Standard Forms of 3-Braid 2-Knots and their Alexander Polynomials

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By a *surface link* we mean a closed oriented locally flat surface F in 4-space $\mathbf{R}^4$. It is called a *closed 2-dimensional braid* of degree m if it is contained in a tubular neighborhood $N(S^2) \cong D^2 \times S^2$ of a standard 2-sphere S^2 in $\mathbf{R}^4$ such that the restriction to F of the projection $D^2 \times S^2 \to S^2$ is a degree-m simple branched covering map from F to S^2. Viro [V; cf. K2; CS] proved that every surface link is ambient isotopic to a closed 2-dimensional braid of degree m for some m. The *braid index* of F, denoted by $\mathrm{Braid}(F)$, is the minimum degree among all closed 2-dimensional braids ambient isotopic to F.

By definition, $\mathrm{Braid}(F) = 1$ if and only if F is an unknotted 2-sphere (i.e., ambient isotopic to the standard 2-sphere in $\mathbf{R}^4$). It is easily seen that $\mathrm{Braid}(F) = 2$ if and only if F is an unknotted surface link in $\mathbf{R}^4$ that is a connected surface with nonnegative genus or a pair of 2-spheres; cf. [K1]. (A surface link is *unknotted* if it bounds mutually disjoint locally flat 3-balls or handlebodies in $\mathbf{R}^4$. This condition is equivalent to its being isotoped into a hyperplane of $\mathbf{R}^4$; see [HK].) In particular, there exist no 2-knots of braid index 2.

Our interest is 3-braid 2-knots, that is, 2-spheres in $\mathbf{R}^4$ of braid index 3. The spun 2-knot of a $(2, q)$-type torus knot is a 3-braid 2-knot unless $q = \pm 1$. Of course, there exist infinitely many 3-braid 2-knots which are not spun 2-knots.

Few results on 3-braid 2-knots are known. For example, all 3-braid 2-knots—and all surface links of braid index 3 or less—are ribbon [K1]. (A surface link is said to be *ribbon* if it is obtained from a split union of unknotted 2-spheres by surgery along some 1-handles attached to them.) Thus the 2-twist spun 2-knot of a trefoil knot is not a 3-braid 2-knot.

The purpose of this paper is to prove that a 3-braid 2-knot can always be deformed into a certain kind of configuration, called a *standard form* (Section 1). In Section 2 we investigate Alexander polynomials of 3-braid 2-knots by use of standard forms. Our main theorem (Theorem 2.3) regards a strong relationship between standard forms and the spans of the Alexander polynomials. (The *span* means the maximal degree minus the minimal.) Using it, we obtain some results on Alexander polynomials of 3-braid 2-knots; for instance, nontriviality of them. Standard forms (and Alexander polynomials) are quite useful for distinguishing

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the knot types (Section 3). As an application, we shall give a complete table of 3-braid 2-knots whose Alexander polynomials have spans less than 10. There are $1 + 1 + 2 + 3 + 7 + 12 + 24 + 45 = 95$ knot types up to mirror images. They are completely classified by standard forms. Moreover, standard forms bring us plenty of (and a series of) examples of 2-knots, most of which are not spun 2-knots; these would be helpful for research on 2-knot theory.

Standard forms (and Alexander polynomials) are also useful for examining whether or not a 3-braid 2-knot is amphicheiral—that is, ambient isotopic to the mirror image of itself (Section 3). (Recall that a 3-braid 2-knot is ribbon, so it is amphicheiral if and only if it is invertible.)

In order to present a ribbon-closed 2-dimensional braid we shall use a notation due to Rudolph [R1; R2] and Viro [V]. Then the standard forms are defined in terms of Murasugi's principal 3-braids, which are used in [Mu] for investigation of closed 3-braids in 3-space $\mathbf{R}^3$. He proved that 3-braids are decomposed into principal parts (so-called alternating parts) and torus-like parts, and calculated Alexander polynomials of them. For further investigation on closed 3-braids in $\mathbf{R}^3$, refer to [B2; BM; T].

For the sake of argument, we treat not only 3-braid 2-knots but also all surface links F with the Euler characteristic $\chi(F) = 2$ and $\mathrm{Braid}(F) \le 3$. Such a surface link is an unknotted 2-knot, a 3-braid 2-knot or a 3-braid surface link that is a union of a 2-sphere and a torus in $\mathbf{R}^4$. In the last case, each component is unknotted, for its braid index is 1 or 2. We work in the piecewise linear (or smooth) category.

1. Standard Forms of 3-Braid 2-Knots

First we introduce Rudolph and Viro's notation to present a ribbon-closed 2-dimensional braid. The 4-space $\mathbf{R}^4$ is regarded as the union of parallel hyperplanes $\mathbf{R}^3_t$ $(t \in \mathbf{R})$. Let $b_1, \dots, b_n$ be m-braids and

$$c_1, \dots, c_n \in \{\sigma_1, \sigma_1^{-1}, \dots, \sigma_{m-1}, \sigma_{m-1}^{-1}\},$$

where $\sigma_1, \dots, \sigma_{m-1}$ are standard generators of the m-braid group B_m (cf. [B1]). Consider a closed 2-dimensional m-braid F satisfying the following conditions.

(1) $F \cap \mathbf{R}^3_t$ is empty for $t \in (-\infty, -2)$.

(2) $F \cap \mathbf{R}^3_{-2}$ consists of m disks.

(3) For each $t \in (-2, -1)$, $F \cap \mathbf{R}^3_t$ is a trivial closed m-braid. In addition, if t is near -1, it is a closed m-braid l represented by $b_1 b_1^{-1} \dots b_n b_n^{-1}$.

(4) $F \cap \mathbf{R}^3_{-1}$ is l together with n saddle bands each of which is a half-twisted band corresponding to c_i located between b_i and b_i^{-1}.

(5) For $t \in (-1, 0]$, $F \cap \mathbf{R}^3_t$ is a closed m-braid represented by $b_1 c_1 b_1^{-1} \dots b_n c_n b_n^{-1}$.

(6) F is symmetric with respect to the hyperplane $\mathbf{R}^3_0$.

(The case of $m = 3$, $n = 2$, and $c_1 = c_2 = \sigma_1^{-1}$ is illustrated in Figure 1.) We denote this closed 2-dimensional m-braid by $F[b_1, c_1 | \dots | b_n, c_n]_m$. If $n = 0$, let

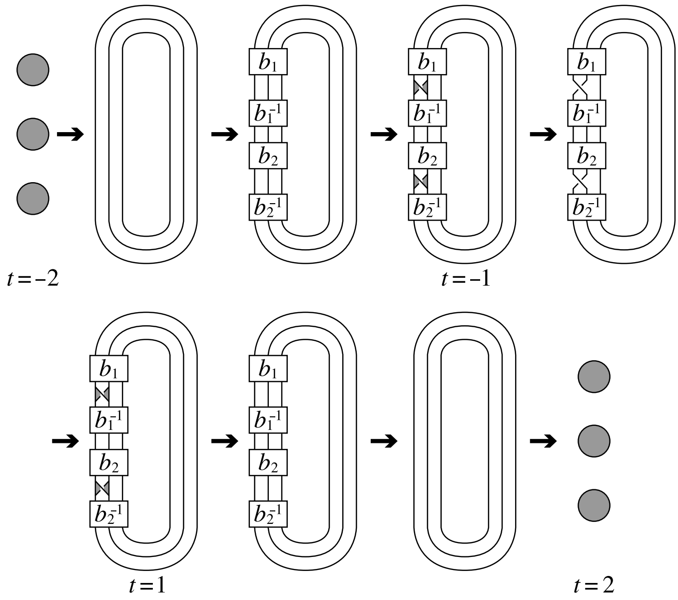

Figure 1

$F[\emptyset]_m$ denote a trivial closed 2-dimensional m-braid, namely, m parallel copies of the standard 2-sphere in $\mathbf{R}^4$.

The following theorem was proved by Rudolph [R1; R2]. (The surface link $F[b_1, c_1|\dots|b_n, c_n]_m$ is the double of a braided surface in the lower half-space $\mathbf{R}^4_-$ associated with a band representation $S(b_1c_1b_1^{-1}, \dots, b_nc_nb_n^{-1})$ in the sense of [R1; R2]. An alternative proof is given in [K1; K2].)

