,$$

whence

$$K_{X'\wedge Z'}(D) \ge c/2 > 0$$

for all sufficiently small λ.

In what follows we will keep the notation of the proof of Proposition 13. By (29) in [4],

$$\lim_{\lambda\to 0} \frac{\alpha - \alpha_0}{\sigma_\lambda} = \lim_{\lambda\to 0} \frac{\beta - \beta_0}{\tilde{\sigma}_\lambda} = \frac{1}{3} K_{X'\wedge Z'}(D) > 0,$$

where σ_λ is the area of the Euclidean triangle $P'B'D'$ and $\tilde{\sigma}_\lambda$ is the area of the Euclidean triangle $B'D'E'$. Thus, inequality (4) holds for sufficiently small λ. Then we repeat the remaining part of Proposition 13 to prove the inequality $\operatorname{cosq}\big(\overrightarrow{PB}, \overrightarrow{CE}\big) > 1$.

The proof of the proposition is complete. $\square$

3.5. An Extremal Property of cosq

In a space where $|\operatorname{cosq}| \leq 1$, in particular in an $\Re_0$ domain, if two geodesic segments $\mathcal{L}$ and $\mathcal{N}$ satisfy $\operatorname{cosq}(\mathcal{L}, \mathcal{N}) = 1$ then the convex hull of $\mathcal{L}$ and $\mathcal{N}$ is either a quadrilateral in $\mathbb{E}^2$ or a section of a geodesic. This will be proved with the averaging and comparison techniques. Theorem 15 is yet another version of the ubiquitous parallelogram law.

THEOREM 15. *Let $(\mathcal{M}, \rho)$ be a metric space such that every pair of points can be joined by a geodesic segment. For every quadruple of points $P, X, Q, Y \in \mathcal{M}$ $(P \neq X,\ Q \neq Y)$, let*

$$\big|\operatorname{cosq}\big(\overrightarrow{PX}, \overrightarrow{QY}\big)\big| \leq 1.$$

Let A, B, C, D be a quadruple of points in $\mathcal{M}$ such that $A \neq B$, $C \neq D$, and

$$\operatorname{cosq}\big(\overrightarrow{AB}, \overrightarrow{CD}\big) = 1.$$

Then the convex hull $\mathcal{GC}[A, B, C, D]$ of the set $\{A, B, C, D\}$ is either isometric to a quadrilateral in a Euclidean plane $\mathbb{E}^2$ or a segment of straight line.

We preface the proof of the theorem by noting that, since $|\operatorname{cosq}| \leq 1$, any pair of points in $\mathcal{M}$ can be joined by at most one geodesic segment (see Remark 7). The proof of the theorem will be done in several steps.

In what follows we assume that the set $\{A, B, C, D\}$ cannot be isometrically embedded into $\mathbb{R}$, since otherwise $\mathcal{GC}[A, B, C, D]$ is isometric to a segment of straight line.

3.5.1. Averaging Principle. Let A, B be the endpoints of a geodesic segment $\mathcal{L}$, and let C, D be the endpoints of the geodesic segment $\mathcal{N}$ in $\mathcal{M}$, such that $A \neq B$ and $C \neq D$. Then

$$\operatorname{cosq}\big(\overrightarrow{AB}, \overrightarrow{CD}\big) = 1 \quad \text{and} \quad |\operatorname{cosq}| \leq 1$$

imply $\operatorname{cosq}\big(\overrightarrow{\tilde{A}\tilde{B}}, \overrightarrow{\tilde{C}\tilde{D}}\big) = 1$ for any $\tilde{A}, \tilde{B} \in \mathcal{L}$ $(\tilde{A} \neq \tilde{B})$ and $\tilde{C}, \tilde{D} \in \mathcal{N}$ $(\tilde{C} \neq \tilde{D})$ such that $AB = A\tilde{A} + \tilde{A}\tilde{B} + \tilde{B}B$ and $CD = C\tilde{C} + \tilde{C}\tilde{D} + \tilde{D}D$.

Proof. Referring to the definition of the sum of ordered pairs (Section 3.1), we see that $\overrightarrow{AB} = \overrightarrow{A\tilde{A}} + \overrightarrow{\tilde{A}\tilde{B}} + \overrightarrow{\tilde{B}B}$, $\overrightarrow{CD} = \overrightarrow{C\tilde{C}} + \overrightarrow{\tilde{C}\tilde{D}} + \overrightarrow{\tilde{D}D}$. By the averaging property (Lemma 2),

$$\begin{aligned} 1 = \operatorname{cosq}\big(\overrightarrow{AB}, \overrightarrow{CD}\big) &= \operatorname{cosq}\big(\overrightarrow{AB}, \overrightarrow{C\tilde{C}} + \overrightarrow{\tilde{C}\tilde{D}} + \overrightarrow{\tilde{D}D}\big) \\ &= \operatorname{cosq}\big(\overrightarrow{AB}, \overrightarrow{C\tilde{C}}\big)\frac{C\tilde{C}}{CD} + \operatorname{cosq}\big(\overrightarrow{AB}, \overrightarrow{\tilde{C}\tilde{D}} + \overrightarrow{\tilde{D}D}\big)\frac{\tilde{C}D}{CD}. \end{aligned}$$

Since $|\mathrm{cosq}| \leq 1$ and $C\tilde{C}/CD + \tilde{C}D/CD = 1$, it follows that $\mathrm{cosq}\big(\overrightarrow{AB}, \overrightarrow{\tilde{C}\tilde{D}} + \overrightarrow{\tilde{D}D}\big) = 1$. Lemma 2 is used in a similar way to yield $\mathrm{cosq}\big(\overrightarrow{AB}, \overrightarrow{\tilde{C}D}\big) = 1$ and then $\mathrm{cosq}\big(\overrightarrow{\tilde{B}B}, \overrightarrow{\tilde{C}\tilde{D}}\big) = 1$ as claimed. □

3.5.2. *An Isometric Embedding into $\mathbb{E}^2$ as a Parallelogram*

LEMMA 16. *Let $AB = CD$, $\mathrm{cosq}\big(\overrightarrow{AB}, \overrightarrow{CD}\big) = 1$, and $|\mathrm{cosq}| \leq 1$. Then the set $\{A, B, C, D\}$ can be* isometrically embedded into $\mathbb{E}^2$ as a parallelogram; *that is, there is a parallelogram $A'B'C'D'$ in $\mathbb{E}^2$ such that*

$$AB = A'B', \quad AC = A'C', \quad BD = B'D',$$
$$CD = C'D', \quad BC = B'C', \quad AD = A'D'.$$

Proof. Since $\mathrm{cosq}\big(\overrightarrow{AB}, \overrightarrow{CD}\big) = 1$,

$$AD^2 + BC^2 - BD^2 - AC^2 = 2AB^2.$$

If $BD \neq AC$ then $BD^2 + AC^2 > 2BD \cdot AC$, and so

$$AD^2 + BC^2 - 2AB^2 > 2BD \cdot AC,$$

whence $\mathrm{cosq}\big(\overrightarrow{AC}, \overrightarrow{BD}\big) > 1$, a contradiction. Hence $BD = AC$.