THEOREM 1.1. *A surface link is ribbon if and only if it is ambient isotopic to a closed 2-dimensional m-braid $F[b_1, c_1|\dots|b_n, c_n]_m$ for some m.*

Let τ be the automorphism of B_m with $\tau(\sigma_i) = \sigma_{m-i}$ for $i = 1, \dots, m-1$. We shall denote it by $F \cong F'$ if two surface links F and F' are ambient isotopic.

LEMMA 1.2. *For $F = F[b_1, c_1|\dots|b_n, c_n]_m$, the following statements hold*:

(1) $F \cong F[b_2, c_2|\dots|b_n, c_n|b_1, c_1]_m$;
(2) $F \cong F[bb_1, c_1|\dots|bb_n, c_n]_m$ *for any* $b \in B_m$;
(3) $F \cong F[b_1, c_1|\dots|b_i', c_i'|\dots|b_n, c_n]_m$ *for any* $i \in \{1, \dots, n\}$ *and* $b_i' \in B_m$ *and* $c_i' \in \{\sigma_1, \sigma_1^{-1}, \dots, \sigma_{m-1}, \sigma_{m-1}^{-1}\}$ *with* $b_ic_ib_i^{-1} = b_i'c_i'b_i'^{-1}$;
(4) $F \cong F[\tau(b_1), \tau(c_1)|\dots|\tau(b_n), \tau(c_n)]_m$;
(5) $F \cong F[b_1, c_1|\dots|b_i, c_i^{-1}|\dots|b_n, c_n]_m$ *for any* $i \in \{1, \dots, n\}$.

Proof. Statements (1)–(4) are easily verified from the definition. Assertion (5) follows from the fact that a surface G in a 4-ball $B^3 \times [-2, 2]$, illustrated as in

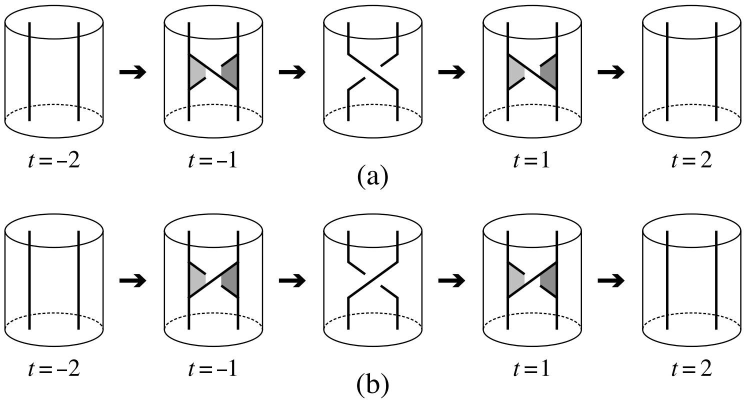

Figure 2

(a) of Figure 2, is ambient isotopic to a surface G' as in (b). For let T be a trivial 2-braid in B^3 and let B (resp. B') be a half-twisted band in B^3 corresponding to σ_1 (resp. σ_1^{-1}). Then G and G' are obtained from $T \times [-2, 2]$ by surgery along 1-handles $B \times [-1, 1]$ and $B' \times [-1, 1]$ respectively. Because these 1-handles have the same core, they are ambient isotopic [Bo; HK]. $\square$

For a 3-braid b we denote by $F(b)$ the surface link $F[1, \sigma_1^{-1} \mid b, \sigma_1^{-1}]_3$ (see Figure 3). Let μ be an automorphism of B_3 with $\mu(\sigma_i) = \sigma_i^{-1}$ $(i = 1, 2)$.

LEMMA 1.3.

(1) *Every surface link F with $\chi(F) = 2$ and* $\mathrm{Braid}(F) \leq 3$ *is ambient isotopic to $F(b)$ for some $b \in B_3$.*
(2) $F(b) \cong F(b^{-1})$.
(3) $F(b) \cong F(b')$ *if* $b\sigma_1^{-1}b^{-1} = b'\sigma_1^{-1}b'^{-1}$.
(4) *The mirror image of $F(b)$ is equivalent to $F(\mu(b))$.*

Proof. (1) Since $\mathrm{Braid}(F) \leq 3$, F is ribbon and hence ambient isotopic to some $F[b_1, c_1|\ldots|b_n, c_n]_3$ (Theorem 1.1). Since $\chi(F) = 2$, we have $n = 2$. By Lemma 1.2 it is deformed into $F(b)$ for some $b \in B_3$. Assertions (2)–(4) are easily verified by Lemma 1.2. $\square$

For a surface link F with $\chi(F) = 2$ and $\mathrm{Braid}(F) \leq 3$, we denote by $J(F)$ the subset of B_3 consisting of all 3-braids b with $F(b) \cong F$. By Lemma 1.3(1), this subset is not empty (it actually consists of infinitely many elements).

The *length* of a 3-braid b is the minimum length of a word expression of b on $\{\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1}\}$. A 3-braid word $s(1)\ldots s(n)$ is *principal* if all $s(1), \ldots, s(n)$ are either in $\{\sigma_1^{-1}, \sigma_2\}$ or in $\{\sigma_1, \sigma_2^{-1}\}$. In other words, the corresponding link (tangle) diagram is alternating. An *oddly principal* 3-braid word is a principal one whose initial and terminal letters are σ_2 or σ_2^{-1}. We call a 3-braid b a *principal*

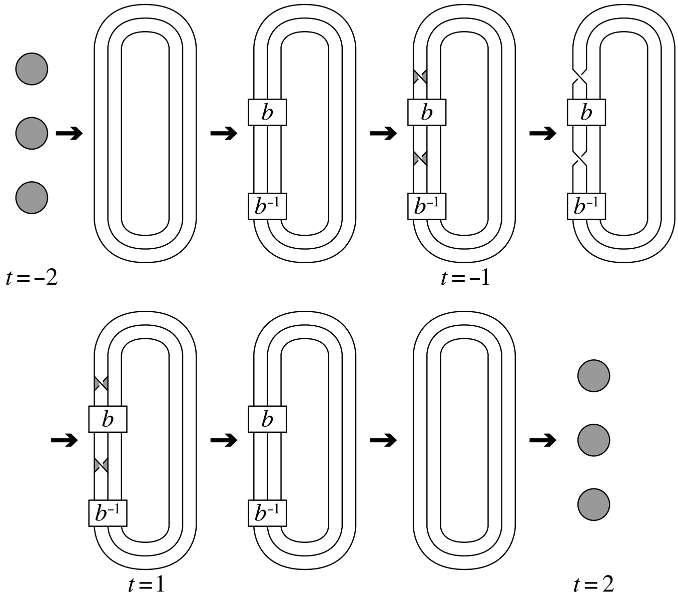

Figure 3

(resp. *oddly principal*) 3-braid if it has a word expression that is principal (resp. oddly principal). We assume that the empty word is oddly principal and so is the identity element $1 \in B_3$.

LEMMA 1.4. *Let b be a principal 3-braid, and let $s(1)\dots s(n)$ be a word expression of b. The word expression is principal if and only if n is the length of b.*

Proof. Every element of B_3 is expressed uniquely in the form

$$x^{2n}x^{a_1}y^{b_1}x^{a_2}y^{b_2}\dots x^{a_k}y^{b_k}$$

in an alternative group presentation $\{x, y \mid x^2 = y^3\}$ of B_3 with $\sigma_1 \longleftrightarrow y^{-1}x$ and $\sigma_2 \longleftrightarrow x^{-1}y^2$, where $n, a_1, \dots, a_k, b_1, \dots, b_k$ are integers satisfying a certain condition (cf. [MKS, p. 46]). Using this condition, we obtain the result. $\square$

LEMMA 1.5. *Let F be a surface link with $\chi(F) = 2$ and* Braid$(F) \leq 3$. *If $b \in J(F)$ has the minimum length among $J(F)$, then it is oddly principal.*

This is our key lemma, which is strengthened as Theorem 2.3 in the next section. We say that a surface link F with $\chi(F) = 2$ and Braid$(F) \leq 3$ is in a *standard form* if it is $F(b)$ for some $b \in J(F)$ as in Lemma 1.5 (or Theorem 2.3).