Now we consider a Euclidean parallelogram having vertices A', B', C', D' with all distances except possibly AD preserved; that is, $AB = A'B'$, $AC = A'C'$, and $BC = B'C'$. Since evidently

$$\mathrm{cosq}\big(\overrightarrow{AB}, \overrightarrow{CD}\big) = 1 = \cos\big(\overrightarrow{A'B'}, \overrightarrow{C'D'}\big) = \mathrm{cosq}\big(\overrightarrow{A'B'}, \overrightarrow{C'D'}\big)$$

and $BD = AC = A'C' = B'D'$, we see that $A'D'$ is forced: $AD = A'D'$. □

3.5.3. Proof of Theorem 15 when $AB = CD$. Partition evenly the geodesic segments $\mathcal{AB}$ and $\mathcal{CD}$ into n arcs by points $A_0 = A, A_1, \ldots, A_{n-1}, A_n = B$, and $C_0 = C, C_1, \ldots, C_{n-1}, C_n = D$, respectively. Recall that there is unique geodesic segment joining points A_i and C_i. Partition evenly every geodesic segment $\mathcal{A}_i\mathcal{C}_i$ into n arcs by points $A_{i,0} = A_i, A_{i,1}, \ldots, A_{i,n-1}$, and $A_{i,n} = C_i$. Let $\mathcal{S}_n$ denote the set $\{\, A_{i,j} \mid i, j = 0, 1, \ldots, n \,\}$.

By Lemma 16, the set $\{A, B, C, D\}$ can be isometrically embedded into $\mathbb{E}^2$ as a parallelogram $A'B'C'D'$. In a similar way we construct the set $\mathcal{S}_n'$ of points $A'_{i,j}$ $(i, j = 1, 2, \ldots, n)$ in the convex hull $\mathcal{P}'$ of the points A', B', C', D', which divide the complete parallelogram $\mathcal{P}'$ into n^2 similar parallelograms. Define the map $f_n \colon \mathcal{S}_n' \to \mathcal{S}_n$ by

$$f_n(A'_{i,j}) = A_{i,j}.$$

For a pair of points $X, Y \in \mathcal{S}_n'$, set $\rho_E(X,Y) = |X - Y|$.

A. CLAIM. *The map $f_n \colon (\mathcal{S}_n', \rho_E) \to (\mathcal{S}_n, \rho)$ is an isometry.*

Proof. (I) We first see that $\rho(A_{i,0}, A_{i,n}) = \rho(A_{0,0}, A_{0,n}) = \rho(A, C)$. Indeed, by the averaging principle $\operatorname{cosq}\big(\overrightarrow{A_{0,0}A_{i,0}}, \overrightarrow{A_{0,n}A_{i,n}}\big) = 1$ and $A_{0,0}A_{i,0} = A_{0,n}A_{i,n}$, so by Lemma 16 it follows that $A_{0,0}, A_{i,0}, A_{i,n}, A_{0,n}$ embed as vertices of a parallelogram and $\rho(A_{i,0}, A_{i,n}) = \rho(A_{0,0}, A_{0,n})$ as claimed. Hence, for every $i, j = 0, 1, \ldots, n-1$,

$$\rho(A_{i,j}, A_{i,j+1}) = \rho(A_{0,j}, A_{0,j+1}) = \frac{\rho(A, C)}{n}.$$

That is, all horizontal distances from edge to edge in Figure 5 are equal. Next, since $\operatorname{cosq}\big(\overrightarrow{A_{i,j}A_{i,j+1}}, \overrightarrow{A_{k,j}A_{k,j+1}}\big) = 1$ and $\rho(A_{i,j}, A_{i+1,j}) = \rho(A_{k,j}, A_{k,j+1})$, we see by Lemma 16 that $\rho(A_{i,j}, A_{k,j}) = \rho(A_{i,j+1}, A_{k,j+1})$. In particular,

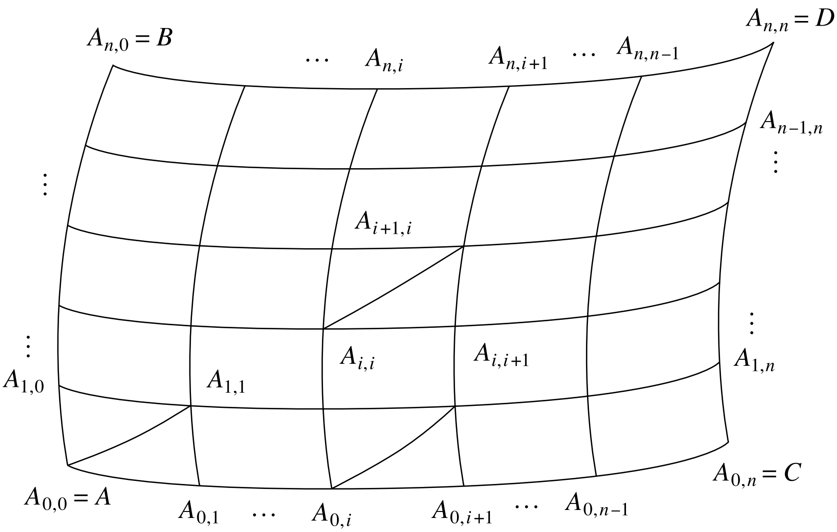

Figure 5

$$\rho(A_{i,j}, A_{i+1,j}) = \rho(A_{i,0}, A_{i+1,0}) = \frac{\rho(A, B)}{n}$$

and similarly $\rho(A_{0,j}, A_{n,j}) = \rho(A_{0,0}, A_{n,0}) = \rho(A, B)$. Hence the polygonal line with vertices $A_{0,k}, A_{1,k}, \ldots, A_{n,k}$ is a shortest path and thus a geodesic segment. That is, all slant vertical geodesic segments from base $A_{0,k}$ to $A_{n,k}$ pass through the division points and are of length $\rho(A, B)$.

Next we see by Lemma 16 that the set of points $\{A_{0,i}, A_{n,i}, A_{n,i+1}, A_{0,i+1}\}$ can be embedded as a parallelogram into $\mathbb{E}_2$, whence $\operatorname{cosq}(\overrightarrow{A_{0,i}A_{n,i}}, \overrightarrow{A_{0,i+1}A_{n,i+1}}\big) =$ 1 and, by the averaging principle, $\operatorname{cosq}\big(\overrightarrow{A_{0,i}A_{i,i}}, \overrightarrow{A_{1,i+1}A_{i+1,i+1}}\big) = 1$ (see Figure 5). Since clearly $A_{0,i}A_{i,i} = A_{1,i+1}A_{i+1,i+1}$, we have by Lemma 16 that $\rho(A_{i,i}, A_{i+1,i+1}) = \rho(A_{0,i}, A_{1,i+1})$. In a similar way we see that

$$\operatorname{cosq}\big(\overrightarrow{A_{0,0}A_{0,i}}, \overrightarrow{A_{1,1}A_{1,i+1}}\big) = 1, \qquad \rho(A_{0,0}, A_{0,i}) = \rho(A_{1,1}, A_{1,i+1}),$$

and, by Lemma 16, $\rho(A_{0,0}, A_{1,1}) = \rho(A_{0,i}, A_{1,i+1})$. Thus,

$$\rho(A_{i,i}, A_{i+1,i+1}) = \rho(A_{0,0}, A_{1,1}), \quad i = 0, 1, \ldots, n-1.$$

One can likewise see that

$$\rho(A_{i+1,i}, A_{i,i+1}) = \rho(A_{n,0}, A_{n-1,1}), \quad i = 0, 1, \ldots, n-1,$$

and $\rho(A_{n,0}, A_{n-1,1}) = \rho(A_{1,0}, A_{0,1})$.