Proof of Lemma 1.5. Let α be the length of b. If $\alpha = 0$ then $b = 1$. If $\alpha \neq 0$, put $b = s(1)\dots s(\alpha)$ where $s(i) \in \{\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1}\}$ $(i = 1, \dots, \alpha)$. By

Lemma 1.3(2)(3) and the minimality of α, we see that $s(1), s(\alpha) \in \{\sigma_2, \sigma_2^{-1}\}$. The case $\alpha = 1$ is trivial. Assume $\alpha \geq 2$. We assert that if $s(1) = \sigma_2$ then $s(1), \dots, s(\alpha) \in \{\sigma_1^{-1}, \sigma_2\}$. Suppose that there exists an integer k with $1 \leq k < \alpha$ such that $s(1), \dots, s(k) \in \{\sigma_1^{-1}, \sigma_2\}$ and $s(k+1) \in \{\sigma_1, \sigma_2^{-1}\}$. Put $x = s(1) \dots s(k+1)$ and $y = s(k+2) \dots s(\alpha)$. There are three cases,

(1) $x = \sigma_2^{a_1}\sigma_1$,
(2) $x = \sigma_2^{a_1}\sigma_1^{-a_2}\sigma_2^{a_3}\sigma_1^{-a_4} \dots \sigma_1^{-a_{n-1}}\sigma_2^{a_n}\sigma_1$ ($n > 1$, odd),
(3) $x = \sigma_2^{a_1}\sigma_1^{-a_2}\sigma_2^{a_3}\sigma_1^{-a_4} \dots \sigma_2^{a_{n-1}}\sigma_1^{-a_n}\sigma_2^{-1}$ ($n > 0$, even),

where $a_1, \dots, a_n$ are positive integers. According to (1)–(3), let x' be a 3-braid expressed by

(1) $x' = \sigma_2^{-1}\sigma_1^{a_1-2}\sigma_2^{-1}$,
(2) $x' = \sigma_2^{-1}\sigma_1^{a_1-1}\sigma_2^{-a_2}\sigma_1^{a_3}\sigma_2^{-a_4} \dots \sigma_2^{-a_{n-1}}\sigma_1^{a_n-1}\sigma_2^{-1}$,
(3) $x' = \sigma_2^{-1}\sigma_1^{a_1-1}\sigma_2^{-a_2}\sigma_1^{a_3}\sigma_2^{-a_4} \dots \sigma_1^{a_{n-1}}\sigma_2^{-a_n+1}\sigma_1$.

Since $x^{-1}\sigma_1^{-1}x = x'^{-1}\sigma_1^{-1}x'$, by Lemma 1.3 we have $x'y \in J(F)$. Note that the length of $x'y$ is smaller than α unless $x = \sigma_2\sigma_1$. Hence $x = \sigma_2\sigma_1$, $\alpha \geq 3$, and $s(3)$ is σ_1 or $\sigma_2^{\pm 1}$. Put $z = s(4) \dots s(\alpha)$. If $s(3) = \sigma_1$ then $b^{-1}\sigma_1^{-1}b = (\sigma_2^{-1}z)^{-1}\sigma_1^{-1}(\sigma_2^{-1}z)$ and hence $(\sigma_2^{-1}z) \in J(F)$. This is a contradiction, for the length of $\sigma_2^{-1}z$ is smaller than α. If $s(3) = \sigma_2^{\pm 1}$ then

$$b^{-1}\sigma_1^{-1}b = (\sigma_2\sigma_1 z)^{-1}\sigma_1^{-1}(\sigma_2\sigma_1 z),$$

which also yields a contradiction. Thus we have the assertion. For the case $s(1) = \sigma_2^{-1}$, apply the above argument to the mirror image $F(\mu(b))$ of $F(b)$ (Lemma 1.3). $\square$

2. Alexander Polynomials

In this section, we investigate Alexander polynomials of 3-braid 2-knots by use of standard forms.

Let F be a surface link and let $E = \mathbf{R}^4 \backslash F$. A homomorphism $H_1(E; \mathbf{Z}) \to \mathbf{Z}$ sending each oriented meridian of F to $1 \in \mathbf{Z}$ determines an infinite cyclic covering $\tilde{E} \to E$, and $H_1(\tilde{E}; \mathbf{Z})$ is a Λ-module in a natural way where $\Lambda = \mathbf{Z}[t, t^{-1}]$. The *Alexander polynomial* of F is the greatest common divisor of the elements of its zeroth elementary ideal, which is unique up to multiplication of units of Λ. (In case the polynomial is zero, we assume the span is -1.)

Let $\lambda \in \Lambda$ and let $A = (a_{ij})$ be an (m, n)-matrix over Λ. We denote it by $A \in L_{m \times n}(\lambda)$ if there exists a not necessarily strictly increasing function $f : \{1, \dots, n\} \to \{1, \dots, m\}$ such that $a_{ij} = \lambda$ if $i = f(j)$ and $a_{ij} = 0$ otherwise.

LEMMA 2.1. *Let $b = s(1) \dots s(n)$ be an oddly principal 3-braid such that $s(1), \dots, s(n) \in \{\sigma_1^{-1}, \sigma_2\}$ and $n \geq 2$. Let u and v be numbers of σ_1^{-1}'s and σ_2's appearing in b. Then, for any surface link F with $F \cong F(b)$, $H_1(\tilde{E}; \mathbf{Z})$ has a square Λ-presentation matrix*

$$\left[\begin{array}{cccc|ccccc}
1 & & & & -t & & & & 0 \\
t & 1 & & & 0 & & & & 0 \\
& t & \ddots & & \vdots & & A_1 & & \vdots \\
& & \ddots & 1 & 0 & & & & 0 \\
& & & t & 0 & & & & -t \\
\hline
& & & & 1 & t & & & \\
& & & & & 1 & t & & \\
& & A_2 & & & & \ddots & \ddots & \\
& & & & & & & 1 & t
\end{array}\right]$$

of size n $(= u + v)$, where $A_1 \in L_{(u+1)\times(v-2)}(-t)$ and $A_2 \in L_{(v-1)\times u}(-1)$.

Proof. Let R_j $(j = 1, 2, 3)$ be a rectangle $\{\,(x, y, z) \in \mathbf{R}^3_+ \mid 0 \leq x \leq 1,\ y = j,\ 0 \leq z \leq 1\}$ in $\mathbf{R}^3_+ = \{\,(x, y, z) \in \mathbf{R}^3 \mid z \geq 0\,\}$. Let $h_0, h_1, \ldots, h_{n+1}$ be half-twisted bands attached to the $x = 1$ boundary of $R_1 \cup R_2 \cup R_3$ in this order (from the top) such that each band h_i corresponds to $s(i)$ if $i \in \{1, \ldots, n\}$ and to σ_1^{-1} if $i \in \{0, n + 1\}$. For example, in case $b = \sigma_2\sigma_1^{-1}\sigma_2\sigma_2$, the bands $h_0, \ldots, h_5$ are as in Figure 4. For $\theta \in (-\pi, \pi]$, let $\rho_\theta \colon \mathbf{R}^3_+ \to \mathbf{R}^4$ be a map with $\rho_\theta(x, y, z) = (x, y, z\cos\theta, z\sin\theta)$. Put $M_0 = \bigcup_{\theta\in(-\pi,\pi]} \rho_\theta(R_1 \cup R_2 \cup R_3)$, $H_i = \bigcup_{\theta\in(-\pi,\pi]} \rho_\theta(h_i)$ for $i \in \{1, \ldots, n\}$, and $H_i = \bigcup_{\theta\in[-\varepsilon,\varepsilon]} \rho_\theta(h_i)$ for $i \in \{0, n + 1\}$, where ε is a small positive number. Then F is ambient isotopic to the boundary of a 3-manifold $M = M_0 \cup H_0 \cup \cdots \cup H_{n+1}$. Let $j_+, j_- \colon H_1(M; \mathbf{Z}) \to H_1(\mathbf{R}^4 \backslash M; \mathbf{Z})$ be homomorphisms obtained by sliding 1-cycles in M in the positive and negative normal directions of M, respectively. By the Mayer–Vietoris theorem, we have a Λ-isomorphism

$$H_1(\tilde{E}; \mathbf{Z}) \cong H_1(\mathbf{R}^4 \backslash M; \mathbf{Z}) \otimes_{\mathbf{Z}} \Lambda/(j_+ \otimes t - j_- \otimes 1)(H_1(M; \mathbf{Z}) \otimes_{\mathbf{Z}} \Lambda).$$

Let Σ be $R_1 \cup R_2 \cup R_3 \cup h_0 \cup \cdots \cup h_{n+1}$. Rename bands $h_0, \ldots, h_{n+1}$ by $A_1, \ldots, A_{u+2}, B_1, \ldots, B_v$ as in Figure 4 such that $A_1, \ldots, A_{u+2}$ (resp. $B_1, \ldots, B_v$) are attached to $R_1 \cup R_2$ (resp. $R_2 \cup R_3$). Define 1-cycles $a_1, \ldots, a_{u+1}, b_1, \ldots, b_{v-1}$ in Σ as follows: For each $i = 1, \ldots, u + 1$ (resp. $j = 1, \ldots, v - 1$), the 1-cycle a_i (resp. b_j) consists of cores of A_i and A_{i+1} (resp. B_j and B_{j+1}) and two straight segments in $R_1 \cup R_2$ (resp. $R_2 \cup R_3$) connecting end-points of the cores. Assign a_i (resp. b_j) an orientation whose restriction to the core of A_i (resp. B_j) is from R_1 to R_2 (resp. R_2 to R_3); see Figure 4. Then $H_1(\Sigma; \mathbf{Z})$ is a free abelian group with basis $\{a_1, \ldots, a_{u+1}, b_1, \ldots, b_{v-1}\}$, where we use the same symbols for 1-cycles and their homology classes. Let $\{\alpha_1, \ldots, \alpha_{u+2}, \beta_1, \ldots, \beta_v\}$ be a basis of $H_1(\mathbf{R}^3_+ \backslash \Sigma; \mathbf{Z})$ such that α_i $(i = 1, \ldots, u + 2)$ and β_j $(j = 1, \ldots, v)$ are represented by small loops around A_i and B_j with $\mathrm{lk}(\alpha_i, a_i) = 1$ and $\mathrm{lk}(\beta_j, b_j) = 1$ respectively, where $\mathrm{lk}(\cdot, \cdot)$ is the linking number. Let $k_+, k_- \colon H_1(\Sigma; \mathbf{Z}) \to$

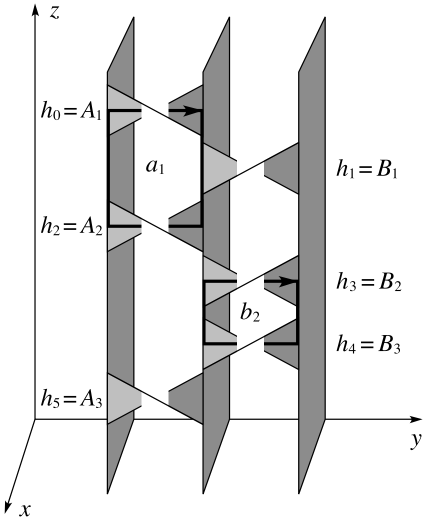

Figure 4

$H_1(\mathbf{R}^3_+ \backslash \Sigma; \mathbf{Z})$ be homomorphisms obtained by sliding 1-cycles in Σ in the positive and negative normal directions of Σ. By construction, the following statements hold.

- (I_1) For each i ($i = 1, \dots, u+1$), α_i is involved in $k_+(a_i)$ and never in $k_+(a_{i'})$ for $i' \neq i$. α_{u+2} does not appear in $k_+(a_i)$ for any i.
- (I_2) For each j ($j = 1, \dots, v$), the term on β_j appears as $-\beta_j$ in $k_+(a_i)$ for a unique $i = i(j)$. If $j_1 < j_2$ then $i(j_1) \leq i(j_2)$.
- (I_3) $k_+(a_1)$ involves $-\beta_1$ and $k_+(a_{u+1})$ involves $-\beta_v$.
- (II) For each j ($j = 1, \dots, v-1$), $k_+(b_j) = \beta_{j+1}$.
- (III) For each i ($i = 1, \dots, u+1$), $k_-(a_i) = -\alpha_{i+1}$.
- (IV_1) For each j ($j = 1, \dots, v-1$), the term on β_j appears as $-\beta_j$ in $k_-(b_j)$ and never in $k_-(b_{j'})$ for $j' \neq j$. β_v is not involved in $k_-(b_j)$ for any j.
- (IV_2) For each i ($i = 2, \dots, u+1$), α_i appears in $k_-(b_j)$ for a unique $j = j(i)$. If $i_1 < i_2$ then $j(i_1) \leq j(i_2)$.
- (IV_3) α_1 and α_{u+2} are not involved in $k_-(b_j)$ for any j.

The map $\rho_0 \colon \mathbf{R}^3_+ \to \mathbf{R}^4$ induces homomorphisms

$$\rho_{0*} \colon H_1(\Sigma; \mathbf{Z}) \to H_1(M; \mathbf{Z})$$

and

$$\rho_{0*} \colon H_1(\mathbf{R}^3_+ \backslash \Sigma; \mathbf{Z}) \to H_1(\mathbf{R}^4 \backslash M; \mathbf{Z}).$$

We use the same symbols for the images of $a_i, b_j, \alpha_i, \beta_j$ under ρ_{0*}. By construction of M, $H_1(M; \mathbf{Z})$ and $H_1(\mathbf{R}^4 \backslash M; \mathbf{Z})$ are free abelian groups with basis $\{a_1, \dots, a_{u+1}, b_1, \dots, b_{v-1}\}$ and $\{\alpha_2, \dots, \alpha_{u+1}, \beta_1, \dots, \beta_v\}$. Notice that

$$\rho_{0*}(\alpha_1) = \rho_{0*}(\alpha_{u+2}) = 0.$$

From the commutative diagram

$$\begin{array}{ccc} H_1(\Sigma; \mathbf{Z}) & \xrightarrow{k_+, k_-} & H_1(\mathbf{R}^3_+ \backslash \Sigma; \mathbf{Z}) \\ \cong \Big\downarrow \rho_{0*} & & \Big\downarrow \rho_{0*} \\ H_1(M; \mathbf{Z}) & \xrightarrow{j_+, j_-} & H_1(\mathbf{R}^4 \backslash M; \mathbf{Z}), \end{array}$$

we see that $H_1(\tilde{E}; \mathbf{Z})$ has the desired Λ-presentation matrix. □

EXAMPLE. Let $b = \sigma_2 \sigma_1^{-1} \sigma_2 \sigma_2$ and $F = F(b)$, with Σ as in Figure 4. Then

$$\begin{aligned} k_+(a_1) &= \alpha_1 - \beta_1, & k_-(a_1) &= -\alpha_2, \\ k_+(a_2) &= \alpha_2 - \beta_2 - \beta_3, & k_-(a_2) &= -\alpha_3, \\ k_+(b_1) &= \beta_2, & k_-(b_1) &= \alpha_2 - \beta_1, \\ k_+(b_2) &= \beta_3, & k_-(b_2) &= -\beta_2, \end{aligned}$$

and

$$\begin{aligned} tj_+(a_1) - j_-(a_1) &= t(-\beta_1) - (-\alpha_2), \\ tj_+(a_2) - j_-(a_2) &= t(\alpha_2 - \beta_2 - \beta_3) - 0, \\ tj_+(b_1) - j_-(b_1) &= t\beta_2 - (\alpha_2 - \beta_1), \\ tj_+(b_2) - j_-(b_2) &= t\beta_3 - (-\beta_2). \end{aligned}$$

Thus we obtain the presentation matrix

$$\left[\begin{array}{c|ccc} 1 & -t & & \\ t & & -t & -t \\ \hline -1 & 1 & t & \\ & & 1 & t \end{array}\right];$$

the determinant, which is the Alexander polynomial of F, is $t^4 - t^3 + 2t^2 - t$. Hence it is not a spun 2-knot.