We claim that

$$\rho(A_{0,0}, A_{1,1}) = \frac{\rho(A, D)}{n} \quad \text{and} \quad \rho(A_{n,0}, A_{n-1,1}) = \frac{\rho(B, C)}{n}.$$

Lemma 16 ensures that, for $i = 0, 1, \ldots, n-1$, the set $\{A_{i,i}, A_{i+1,i}, A_{i+1,i+1}, A_{i,i+1}\}$ can be embedded into $\mathbb{E}_2$ as a parallelogram. However, we do not yet know if this is the Euclidean parallelogram $A'_{i,i}A'_{i+1,i}A'_{i+1,i+1}A'_{i,i+1}$ in $\mathcal{P}'$.

Invoking the classical parallelogram law yields

$$\begin{aligned} n^2[\rho^2(A_{0,0}, A_{1,1}) + \rho^2(A_{n,0}, A_{n-1,1})] &= n^2[\rho^2(A_{0,0}, A_{1,1}) + \rho^2(A_{1,0}, A_{0,1})] \\ &= 2n^2[\rho^2(A_{0,0}, A_{0,1}) + \rho^2(A_{0,0}, A_{1,0})]. \end{aligned}$$

Hence this law implies that

$$n^2[\rho^2(A_{0,0}, A_{1,1}) + \rho^2(A_{n,0}, A_{n-1,1})] = \rho^2(A, D) + \rho^2(B, C). \tag{5}$$

By the triangle inequality,

$$n\rho(A_{0,0}, A_{1,1}) = \sum_{i=0}^{n-1} \rho(A_{i,i}, A_{i+1,i+1}) \geq \rho(A, D).$$

If $n\rho(A_{0,0}, A_{1,1}) > \rho(A, D)$, then invoking (5) yields $n\rho(A_{n,0}, A_{n-1,1}) < \rho(B, C)$. This is impossible since, by the triangle inequality, $n\rho(A_{n,0}, A_{n-1,1}) = \sum_{i=0}^{n-1} \rho(A_{i+1,i}, A_{i,i+1}) \geq \rho(B, C)$. Hence the equality $n\rho(A_{0,0}, A_{1,1}) = \rho(A, D)$ holds as claimed. Of course our entire argument applies equally well to the cross-diagonals and so $n\rho(A_{1,0}, A_{0,1}) = \rho(B, C)$.

We now have partitioned our "metric parallelogram" (i.e., the four points can be embedded as a Euclidean parallelogram into $\mathbb{E}_2$) into n^2 identical parallelograms each similar to the parallelogram with vertices A, B, C, D, so we have

$$\rho(A_{i,j}, A_{i,j+1}) = \rho_E(A'_{i,j}, A'_{i,j+1}) \text{ et seq.}$$

That is, the complete parallelogram formed by $\{A_{i,j}, A_{i+1,j}, A_{i,j+1}, A_{i+1,j+1}\}$ is isometric to the Euclidean parallelogram with vertices $A'_{i,j}, A'_{i+1,j}, A'_{i,j+1}, A'_{i+1,j+1}$.

(II) Finally we show that

$$\rho(A_{i,j}, A_{k,l}) = \rho_E(A'_{i,j}, A'_{k,l}), \quad i, j, k, l = 0, 1, \ldots, n \tag{6}$$

by reducing the calculation to a known situation.

The proof is by induction on $|k - i| + |l - j|$. Equation (6) is clearly true when $|k - i| + |l - j| = 0$. Assume (6) holds when $|k - i| + |l - j| < m$. Let

$|k-i|+|l-j|=m$. Since (6) is established when $|k-i|\le 1$ and $|l-j|\le 1$, it remains to prove (6) when either $|k-i|>1$ or $|l-j|>1$.

Indeed, assume without loss of generality that $k-i\ge 2$ and $k-i+|l-j|=m$. By the assumption of induction,

$$\rho(A_{i,j},A_{i+1,j})=\rho_E(A'_{i,j},A'_{i+1,j}),\qquad \rho(A_{i+1,j},A_{k,l})=\rho_E(A'_{i+1,j},A'_{k,l}),$$
$$\rho(A_{k,l},A_{k-1,l})=\rho_E(A'_{k,l},A'_{k-1,l}),\qquad \rho(A_{k-1,l},A_{i,j})=\rho_E(A'_{k-1,l},A'_{i,j}),$$

and

$$\rho(A_{i+1,j},A_{k-1,l})=\rho_E(A'_{i+1,j},A'_{k-1,l}).$$

See Figure 6.

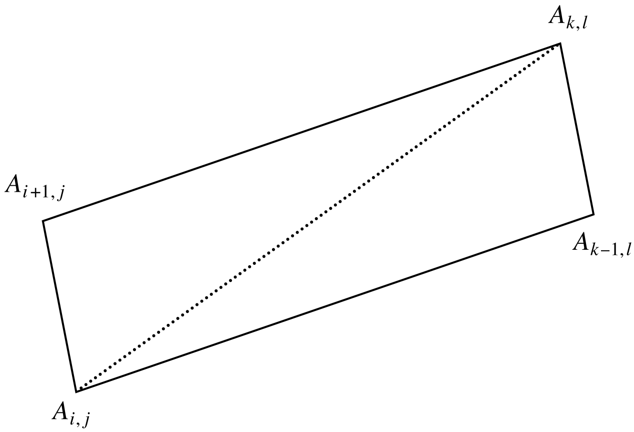

Figure 6

We established in (I) that

$$\operatorname{cosq}\big(\overrightarrow{A_{0,j}A_{n,j}},\overrightarrow{A_{0,k}A_{n,k}}\big)=1.$$

By the averaging principle, $\operatorname{cosq}\big(\overrightarrow{A_{i,j}A_{i+1,j}},\overrightarrow{A_{k-1,l}A_{k,l}}\big)=1$; moreover,

$$\rho(A_{i,j},A_{i+1,j})=\rho_E(A'_{i,j},A'_{i+1,j})=\rho_E(A'_{k,i},A'_{k-1,i})=\rho(A_{k,i},A_{k-1,i}).$$

We see by Lemma 16 that the set of points $\{A_{i,j},A_{i+1,j},A_{k,l},A_{k-1,l}\}$ can be embedded into $\mathbb{E}_2$ as a parallelogram. Therefore the parallelogram law holds for the set $\{A_{i,j},A_{i+1,j},A_{k,l},A_{k-1,l}\}$ as well as for the set $\{A'_{i,j},A'_{i+1,j},A'_{k,l},A'_{k-1,l}\}$. By invoking the parallelogram law we obtain

$$\begin{aligned}
&\rho^2(A_{i,j},A_{k,l})\\
&\quad=2[\rho^2(A_{i,j},A_{i+1,j})+\rho^2(A_{i+1,j},A_{k,l})]-\rho^2(A_{i+1,j},A_{k-1,l})\\
&\quad=2[\rho_E^2(A'_{i,j},A'_{i+1,j})+\rho_E^2(A'_{i+1,j},A'_{k,l})]-\rho_E^2(A'_{i+1,j},A'_{k-1,l})\\
&\quad=\rho_E^2(A'_{i,j},A'_{k,l}),
\end{aligned}$$

as claimed. The proof of the claim is complete.

B. Let $\mathcal{P}$ be the closure of $\bigcup_{n=0}^{\infty} S_n$. Evidently $\mathcal{P}'$ coincides with the closure of $\bigcup_{n=0}^{\infty} S_n'$. By invoking Claim A, the isometries $f_n : S_n' \to S_n$ can be extended to an isometry $f : \mathcal{P}' \to \mathcal{P}$. Since $\mathcal{P}'$ is the convex hull of the set $\{A', B, C', D'\}$ in $\mathbb{E}^2$, $\mathcal{P}$ coincides with the convex hull of the set $\{A, B, C, D\}$ in $\mathcal{M}$. This completes the proof of Theorem 15 when $AB = CD$. $\square$

3.5.4. General Case. Let $AB \neq CD$. Without loss of generality, suppose that $AB < CD$. We let U and T be a pair of points on the geodesic segment $\mathcal{CD}$ such that $AB = CT$ and $AB = UD$. By the averaging principle,

$$\operatorname{cosq}\big(\overrightarrow{AB}, \overrightarrow{CT}\big) = 1 \quad \text{and} \quad \operatorname{cosq}\big(\overrightarrow{AB}, \overrightarrow{UD}\big) = 1.$$

By referring to Section 3.5.3, $\mathcal{GC}[A, B, T, C]$ is isometric to a complete parallelogram $A'B'T'C'$ and $\mathcal{GC}[A, B, D, U]$ is isometric to a complete parallelogram $A'B'D'U'$. Let $A'B'D'C'$ be a complete Euclidean trapezoid that is the union of complete parallelograms $A'B'D'U'$ and $A'B'T'C'$, as shown on Figure 7. Since we decompose the convex hull $\mathcal{GC}[A, B, C, D]$ into a union of complete Euclidean parallelograms $\mathcal{GC}[A, B, T, C]$ (isometric to $A'B'T'C'$) and $\mathcal{GC}[A, B, D, U]$ (isometric to $A'B'D'U'$) coinciding on the overlap (isometric to the complete trapezoid $A'B'T'U'$), the convex hull $\mathcal{GC}[A, B, C, D]$ is isometric to the complete Euclidean trapezoid $A'B'D'C'$.

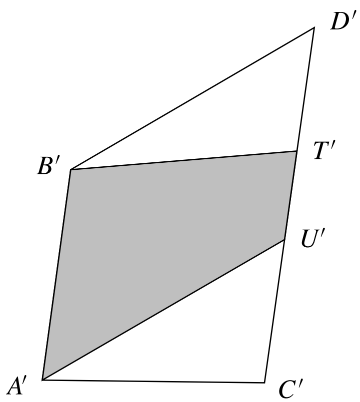

Figure 7

The proof of Theorem 15 is complete. $\square$

3.6. *Stability of the Quadrilateral Cosine*

The following important lemma is an analog of [11, Lemma 3].

LEMMA 17. *Let $\mathcal{PP}_1$, $\mathcal{PP}_1'$, and $\mathcal{QQ}_1$ be geodesic segments in an $\Re_K$ domain. Let $X \in \mathcal{PP}_1$, $X' \in \mathcal{PP}_1'$, $Y \in \mathcal{QQ}_1$ be points such that*

(1) *each of the sets* $\{P, X, X', Y\}$ *and* $\{Q, X, X', Y\}$ *consist of distinct points*; *and*
(2) $PX = PX' = x$ *and* $QY = y$.

Denote by ε *the angle* $\angle P_1PP_1'$. *Then*

$$\overline{\lim_{X,X'\to P;\, Y\to Q}} \left|\operatorname{cosq}\left(\overrightarrow{PX}, \overrightarrow{QY}\right) - \operatorname{cosq}\left(\overrightarrow{PX'}, \overrightarrow{QY}\right)\right| \leq \begin{cases} \varepsilon(k\hat{z}/(\sin k\hat{z})) & \textit{if } K = k^2 > 0, \\ \varepsilon & \textit{if } K \leq 0, \end{cases} \tag{7}$$

uniformly with respect to Q_1. *Here* $\hat{z}$ *denotes the largest distance in the configurations of the sets* $\{P, P_1, Q, Q_1\}$ *and* $\{P, P_1', Q, Q_1\}$.

Proof. The statement of the Lemma 17 is obvious if $P = Q$. Let $P \neq Q$. Observe that

$$\operatorname{cosq}\left(\overrightarrow{PX}, \overrightarrow{QY}\right) - \operatorname{cosq}\left(\overrightarrow{PX'}, \overrightarrow{QY}\right) = \frac{XX'}{x} \operatorname{cosq}\left(\overrightarrow{XX'}, \overrightarrow{QY}\right).$$

Clearly,

$$\lim_{x\to 0} \frac{XX'}{x} = 2\sin\frac{\varepsilon}{2} \leq \varepsilon.$$

Thus, Corollary 11 or 12 yields (7). □

We derive with little effort the following extension of Lemma 17.

LEMMA 18. *Let* P, Q, X, X', Y, Y' *be points in an* $\Re_K$ *domain, and let* $0 < \varepsilon_1, \varepsilon_2 \leq 1$ *such that* $PQ > 0$, $XY = h > 0$, $XX' = \varepsilon_1 h$, *and* $YY' = \varepsilon_2 h$. *Suppose that* $PX, PX' > 0$ *and* $h < \pi/2\sqrt{K}$ *if* $K > 0$. *Then necessarily*

$$\left|\operatorname{cosq}\left(\overrightarrow{PQ}, \overrightarrow{XY}\right) - \operatorname{cosq}\left(\overrightarrow{PQ}, \overrightarrow{X'Y'}\right)\right| \leq \begin{cases} 2(\varepsilon_1 + \varepsilon_2)(kz_+/(\sin kz_+)) & \textit{if } K = k^2 > 0, \\ 2(\varepsilon_1 + \varepsilon_2) & \textit{if } K \leq 0, \end{cases}$$

where $z_+ = \max\left\{\hat{z}\left(\overrightarrow{PQ}, \overrightarrow{YY'}\right), \hat{z}\left(\overrightarrow{PQ}, \overrightarrow{X'X}\right)\right\}$.