COROLLARY 2.2. *Let F be a surface link with $\chi(F) = 2$ and* $\mathrm{Braid}(F) \leq 3$. *If $b \in J(F)$ is oddly principal then the length of b is the span of the Alexander polynomial of F plus one.*

Proof. If $b = 1$ then F is the unknotted surface link $S^2 \amalg T^2$ whose Alexander polynomial is zero. If $b = \sigma_2$ or σ_2^{-1}, then F is an unknotted 2-knot whose Alexander polynomial is unity. In case the length n of b exceeds unity, by Lemmas 1.3(4) and 1.4 we may assume that F and b are as in Lemma 2.1. The Alexander polynomial of F is the determinant of a square matrix of size n, as in the lemma, whose span is $n - 1$. □

Our main theorem is as follows.

THEOREM 2.3. *Let F be a surface link with $\chi(F) = 2$ and* $\mathrm{Braid}(F) \leq 3$. *For $b \in J(F)$, the following three conditions are mutually equivalent.*

(1) *b is oddly principal.*
(2) *b has the minimum length among $J(F)$.*
(3) *The length of b is the span of the Alexander polynomial of F plus one.*

Proof. It is a consequence of Lemma 1.5 and Corollary 2.2. □

THEOREM 2.4. *Every* 3*-braid* 2*-knot has a nontrivial Alexander polynomial.*

Proof. By Theorem 2.3, a 3-braid 2-knot with a trivial Alexander polynomial is ambient isotopic to $F(\sigma_2)$ or $F(\sigma_2^{-1})$. It is an unknotted 2-knot of braid index 1, a contradiction. □

THEOREM 2.5. *For any* 3*-braid* 2*-knot, the coefficients of the terms of the Alexander polynomial of maximal and minimal degree are* ± 1.

Proof. This follows from Lemmas 1.3(4), 1.5, and 2.1. □

THEOREM 2.6. *The number of* 3*-braid* 2*-knot types such that the spans of their Alexander polynomials are the same is finite.*

Proof. Since there are finitely many oddly principal 3-braids with a given length, the result follows from Theorem 2.3. □

3. Tabulation of 3-Braid 2-Knots

Throughout this section, F denotes a surface link with $\chi(F) = 2$ and $\mathrm{Braid}(F) \leq 3$. Let $\alpha(F)$ stand for the length of $b \in J(F)$ as in Theorem 2.3, which is the span of the Alexander polynomial of F plus one. We shall denote by $[F]$ (resp. $[F]^*$) the knot type—that is, the ambient isotopy class—of F (resp. the knot type modulo mirror images).

For each nonnegative integer α, let H_α (resp. H_α^*) be the set of knot types (resp. knot types modulo mirror images) of F's such that $\alpha(F) = \alpha$. Both H_0 and H_0^* consist of the class of an unknotted surface link being $S^2 \amalg T^2$. H_1 and H_1^* consist of the class of an unknotted 2-knot.

For each integer $\alpha \geq 2$, let G_α be the power set of $\{1, 2, \ldots, \alpha - 2\}$ and define a map

$$\varphi\colon G_\alpha \to B_3$$

by $\varphi(g) = s(1) \ldots s(\alpha - 2)$ with $s(i) = \sigma_1^{-1}$ if $i \in g$ and $s(i) = \sigma_2$ otherwise. By Theorem 2.3 and Lemma 1.3(4) we have a surjection,

$$G_\alpha \to H_\alpha \to H_\alpha^*, \quad g \mapsto [F(\sigma_2\varphi(g)\sigma_2)] \mapsto [F(\sigma_2\varphi(g)\sigma_2)]^*;$$

in other words, if $\alpha = \alpha(F) \geq 2$ then F is ambient isotopic to $F(\sigma_2\varphi(g)\sigma_2)$ for some $g \in G_\alpha$ or its mirror image.

For $g \in G_\alpha$, let $g^{\mathrm{co}} = \{1, \ldots, \alpha - 2\} \setminus g$ and $g^{\mathrm{op}} = \{\alpha - 1 - j \mid j \in g\}$.

LEMMA 3.1. *For any $g \in G_\alpha$ ($\alpha \geq 2$), both $F(\sigma_2\varphi(g^{\mathrm{co}})\sigma_2)$ and $F(\sigma_2\varphi(g^{\mathrm{op}})\sigma_2)$ are ambient isotopic to the mirror image of $F(\sigma_2\varphi(g)\sigma_2)$.*

Proof. Note that

$$\varphi(g^{\mathrm{co}}) = \tau \circ \mu(\varphi(g)) \quad \text{and} \quad \varphi(g^{\mathrm{op}}) = \mu(\varphi(g))^{-1},$$

where τ and μ are as before. By Lemma 1.3, the mirror image of $F(\sigma_2\varphi(g)\sigma_2)$ is $F(\sigma_2^{-1}\mu(\varphi(g))\sigma_2^{-1})$. By Lemma 1.2,

$$\begin{aligned}
F(\sigma_2^{-1}\mu(\varphi(g))\sigma_2^{-1}) &= F[1, \sigma_1^{-1} \mid \sigma_2^{-1}\mu(\varphi(g))\sigma_2^{-1}, \sigma_1^{-1}] \\
&\cong F[\sigma_1\sigma_2, \sigma_1^{-1} \mid \sigma_1\mu(\varphi(g))\sigma_2^{-1}, \sigma_1^{-1}] \\
&\cong F[1, \sigma_2^{-1} \mid \sigma_1\mu(\varphi(g))\sigma_2^{-1}, \sigma_1^{-1}] \\
&\cong F[1, \sigma_2^{-1} \mid \sigma_1\mu(\varphi(g))\sigma_1, \sigma_2^{-1}] \\
&\cong F[1, \sigma_1^{-1} \mid \sigma_2\tau \circ \mu(\varphi(g))\sigma_2, \sigma_1^{-1}] \\
&= F(g^{\mathrm{co}})
\end{aligned}$$

and

$$\begin{aligned}
F(\sigma_2^{-1}\mu(\varphi(g))\sigma_2^{-1}) &= F[1, \sigma_1^{-1} \mid \sigma_2^{-1}\mu(\varphi(g))\sigma_2^{-1}, \sigma_1^{-1}] \\
&\cong F[\sigma_2^{-1}\mu(\varphi(g))\sigma_2^{-1}, \sigma_1^{-1} \mid 1, \sigma_1^{-1}] \\
&\cong F[1, \sigma_1^{-1} \mid \sigma_2\mu(\varphi(g))^{-1}\sigma_2, \sigma_1^{-1}] \\
&= F(g^{\mathrm{op}}).
\end{aligned}$$

□

COROLLARY 3.2. *If $g = g^{\mathrm{op}}$ then $F(\sigma_2\varphi(g)\sigma_2)$ is amphicheiral.*

Define an equivalence relation $\sim$ on G_α by $g \sim g^{\mathrm{co}} \sim g^{\mathrm{op}} \sim g^{\mathrm{coop}} = g^{\mathrm{opco}}$. We denote by $[g]^*$ the equivalence class of g and by G_α^* the quotient set of G_α. By Lemma 3.1, the surjection $G_\alpha \to H_\alpha^*$ induces a surjection

$$\Phi_\alpha\colon G_\alpha^* \to H_\alpha^*, \quad [g]^* \mapsto [F(\sigma_2\varphi(g)\sigma_2)]^*.$$