Proof. By Corollary 5,

$$0 = h\operatorname{cosq}\left(\overrightarrow{PQ}, \overrightarrow{XY}\right) + \varepsilon_2 h \operatorname{cosq}\left(\overrightarrow{PQ}, \overrightarrow{YY'}\right) + X'Y' \operatorname{cosq}\left(\overrightarrow{PQ}, \overrightarrow{Y'X'}\right) + \varepsilon_1 h \operatorname{cosq}\left(\overrightarrow{PQ}, \overrightarrow{X'X}\right).$$

By the triangle inequality, $|X'Y' - h| \leq (\varepsilon_1 + \varepsilon_2)h$. Thus,

$$X'Y' = h + \eta h \quad \text{where } |\eta| \leq \varepsilon_1 + \varepsilon_2, \tag{8}$$

and we have

$$\left|\operatorname{cosq}\left(\overrightarrow{PQ}, \overrightarrow{XY}\right) - (1+\eta)\operatorname{cosq}\left(\overrightarrow{PQ}, \overrightarrow{X'Y'}\right)\right| \leq \varepsilon_1\left|\operatorname{cosq}\left(\overrightarrow{PQ}, \overrightarrow{X'X}\right)\right| + \varepsilon_2\left|\operatorname{cosq}\left(\overrightarrow{PQ}, \overrightarrow{YY'}\right)\right|.$$

By Corollary 11 or 12,

$$\left|\operatorname{cosq}(\overrightarrow{PQ},\overrightarrow{XY}) - (1+\eta)\operatorname{cosq}(\overrightarrow{PQ},\overrightarrow{X'Y'})\right| \le \begin{cases} (\varepsilon_1+\varepsilon_2)(kz_+/(\sin kz_+)) & \text{if } K = k^2 > 0, \\ (\varepsilon_1+\varepsilon_2) & \text{if } K \le 0. \end{cases}$$

The last estimate together with (8) yields the statement of the lemma. □

3.7. Stability of the Quadrilateral Sine

LEMMA 19. *Let P, Q, Y be a triple of distinct points in an $\Re_K$ domain and let $\mathcal{L}, \mathcal{L}' \in \Pi^{-1}(\boldsymbol{\xi})$ be fixed, where $\boldsymbol{\xi}$ is a direction at P. Then, given $n = 0, 1, 2, \ldots$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that, for $X \in \mathcal{L}$ and $X' \in \mathcal{L}'$ where $0 < XP = X'P = h < \delta$, the Hausdorff distance between $\mathcal{G}^n[P, Q, X, Y]$ and $\mathcal{G}^n[P, Q, X', Y]$ is finite and is not greater than $\varepsilon \cdot h$.*

Proof. (I) We note that, for every n, the set $\mathcal{G}^n[P, Q, X, Y]$ is a compact subspace of $\Re_K$, since geodesic segments depend continuously on their ends [1]. Thus,

$$d_H(\mathcal{G}^n[P, Q, X, Y], \mathcal{G}^n[P, Q, X', Y]) < +\infty.$$

(II) Let $K \le 0$. We introduce the following notation:

$$\alpha_0(h) = 2 \arcsin \frac{XX'}{2h}.$$

Our hypothesis that $\mathcal{L}$ and $\mathcal{L}'$ form angle zero at P implies that

$$\lim_{h\to 0} \alpha_0(h) = 0.$$

We claim that, for $n = 0, 1, \ldots,$

$$\sup_{Z\in\mathcal{G}^n[P,Q,X,Y]} \ \inf_{W\in\mathcal{G}^n[P,Q,X',Y]} \rho(Z, W) \le 2h \sin \frac{\alpha_0(h)}{2}. \tag{9}$$

The estimate (9) is obvious when $n = 0$. Suppose that (9) is true when $n = k$. Let $A \in \mathcal{G}^{k+1}[P, Q, X, Y]$. By definition, there exist points $B, D \in \mathcal{G}^k[P, Q, X, Y]$ such that $A \in \mathcal{BD}$. By (9), for $n = k$ there are points $B', D' \in \mathcal{G}^k[P, Q, X', Y]$ such that

$$BB', DD' \le 2h \sin \frac{\alpha_0(h)}{2}.$$

Let $A' \in \mathcal{B}'\mathcal{D}'$ and $B'A' = t \cdot B'D'$, where $t = BA/BD$. The property of K-concavity enables us to write the estimate

$$AA' \le 2h \sin \frac{\alpha_0(h)}{2}$$

(see [1] or [3, Prop. 5.3]), which completes the proof of (9). In a similar way one can prove that, for each $n = 0, 1, \ldots,$

$$\sup_{Z\in\mathcal{G}^n[P,Q,X',Y]} \ \inf_{W\in\mathcal{G}^n[P,Q,X,Y]} \rho(Z, W) \le 2h \sin \frac{\alpha_0(h)}{2}.$$

Thus,

$$d_H(\mathcal{G}^n[P,Q,X,Y],\mathcal{G}^n[P,Q,X',Y]) \le 2h\sin\frac{\alpha_0(h)}{2}.$$

Lemma 19 immediately follows from the last inequality.

The case when $K > 0$ is treated in a similar way except that, with each succeeding n, K-concavity gives us a constant greater than 1. $\square$ **?** [MS p.30]

LEMMA 20. *Let P and Q be points in an $\Re_K$ domain. Let $\boldsymbol{\xi} \in \Omega_P(\Re_K)$ and $\boldsymbol{\zeta} \in \Omega_Q(\Re_K)$. Then, for every $\mathcal{L} \in \Pi^{-1}(\boldsymbol{\xi})$, and $\mathcal{N} \in \Pi^{-1}(\boldsymbol{\zeta})$,*

$$\operatorname{sinq}(\boldsymbol{\xi},\boldsymbol{\zeta}) = \operatorname{sinq}(\mathcal{L},\mathcal{N}).$$

Proof. Let $\mathcal{L}' \in \Pi^{-1}(\boldsymbol{\xi})$. To prove Lemma 20 we need to establish that

$$\operatorname{sinq}(\mathcal{L},\mathcal{N}) = \operatorname{sinq}(\mathcal{L}',\mathcal{N}).$$

First we show that

$$\operatorname{sinq}(\mathcal{L},\mathcal{N}) \le \operatorname{sinq}(\mathcal{L}',\mathcal{N}). \tag{10}$$

Let $X \in \mathcal{L}$, $X' \in \mathcal{L}'$, and $Y \in \mathcal{N}$ be the points such that $PX = PX' = QY = h > 0$. Clearly the last inequality is a corollary of the following claim: Given $a > 0$, $n = 1, 2, \ldots$, and $\varepsilon > 0$, there is a $\delta > 0$ such that for each $0 < h \le \delta$ and every pair of distinct points A, $B \in \mathcal{G}^n[P,Q,X,Y]$ $(AB = a\cdot h)$ there exists a pair of distinct points A', $B' \in \mathcal{G}^n[P,Q,X',Y]$ such that

$$A'B' = a\cdot h \tag{11}$$

and

$$\left|\operatorname{sinq}\big(\overrightarrow{PX},\overrightarrow{QY};\overrightarrow{AB}\big) - \operatorname{sinq}\big(\overrightarrow{PX'},\overrightarrow{QY};\overrightarrow{A'B'}\big)\right| < \varepsilon. \tag{12}$$

Observe that

$$\begin{aligned}
&\operatorname{sinq}\big(\overrightarrow{PX},\overrightarrow{QY};\overrightarrow{AB}\big) - \operatorname{sinq}\big(\overrightarrow{PX'},\overrightarrow{QY};\overrightarrow{A'B'}\big) \\
&\quad = \big\{\operatorname{cosq}\big(\overrightarrow{PX},\overrightarrow{AB}\big) - \operatorname{cosq}\big(\overrightarrow{PX'},\overrightarrow{AB}\big)\big\} \\
&\qquad + \big\{\operatorname{cosq}\big(\overrightarrow{PX'},\overrightarrow{AB}\big) - \operatorname{cosq}\big(\overrightarrow{PX'},\overrightarrow{A'B'}\big)\big\} \\
&\qquad - \big\{\operatorname{cosq}\big(\overrightarrow{QY},\overrightarrow{AB}\big) - \operatorname{cosq}\big(\overrightarrow{QY},\overrightarrow{A'B'}\big)\big\}.
\end{aligned} \tag{13}$$

We remark that, by Lemma 19, given points A, $B \in \mathcal{G}^n[P,Q,X,Y]$ there exist points A', $B' \in \mathcal{G}^n[P,Q,X',Y]$ such that AA', $BB' = o(h)$. By taking an arc of the geodesic segment $\mathcal{A}'\mathcal{B}'$ we can always achieve that AA', $BB' = o(h)$ and (11) holds.

Inequality (12) then follows easily from Lemmas 19 and 17. Indeed, the first summand in the right-hand side of (13) can be made less than $\varepsilon/3$ if $0 < h \le \delta_1(\varepsilon/3)$. By Lemma 19, the second and third summands can be made less than $\varepsilon/3$ if $0 < h < \delta_2(\varepsilon/3)$ by invoking Lemma 18. Thus, we obtain (12) for $0 < h \le \min\{\delta_1(\varepsilon/3), \delta_2(\varepsilon/3, a)\}$. $\square$

REMARK 8. Let γ be a geodesic segment in an $\Re_K$ domain, and let $\dot\gamma$ denote the field of directions tangent to γ. By Lemma 20, one can compute $\operatorname{sinq}(\dot\gamma(s),\dot\gamma(t))$

with the help of the corresponding arcs of γ. Since the test ordered pair $\overrightarrow{AB}$ in the definition of the quadrilateral sine belongs to the convex hull, in our case points A and B belong to γ. An easy computation shows that $\operatorname{cosq}\big(\dot\gamma(s), \overrightarrow{AB}\big) = \operatorname{cosq}\big(\dot\gamma(t), \overrightarrow{AB}\big) = 1$. Thus,

$$\operatorname{sinq}(\dot\gamma(s), \dot\gamma(t)) = 0.$$

Therefore, in the definition of the quadrilateral sine we took a test vector from the convex hull of curves $\mathcal{L}, \mathcal{N}$ that specify the directions under consideration to ensure that the quadrilateral sine of a pair of directions tangent to a geodesic segment is zero.

The graph of Figure 1 provides an instantaneous example where

$$\operatorname{cosq}\big(\dot\gamma(s'), \overrightarrow{OX}\big) - \operatorname{cosq}\big(\dot\gamma(s''), \overrightarrow{OX}\big) = 2.$$

Indeed, let $\gamma(s)$ be a normal parameterization of the geodesic segment $\mathcal{TY}$. Take s', s'' such that $\gamma(s')$ is an interior point of the geodesic segment $\mathcal{TO}$ and $\gamma(s'')$ is an interior point of the geodesic segment $\mathcal{OY}$. Then we readily see that $\operatorname{cosq}\big(\dot\gamma(s'), \overrightarrow{OX}\big) = 1$ and $\operatorname{cosq}\big(\dot\gamma(s''), \overrightarrow{OX}\big) = -1$. Clearly $\overrightarrow{OX}$ is not an admissible vector since, for $\Delta s > 0$, X is not in the convex hull of the set of points $\{\gamma(s'), \gamma(s''), \gamma(s' + \Delta s), \gamma(s'' + \Delta s)\}$.

3.8. Quadrilateral Sine of Sides of a Triangle

The following theorem tells us that, given two geodesics in an $\Re_K$ domain emanating from a point, there is a neighborhood of that point such that, given any segment in that neighborhood, the quadrilateral sine with arbitrary short segments of the geodesics tested against this segment is bounded above by approximately the $\mathbb{E}_n$ value and, moreover, in that neighborhood segments exist on which the approximation is itself almost attained, again tested against arbitrarily short geodesic segments.

For a pair of geodesic segments $\mathcal{PQ}$ and $\mathcal{PR}$ and for

$$0 \le r \le \min\{PQ, PR\},$$

let X_r be the point on $\mathcal{PQ}$ such that $PX_r = r$ and let Y_r be the point on $\mathcal{PR}$ such that $PY_r = r$. Let

$$\boldsymbol{\xi} = \Pi(\mathcal{PQ}), \qquad \boldsymbol{\zeta} = \Pi(\mathcal{PR}).$$

THEOREM 21. *Let $\mathcal{T} = QPR$ be a triangle in an $\Re_K$ domain and let*

$$0 \le h \le \eta \le \min\{PQ, PR\}.$$

Then (1)

$$\lim_{\eta\to 0}\lim_{h\to 0}\sup_{AP, BP<\eta} \operatorname{sinq}\big(\overrightarrow{PX_h}, \overrightarrow{PY}_h; \overrightarrow{AB}\big) \le 2\sin\frac{\angle(\boldsymbol{\xi}, \boldsymbol{\zeta})}{2}$$

and (2), *for* $\xi \ne \zeta$,

$$\lim_{\eta\to 0}\lim_{h\to 0}\lim_{\nu\to 0}\sup_{\substack{AB\le\nu\\ \mathcal{AB}\ \text{subsegment of}\ \mathcal{X}_\eta\mathcal{Y}_\eta}} \operatorname{sinq}\big(\overrightarrow{PX_h}, \overrightarrow{PY_h}; \overrightarrow{AB}\big) = 2\sin\frac{\angle(\boldsymbol{\xi}, \boldsymbol{\zeta})}{2}$$

(*see Figure 8*).