We provide a list of H_α^* for $\alpha \leq 10$ in Tables 1–5. (All surface links in the list are distinguished by their Alexander polynomials except three pairs: 9_5 and 9_{11}; 10_{19} and 10_{32}; 10_{44} and 10_{57}. For a surface link F and a positive integer d, let $I_d(F)$ be the number of S_d-conjugacy classes of transitive representations of $\pi_1(\mathbf{R}^4 \setminus F)$ to the symmetric group S_d on d letters. Using the computer program "Knot" by Dr. Kouji Kodama, we have a partial list of $I_d(F)$ as in Table 6, which shows $9_5 \ncong 9_{11}$, $10_{19} \ncong 10_{32}$, and $10_{44} \ncong 10_{57}$. To determine whether or not each

	g		Alexander Polynomials	
0_1	—	$T_{0,1}$	0	A
1_1	—	$S_{1,1}$	1	A
2_1	{}	$T_{2,1}$	1, −1	A
3_1	{}	$S_{3,1}$	1, −1, 1	A
4_1	{}	$T_{4,1}$	1, −1, 1, −1	A
4_2	{1}	$S_{4,1}$	1, −1, 2, −1	N
5_1	{}	$S_{5,1}$	1, −1, 1, −1, 1	A
5_2	{1}	$S_{5,2}$	1, −1, 2, −2, 1	N
5_3	{2}	$T_{5,1}$	1, −2, 2, −2, 1	A
6_1	{}	$T_{6,1}$	1, −1, 1, −1, 1, −1	A
6_2	{1}	$S_{6,1}$	1, −1, 2, −2, 2, −1	N
6_3	{2}	$S_{6,2}$	1, −2, 2, −3, 2, −1	N
6_4	{1, 2}	$T_{6,2}$	1, −1, 2, −3, 2, −1	N
6_5	{1, 3}	$S_{6,3}$	1, −2, 3, −3, 3, −1	N
6_6	{1, 4}	$T_{6,3}$	1, −2, 3, −3, 2, −1	A
7_1	{}	$S_{7,1}$	1, −1, 1, −1, 1, −1, 1	A
7_2	{1}	$S_{7,2}$	1, −1, 2, −2, 2, −2, 1	N
7_3	{2}	$T_{7,1}$	1, −2, 2, −3, 3, −2, 1	N
7_4	{3}	$S_{7,3}$	1, −2, 3, −3, 3, −2, 1	A
7_5	{1, 2}	$S_{7,4}$	1, −1, 2, −3, 3, −2, 1	N
7_6	{1, 3}	$T_{7,2}$	1, −2, 3, −4, 4, −3, 1	N
7_7	{1, 4}	$S_{7,5}$	1, −2, 4, −4, 4, −3, 1	N
7_8	{1, 5}	$T_{7,3}$	1, −2, 3, −4, 3, −2, 1	A
7_9	{2, 3}	$S_{7,6}$	1, −2, 3, −4, 4, −2, 1	N
7_{10}	{2, 4}	$S_{7,7}$	1, −3, 4, −5, 4, −3, 1	A

Table 1

F is amphicheiral, we use Corollary 3.2 and the fact that the Alexander polynomial of an amphicheiral surface link must be reciprocal; i.e., $f(t) = \pm t^n f(t^{-1})$ for some n.)

In the first column $\alpha(F)$ $(= \alpha)$ is given. The subscript indicates the order of $[F]^*$ in H^*_α. In the second column an element $g \in G_\alpha$ with $\Phi_\alpha([g]^*) = [F]^*$ is given. Using it, one can recover the configuration of F. For the third column we divide H^*_α into two families, S^*_α and T^*_α. The symbol S (resp. T) means that F is a 2-knot (resp. a surface link that is a union of a 2-sphere and a torus). The first subscript indicates α and the second the order of $[F]^*$ in the subset S^*_α (resp. T^*_α). In the fourth column, the coefficients of an Alexander polynomial of $[F]^*$ are given. (The Alexander polynomial of $[F]^*$ should be considered up to *weak equivalence*: $f(t)$ is *weakly equivalent* to $g(t)$ if $f(t)$ is $\pm t^n g(t)$ or $\pm t^n g(t^{-1})$ for

	g		Alexander Polynomials	
8_1	{}	$T_{8,1}$	1, −1, 1, −1, 1, −1, 1, −1	A
8_2	{1}	$S_{8,1}$	1 − 1, 2, −2, 2, −2, 2, −1	N
8_3	{2}	$S_{8,2}$	1, −2, 2, −3, 3, −3, 2, −1	N
8_4	{3}	$S_{8,3}$	1, −2, 3, −3, 4, −3, 2, −1	N
8_5	{1, 2}	$T_{8,2}$	1, −1, 2, −3, 3, −3, 2, −1	N
8_6	{1, 3}	$S_{8,4}$	1, −2, 3, −4, 5, −4, 3, −1	N
8_7	{1, 4}	$T_{8,3}$	1, −2, 4, −5, 5, −5, 3, −1	N
8_8	{1, 5}	$S_{8,5}$	1, −2, 4, −5, 5, −4, 3, −1	N
8_9	{1, 6}	$T_{8,4}$	1, −2, 3, −4, 4, −3, 2, −1	A
8_{10}	{2, 3}	$T_{8,5}$	1, −2, 3, −4, 5, −4, 2, −1	N
8_{11}	{2, 4}	$S_{8,6}$	1, −3, 4, −6, 6, −5, 3, −1	N
8_{12}	{2, 5}	$T_{8,6}$	1, −3, 5, −6, 6, −5, 3, −1	A
8_{13}	{3, 4}	$T_{8,7}$	1, −2, 4, −5, 5, −4, 2, −1	A
8_{14}	{1, 2, 3}	$S_{8,7}$	1, −1, 2, −3, 4, −3, 2, −1	N
8_{15}	{1, 2, 4}	$S_{8,8}$	1, −2, 3, −5, 5, −5, 3, −1	N
8_{16}	{1, 2, 5}	$S_{8,9}$	1, −2, 4, −5, 6, −5, 3, −1	N
8_{17}	{1, 2, 6}	$S_{8,10}$	1, −2, 3, −5, 5, −4, 2, −1	N
8_{18}	{1, 3, 5}	$T_{8,8}$	1, −3, 5, −7, 7, −6, 4, −1	N
8_{19}	{1, 3, 6}	$S_{8,11}$	1, −3, 5, −6, 7, −5, 3, −1	N
8_{20}	{1, 4, 5}	$S_{8,12}$	1, −2, 5, −6, 6, −5, 3, −1	N

Table 2

some n.) In the last column, "A" (resp. "N") denotes that F is amphicheiral (resp. non-amphicheiral).

Since the spun 2-knot of a figure-eight knot has Alexander polynomial $t^2 - 3t + 1$ which is out of the list, we see that it is not a 3-braid 2-knot.

Concluding Remarks. The surjection $\Phi_\alpha : G^*_\alpha \to H^*_\alpha$ is an injection (i.e. bijection) for $\alpha \leq 10$; in other words, the weak equivalence classes of 3-braid 2-knots whose Alexander polynomials have spans less than 10 are completely classified by standard forms. Is there an integer α such that Φ_α is not injective? For $\alpha \leq 10$, the converse of Corollary 3.2 holds; namely, standard forms determine amphicheirality of 3-braid 2-knots with $\alpha \leq 10$. Is this true for every α?