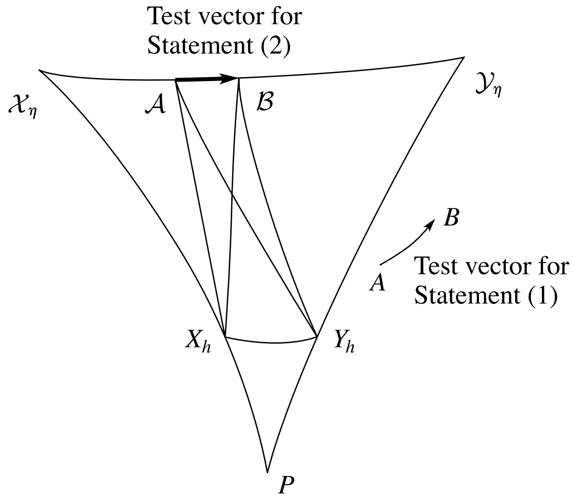

Figure 8

Statement (1) tells us that all test vectors near the vertex yield a sinq, subject to the desired upper bound given by the actual angle. Statement (2) tells us that at a fixed small distance we can approximate the desired sinq with arbitrarily short test vectors chosen from the opposite side of the vertex and with arbitrarily short segments of the vertex configuration.

We emphasize the order of the limits here because—as the segments that define the angle shorten—the test segments are not being pushed back to the vertex, which would be a less interesting result.

Note that, by Lemma 20, it suffices to deal with only geodesic segments rather than curves specifying a direction. Since in an $\Re_K$ domain the geodesic segment joining a pair of points is unique, $U \in \mathcal{PQ} \cap \mathcal{PR} \backslash \{P\}$ implies that $\mathcal{PU} \subset \mathcal{PQ} \cap \mathcal{PR}$; by referring to the first part of Theorem 21, we readily see that in this case $\operatorname{sinq}(\boldsymbol{\xi}, \boldsymbol{\zeta}) = 2 \sin \frac{\angle(\boldsymbol{\xi}, \boldsymbol{\zeta})}{2} = 0$. If $\mathcal{PQ} \cap \mathcal{PR} \backslash \{P\} = \emptyset$, we can apply the second part of Theorem 21. Hence Theorem 21 yields the following corollary.

COROLLARY 22.

$$\operatorname{sinq}(\boldsymbol{\xi}, \boldsymbol{\zeta}) = 2 \sin \frac{\angle(\boldsymbol{\xi}, \boldsymbol{\zeta})}{2}.$$

REMARK 9. Corollary 22 remains true if, in the definition of the quadrilateral sine, we use case (1) rather than case (2) (see Section 2.4). That is, $\overrightarrow{AB}$ was not restricted to any convex hull but was any segment near P in our entire space.

Theorem 21 is obvious if $\angle(\boldsymbol{\xi}, \boldsymbol{\zeta}) = 0$. Hereafter we assume that $\angle(\boldsymbol{\xi}, \boldsymbol{\zeta}) > 0$.

3.8.1. Proof of Statement (1). By Corollary 3 applied to the quadrilateral sine of $\overrightarrow{PX_h}$ and $\overrightarrow{PY_h}$,

$$\operatorname{sinq}\big(\overrightarrow{PX_h}, \overrightarrow{PY_h}; \overrightarrow{AB}\big) = \frac{X_hY_h}{h}\big|\operatorname{cosq}\big(\overrightarrow{X_hY_h}, \overrightarrow{AB}\big)\big|, \tag{14}$$

where $0 < PA, PB \leq \eta$.

By Corollary 11, $\big|\operatorname{cosq}\big(\overrightarrow{X_hY_h}, \overrightarrow{AB}\big)\big|$ is bounded above by 1 if $K \leq 0$ and by $k\hat{z}/(\sin k\hat{z})$ if $K > 0$. Clearly, $\hat{z}$ converges to zero and consequently $k\hat{z}/(\sin k\hat{z})$ converges to unity as η converges to zero. Thus,

$$\varlimsup_{\eta\to 0}\ \varlimsup_{h\to 0}\big|\operatorname{cosq}\big(\overrightarrow{X_hY_h}, \overrightarrow{AB}\big)\big| \leq 1.$$

Now consider XY/h:

$$\frac{X_hY_h}{h} = \frac{X_h^0Y_h^0}{h} = 2\sin\frac{\angle X_h^0P^0Y_h^0}{2}.$$

Since $\angle X_h^0P^0Y_h^0$ converges to $\angle(\boldsymbol{\xi}, \boldsymbol{\zeta})$ as $h \to 0$, we are done. $\square$

3.8.2. Auxiliary Lemma. Lemma 23 is of independent interest. In it we see that segments cutting an isosceles triangle into isosceles subtriangles with the same vertex are approximately parallel in the sense that the quadrilateral cosine between them is approximately 1.

LEMMA 23. *Let $\mathcal{T} = LPM$ be a triangle in an $\Re_K$ domain such that $\angle LPM >$ 0. Let $Q, X \in \mathcal{PL}$ and $R, Y \in \mathcal{PM}$. Let $PQ = PR = t$ and $PX = PY = h$. Then*

$$\lim_{h,t\to 0}\operatorname{cosq}\big(\overrightarrow{QR}, \overrightarrow{XY}\big) = 1,$$

(*see Figure 9*).

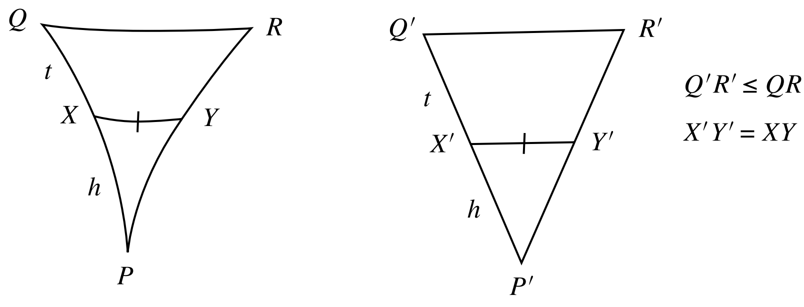

Figure 9

Proof. Without loss of generality we can assume that $t \geq h$. Let $K^+ = \max\{0, K\}$. Consider a triangle $\mathcal{T}^{K^+} = P^{K^+}X^{K^+}Y^{K^+}$. Let $\mathcal{T}' = P'Q'R'$ be a triangle in S_{K^+} such that $P'Q' = PQ$, $P'R' = PR$, and $\angle Q'P'R' = \angle X^{K^+}P^{K^+}Y^{K^+}$. Denote by X', Y' points on geodesic segments $\mathcal{P}'\mathcal{Q}'$ and $\mathcal{P}'\mathcal{R}'$ such that $PX = P'X'$ and $PY = P'Y'$, and so $X'Y' = XY$. We claim that

$$X'R' \leq XR \quad \text{and} \quad Y'Q' \leq YQ. \tag{15}$$

Indeed, by K^+-concavity,

$$\angle R^{K+}P^{K+}X^{K+} \geq \angle X^{K+}P^{K+}Y^{K+} = \angle X'P'Y'.$$

Since $P'X' = P^{K+}X^{K+}$ and $P'R' = P^{K+}R^{K+}$, the foregoing inequality yields the first of inequalities (15). The second inequality of (15) is proved in a similar way.

By (15), $\operatorname{cosq}\big(\overrightarrow{QR}, \overrightarrow{XY}\big) \geq (Q'R'/QR)\operatorname{cosq}\big(\overrightarrow{Q'R'}, \overrightarrow{X'Y'}\big)$, which together with Corollaries 11 and 12 yields

$$\begin{aligned} \frac{Q'R'}{QR}\operatorname{cosq}\big(\overrightarrow{Q'R'}, \overrightarrow{X'Y'}\big) &\leq \operatorname{cosq}\big(\overrightarrow{QR}, \overrightarrow{XY}\big) \\ &\leq \begin{cases} k\hat{z}/(\sin k\hat{z}) & \text{if } K = k^2 > 0, \\ 1 & \text{if } K \leq 0. \end{cases} \end{aligned} \tag{16}$$

Observe that

$$Q'R' = 2PQ\sin\frac{\angle X^0P^0Y^0}{2} \quad \text{and} \quad QR = 2PQ\sin\frac{\angle Q^0P^0R^0}{2}$$

for $K \leq 0$. A similar computation yields that, for $K > 0$,

$$Q'R' = 2PQ(\sin(\angle X^0P^0Y^0/2) + O(t)).$$

Since $\angle(\boldsymbol{\xi}, \boldsymbol{\zeta}) > 0$,

$$\lim_{h,t\to 0}\frac{\angle X^0P^0Y^0}{\angle Q^0P^0R^0} = 1$$

and consequently

$$\lim_{h,t\to 0}\frac{Q'R'}{QR} = 1.$$

Let $K \leq 0$. Then $\operatorname{cosq}\big(\overrightarrow{Q'R'}, \overrightarrow{X'Y'}\big) = 1$ (parallel segments in Euclidean space). Thus, for $K \leq 0$, we have established Lemma 23.

To complete the proof of Lemma 23 for $K > 0$, we need to show that

$$\lim_{h,t\to 0}\operatorname{cosq}\big(\overrightarrow{Q'R'}, \overrightarrow{X'Y'}\big) = 1. \tag{17}$$

By the Gauss–Bonnet theorem, for an angle α of a sufficiently small spherical triangle $\mathcal{T}$ we have

$$\alpha - \alpha_0 = O(\sigma(\mathcal{T})),$$

where $\sigma(\mathcal{T})$ is the area of a Euclidean triangle with the same edge lengths as $\mathcal{T}$. Recall that we denote PQ by t. Then

$$\begin{aligned} Q'Y'^2 &= h^2 + t^2 - 2ht\cos\angle Q'P'R' + O(h^2t^2); \\ R'X'^2 &= h^2 + t^2 - 2ht\cos\angle Q'P'R' + O(h^2t^2). \end{aligned}$$

Thus,

$$\begin{aligned}
&\operatorname{cosq}\bigl(\overrightarrow{Q'R'}, \overrightarrow{X'Y'}\bigr) \\
&\quad = \frac{h^2 + t^2 - 2ht\cos\angle Q'P'R' + h^2 + t^2 - 2ht\cos\angle Q'P'R'}{2Q'R' \cdot X'Y'} \\
&\qquad - \frac{2(t-h)^2 + O(h^2t^2)}{2Q'R' \cdot X'Y'} \\
&\quad = \frac{8ht\sin^2\dfrac{\angle Q'P'R'}{2} + O(h^2t^2)}{2Q'R' \cdot X'Y'} = \frac{\sin^2\dfrac{\angle Q'P'R'}{2} + O(ht)}{\sin\dfrac{\angle Q'_0P'_0R'_0}{2} \cdot \sin\dfrac{\angle X'_0P'_0Y'_0}{2}},
\end{aligned}$$

which immediately yields (17). □

3.8.3. Proof of Statement (2). To prove Statement (2) we need to replace vector $\overrightarrow{QR}$ in Lemma 23 with a tangent vector along $\mathcal{QR}$ of sufficiently small length. Corollary 4 allows us to make this transition. For brevity introduce the following notation: $\vec{u} = \overrightarrow{PX}$, $\vec{v} = \overrightarrow{PY}$, $\vec{w} = \overrightarrow{QR}$.

By Lemma 23, given an $\varepsilon > 0$ there exist $\delta, \mu > 0$ such that if $0 < PQ, PR \leq \delta$, and $0 < h \leq \nu < PQ$ then

$$\operatorname{cosq}(\vec{v} - \vec{u}, \vec{w}) \geq 1 - \varepsilon/2.$$

Let $\mathcal{Q}'\mathcal{R}'$ be a subsegment of the segment $\mathcal{QR}$. In what follows, $\vec{w}_i = \overrightarrow{Q'R'}$. By Corollary 11 we can assume that for the same choice of Q and R,

$$\operatorname{cosq}\bigl(\vec{v} - \vec{u}, \vec{w}_i\bigr) \leq 1 + \varepsilon;$$

in particular,

$$\operatorname{cosq}(\vec{v} - \vec{u}, \vec{w}) \leq 1 + \varepsilon.$$

Let $n \geq 5$ be a natural number such that $0 < h' = QR/n \leq h$. Split the geodesic segment $\mathcal{QR}$ by points

$$Q = A_0 < A_1 < A_2 < \cdots < A_n < A_{n+1} = R$$

into geodesic segments $\mathcal{A}_i\mathcal{A}_{i+1}$ $(i = 0, \ldots, n)$ of length h', and consider vectors $\vec{w}_i = \overrightarrow{A_iA_{i+1}}$. By Corollary 4,

$$\operatorname{cosq}(\vec{v} - \vec{u}, \vec{w}) = \frac{1}{n}\sum_{i=0}^{n-1}\operatorname{cosq}(\vec{v} - \vec{u}, \vec{w}_i).$$

From this it is clear that there is at least one ordered pair $\vec{w}_{i_0}$ such that

$$\operatorname{cosq}(\vec{v} - \vec{u}, \vec{w}_{i_0}) \geq 1 - \varepsilon. \tag{18}$$

We have thus established that, given an $\varepsilon > 0$, there exist $\delta, \mu > 0$ and an ordered pair $\vec{w}_\varepsilon$ of length less than h such that, if $0 < PQ, PR \leq \delta$, and $0 < h \leq \nu < PQ$,

$$1 - \varepsilon \leq \operatorname{cosq}(\vec{v} - \vec{u}, \vec{w}_\varepsilon) \leq 1 + \varepsilon.$$

The last bound and (14) immediately yield the statement of the theorem. □

REMARK 10. In fact, the proof of Theorem 21, by using $\sqrt{\varepsilon}$ in (18), will show that all of the $\vec{w}_i$ except for a set of proportion order $\sqrt{\varepsilon}$ will satisfy (18); this yields a sort of convergence in measure and in integral.

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I. D. Berg
Department of Mathematics
University of Illinois
Urbana, IL 61801

berg@symcom.math.uiuc.edu

I. G. Nikolaev
Department of Mathematics
University of Illinois
Urbana, IL 61801

inik@symcom.math.uiuc.edu