	g		Alexander Polynomials	
9_1	$\{\}$	$S_{9,1}$	$1, -1, 1, -1, 1, -1, 1, -1, 1$	A
9_2	$\{1\}$	$S_{9,2}$	$1, -1, 2, -2, 2, -2, 2, -2, 1$	N
9_3	$\{2\}$	$T_{9,1}$	$1, -2, 2, -3, 3, -3, 3, -2, 1$	N
9_4	$\{3\}$	$S_{9,3}$	$1, -2, 3, -3, 4, -4, 3, -2, 1$	N
9_5	$\{4\}$	$T_{9,2}$	$1, -2, 3, -4, 4, -4, 3, -2, 1$	A
9_6	$\{1, 2\}$	$S_{9,4}$	$1, -1, 2, -3, 3, -3, 3, -2, 1$	N
9_7	$\{1, 3\}$	$T_{9,3}$	$1, -2, 3, -4, 5, -5, 4, -3, 1$	N
9_8	$\{1, 4\}$	$S_{9,5}$	$1, -2, 4, -5, 6, -6, 5, -3, 1$	N
9_9	$\{1, 5\}$	$T_{9,4}$	$1, -2, 4, -6, 6, -6, 5, -3, 1$	N
9_{10}	$\{1, 6\}$	$S_{9,6}$	$1, -2, 4, -5, 6, -5, 4, -3, 1$	N
9_{11}	$\{1, 7\}$	$T_{9,5}$	$1, -2, 3, -4, 4, -4, 3, -2, 1$	A
9_{12}	$\{2, 3\}$	$S_{9,7}$	$1, -2, 3, -4, 5, -5, 4, -2, 1$	N
9_{13}	$\{2, 4\}$	$S_{9,8}$	$1, -3, 4, -6, 7, -7, 5, -3, 1$	N
9_{14}	$\{2, 5\}$	$S_{9,9}$	$1, -3, 5, -7, 8, -7, 6, -3, 1$	N
9_{15}	$\{2, 6\}$	$S_{9,10}$	$1, -3, 5, -7, 7, -7, 5, -3, 1$	A
9_{16}	$\{3, 4\}$	$S_{9,11}$	$1, -2, 4, -5, 6, -6, 4, -2, 1$	N
9_{17}	$\{3, 5\}$	$T_{9,6}$	$1, -3, 5, -7, 8, -7, 5, -3, 1$	A
9_{18}	$\{1, 2, 3\}$	$S_{9,12}$	$1, -1, 2, -3, 4, -4, 3, -2, 1$	N
9_{19}	$\{1, 2, 4\}$	$T_{9,7}$	$1, -2, 3, -5, 6, -6, 5, -3, 1$	N
9_{20}	$\{1, 2, 5\}$	$S_{9,13}$	$1, -2, 4, -6, 7, -7, 6, -3, 1$	N
9_{21}	$\{1, 2, 6\}$	$T_{9,8}$	$1, -2, 4, -6, 7, -7, 5, -3, 1$	N
9_{22}	$\{1, 2, 7\}$	$S_{9,14}$	$1, -2, 3, -5, 6, -5, 4, -2, 1$	N
9_{23}	$\{1, 3, 4\}$	$S_{9,15}$	$1, -2, 4, -5, 7, -7, 5, -3, 1$	N
9_{24}	$\{1, 3, 5\}$	$S_{9,16}$	$1, -3, 5, -8, 9, -9, 7, -4, 1$	N
9_{25}	$\{1, 3, 6\}$	$S_{9,17}$	$1, -3, 6, -8, 10, -9, 7, -4, 1$	N
9_{26}	$\{1, 3, 7\}$	$S_{9,18}$	$1, -3, 5, -7, 8, -8, 5, -3, 1$	N
9_{27}	$\{1, 4, 5\}$	$S_{9,19}$	$1, -2, 5, -7, 8, -8, 6, -3, 1$	N
9_{28}	$\{1, 4, 6\}$	$T_{9,9}$	$1, -3, 6, -9, 10, -9, 7, -4, 1$	N
9_{29}	$\{1, 4, 7\}$	$S_{9,20}$	$1, -3, 6, -8, 9, -8, 6, -3, 1$	A
9_{30}	$\{1, 5, 6\}$	$S_{9,21}$	$1, -2, 5, -7, 8, -7, 5, -3, 1$	N
9_{31}	$\{2, 3, 4\}$	$T_{9,10}$	$1, -2, 3, -5, 6, -6, 4, -2, 1$	N
9_{32}	$\{2, 3, 5\}$	$S_{9,22}$	$1, -3, 5, -7, 9, -8, 6, -3, 1$	N
9_{33}	$\{2, 3, 6\}$	$T_{9,11}$	$1, -3, 6, -8, 9, -9, 6, -3, 1$	N
9_{34}	$\{2, 4, 5\}$	$T_{9,12}$	$1, -3, 5, -8, 9, -8, 6, -3, 1$	N
9_{35}	$\{2, 4, 6\}$	$S_{9,23}$	$1, -4, 7, -10, 11, -10, 7, -4, 1$	A
9_{36}	$\{3, 4, 5\}$	$S_{9,24}$	$1, -2, 4, -6, 7, -6, 4, -2, 1$	A

Table 3

	g		Alexander Polynomials	
10_1	{}	$T_{10,1}$	1, −1, 1, −1, 1, −1, 1, −1, 1, −1	A
10_2	{1}	$S_{10,1}$	1, −1, 2, −2, 2, −2, 2, −2, 2, −1	N
10_3	{2}	$S_{10,2}$	1, −2, 2, −3, 3, −3, 3, −3, 2, −1	N
10_4	{3}	$S_{10,3}$	1, −2, 3, −3, 4, −4, 4, −3, 2, −1	N
10_5	{4}	$S_{10,4}$	1, −2, 3, −4, 4, −5, 4, −3, 2, −1	N
10_6	{1, 2}	$T_{10,2}$	1, −1, 2, −3, 3, −3, 3, −3, 2, −1	N
10_7	{1, 3}	$S_{10,5}$	1, −2, 3, −4, 5, −5, 5, −4, 3, −1	N
10_8	{1, 4}	$T_{10,3}$	1, −2, 4, −5, 6, −7, 6, −5, 3, −1	N
10_9	{1, 5}	$S_{10,6}$	1, −2, 4, −6, 7, −7, 7, −5, 3, −1	N
10_{10}	{1, 6}	$T_{10,4}$	1, −2, 4, −6, 7, −7, 6, −5, 3, −1	N
10_{11}	{1, 7}	$S_{10,7}$	1, −2, 4, −5, 6, −6, 5, −4, 3, −1	N
10_{12}	{1, 8}	$T_{10,5}$	1, −2, 3, −4, 4, −4, 4, −3, 2, −1	A
10_{13}	{2, 3}	$T_{10,6}$	1, −2, 4, −5, 5, −5, 4, −3, 2, −1	N
10_{14}	{2, 4}	$S_{10,8}$	1, −3, 4, −6, 7, −8, 7, −5, 3, −1	N
10_{15}	{2, 5}	$T_{10,7}$	1, −3, 5, −7, 9, −9, 8, −6, 3, −1	N
10_{16}	{2, 6}	$S_{10,9}$	1, −3, 5, −8, 9, −9, 8, −6, 3, −1	N
10_{17}	{2, 7}	$T_{10,8}$	1, −3, 5, −7, 8, −8, 7, −5, 3, −1	A
10_{18}	{3, 4}	$T_{10,9}$	1, −2, 4, −5, 6, −7, 6, −4, 2, −1	N
10_{19}	{3, 5}	$S_{10,10}$	1, −3, 5, −7, 9, −9, 8, −5, 3, −1	N
10_{20}	{3, 6}	$T_{10,10}$	1, −3, 6, −8, 10, −10, 8, −6, 3, −1	A
10_{21}	{4, 5}	$T_{10,11}$	1, −2, 4, −6, 7, −7, 6, −4, 2, −1	A
10_{22}	{1, 2, 3}	$S_{10,11}$	1, −1, 2, −3, 4, −4, 4, −3, 2, −1	N
10_{23}	{1, 2, 4}	$S_{10,12}$	1, −2, 3, −5, 6, −7, 6, −5, 3, −1	N
10_{24}	{1, 2, 5}	$S_{10,13}$	1, −2, 4, −6, 8, −8, 8, −6, 3, −1	N
10_{25}	{1, 2, 6}	$S_{10,14}$	1, −2, 4, −7, 8, −9, 8, −6, 3, −1	N
10_{26}	{1, 2, 7}	$S_{10,15}$	1, −2, 4, −6, 8, −8, 7, −5, 3, −1	N
10_{27}	{1, 2, 8}	$S_{10,16}$	1, −2, 3, −5, 6, −6, 5, −4, 2, −1	N
10_{28}	{1, 3, 4}	$S_{10,17}$	1, −2, 4, −5, 7, −8, 7, −5, 3, −1	N
10_{29}	{1, 3, 5}	$T_{10,12}$	1, −3, 5, −8, 10, −11, 10, −7, 4, −1	N
10_{30}	{1, 3, 6}	$S_{10,18}$	1, −3, 6, −9, 12, −12, 11, −8, 4, −1	N
10_{31}	{1, 3, 7}	$T_{10,13}$	1, −3, 6, −9, 11, −12, 10, −7, 4, −1	N
10_{32}	{1, 3, 8}	$S_{10,19}$	1, −3, 5, −7, 9, −9, 8, −5, 3, −1	N
10_{33}	{1, 4, 5}	$S_{10,20}$	1, −2, 5, −7, 9, −10, 9, −6, 3, −1	N
10_{34}	{1, 4, 6}	$S_{10,21}$	1, −3, 6, −10, 12, −13, 11, −8, 4, −1	N
10_{35}	{1, 4, 7}	$S_{10,22}$	1, −3, 7, −10, 13, −13, 11, −8, 4, −1	N
10_{36}	{1, 4, 8}	$S_{10,23}$	1, −3, 6, −9, 10, −11, 9, −6, 3, −1	N

Table 4

	g		Alexander Polynomials	
10_{37}	{1, 5, 6}	$S_{10,24}$	1, −2, 5, −8, 10, −10, 9, −6, 3, −1	N
10_{38}	{1, 5, 7}	$T_{10,14}$	1, −3, 6, −10, 12, −12, 10, −7, 4, −1	N
10_{39}	{1, 6, 7}	$S_{10,25}$	1, −2, 5, −7, 9, −9, 7, −5, 3, −1	N
10_{40}	{2, 3, 4}	$S_{10,26}$	1, −2, 3, −5, 6, −7, 6, −4, 2, −1	N
10_{41}	{2, 3, 5}	$S_{10,27}$	1, −3, 5, −7, 10, −10, 9, −6, 3, −1	N
10_{42}	{2, 3, 6}	$S_{10,28}$	1, −3, 6, −9, 11, −12, 10, −7, 3, −1	N
10_{43}	{2, 3, 7}	$S_{10,29}$	1, −3, 6, −9, 11, −11, 10, −6, 3, −1	N
10_{44}	{2, 4, 5}	$S_{10,30}$	1, −3, 5, −8, 10, −11, 9, −6, 3, −1	N
10_{45}	{2, 4, 6}	$T_{10,15}$	1, −4, 7, −11, 14, −14, 12, −8, 4, −1	N
10_{46}	{2, 4, 7}	$S_{10,31}$	1, −4, 8, −12, 14, −15, 12, −8, 4, −1	N
10_{47}	{2, 5, 6}	$S_{10,32}$	1, −3, 6, −10, 12, −12, 10, −7, 3, −1	N
10_{48}	{3, 4, 5}	$S_{10,33}$	1, −2, 4, −6, 8, −8, 7, −4, 2, −1	N
10_{49}	{3, 4, 6}	$S_{10,34}$	1, −3, 6, −9, 11, −12, 9, −6, 3, −1	N
10_{50}	{1, 2, 3, 4}	$T_{10,16}$	1, −1, 2, −3, 4, −5, 4, −3, 2, −1	N
10_{51}	{1, 2, 3, 5}	$S_{10,35}$	1, −2, 3, −5, 7, −7, 7, −5, 3, −1	N
10_{52}	{1, 2, 3, 6}	$T_{10,17}$	1, −2, 4, −6, 8, −9, 8, −6, 3, −1	N
10_{53}	{1, 2, 3, 7}	$S_{10,36}$	1, −2, 4, −6, 8, −9, 8, −5, 3, −1	N
10_{54}	{1, 2, 3, 8}	$T_{10,18}$	1, −2, 3, −5, 7, −7, 6, −4, 2, −1	N
10_{55}	{1, 2, 4, 6}	$S_{10,37}$	1, −3, 5, −9, 11, −12, 11, −8, 4, −1	N
10_{56}	{1, 2, 4, 7}	$T_{10,19}$	1, −3, 6, −9, 12, −13, 11, −8, 4, −1	N
10_{57}	{1, 2, 4, 8}	$S_{10,38}$	1, −3, 5, −8, 10, −11, 9, −6, 3, −1	N
10_{58}	{1, 2, 5, 6}	$T_{10,20}$	1, −2, 5, −8, 10, −11, 10, −7, 3, −1	N
10_{59}	{1, 2, 5, 7}	$S_{10,39}$	1, −3, 6, −10, 13, −13, 12, −8, 4, −1	N
10_{60}	{1, 2, 5, 8}	$T_{10,21}$	1, −3, 6, −9, 12, −12, 10, −7, 3, −1	N
10_{61}	{1, 2, 6, 7}	$T_{10,22}$	1, −2, 5, −8, 10, −11, 9, −6, 3, −1	N
10_{62}	{1, 2, 6, 8}	$S_{10,40}$	1, −3, 5, −9, 11, −11, 9, −6, 3, −1	N
10_{63}	{1, 2, 7, 8}	$T_{10,23}$	1, −2, 4, −6, 8, −8, 6, −4, 2, −1	A
10_{64}	{1, 3, 4, 7}	$S_{10,41}$	1, −3, 7, −10, 13, −14, 12, −8, 4, −1	N
10_{65}	{1, 3, 4, 8}	$T_{10,24}$	1, −3, 6, −9, 11, −12, 10, −6, 3, −1	N
10_{66}	{1, 3, 5, 7}	$S_{10,42}$	1, −4, 8, −13, 16, −17, 14, −10, 5, −1	N
10_{67}	{1, 3, 5, 8}	$S_{10,43}$	1, −4, 8, −12, 15, −15, 13, −8, 4, −1	N
10_{68}	{1, 3, 6, 7}	$S_{10,44}$	1, −3, 7, −11, 14, −14, 12, −8, 4, −1	N
10_{69}	{1, 3, 6, 8}	$T_{10,25}$	1, −4, 8, −12, 15, −15, 12, −8, 4, −1	A
10_{70}	{1, 4, 5, 8}	$T_{10,26}$	1, −3, 7, −11, 13, −13, 11, −7, 3, −1	A
10_{71}	{1, 4, 6, 7}	$T_{10,27}$	1, −3, 7, −11, 14, −14, 11, −8, 4, −1	N
10_{72}	{1, 5, 6, 7}	$S_{10,45}$	1, −2, 5, −8, 10, −10, 8, −5, 3, −1	N

Table 5

F	I_2	I_3	I_4	I_5	I_6	I_7	I_8
9_5	3	7	22	37			
9_{11}	3	7	24	47			
10_{19}	1	2	3	2	8	7	10
10_{32}	1	2	3	2	5	7	13
10_{44}	1	2	3	3	9	9	17
10_{57}	1	2	3	3	9	10	17

Table 6

References

[B1] J. S. Birman, *Braids, links, and mapping class groups,* Ann. of Math. Stud., 82, Princeton Univ. Press, Princeton, NJ, 1974.

[B2] ———, *On the Jones polynomial of closed 3-braids,* Invent. Math. 81 (1985), 287–294.

[BM] J. S. Birman and W. W. Menasco, *Studying links via closed braids III: Classifying links which are closed 3-braids,* Pacific J. Math. 161 (1993), 25–113.

[Bo] J. Boyle, *Classifying 1-handles attached to knotted surfaces,* Trans. Amer. Math. Soc. 306 (1988), 475–487.

[CS] J. S. Carter and M. Saito, *Knotted surfaces, braid moves and beyond*, Knots and quantum gravity, pp. 191–229, Oxford Science Publishing, 1994.

[HK] F. Hosokawa and A. Kawauchi, *Proposals for unknotted surfaces in four-spaces,* Osaka J. Math. 16 (1979), 233–248.

[K1] S. Kamada, *Surfaces in* $\mathbf{R}^4$ *of braid index three are ribbon*, J. Knot Theory Ramifications 1 (1992), 137–160.

[K2] ———, *A characterization of groups of closed orientable surfaces in 4-space,* Topology 33 (1994), 113–122.

[MKS] W. Magnus, A. Karrass, and D. Solitar, *Combinatorial group theory,* Wiley, New York, 1966.

[Mu] K. Murasugi, *On closed 3-braids,* Mem. Amer. Math. Soc., 151, Amer. Math. Soc., Providence, RI, 1974.

[R1] L. Rudolph, *Braided surfaces and Seifert ribbons for closed braids,* Comment. Math. Helv. 58 (1983), 1–37.

[R2] ———, *Special positions for surfaces bounded by closed braids,* Rev. Mat. Iberoamericana 1 (1985), 93–133.

[T] M. Takahashi, *Explicit formulas for Jones polynomials of closed 3-braids,* Comment. Math. Univ. St. Paul. 38 (1989), 129–167.

[V] O. Ya. Viro, Lecture given at Osaka City University, September, 1990.

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