Projectivity and Extensions of Hilbert Modules over $\mathbb{A}(\mathbb{D}^N)$

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1. Introduction

Let $\mathbb{A}(\mathbb{D}^N)$ denote the algebra of functions analytic in the polydisk $\mathbb{D}^N$ that extend continuously to its closure, and let $\mathcal{H}$ be the category of all Hilbert $\mathbb{A}(\mathbb{D}^N)$-modules. Briefly, an *object* in $\mathcal{H}$ is a Hilbert space H, together with a continuous bilinear multiplication $\mathbb{A}(\mathbb{D}^N) \times H \to H$. Douglas and Paulsen [5] were the first to approach the category $\mathcal{H}$ as a natural setting for certain questions of operator theory. For example, considering the *operator variables,* the operators of multiplication by $z_1, \dots, z_N$ in H, the transition from 1 to N commuting operators can be viewed as a natural one. In the present note, we deal with several questions about Hilbert modules over $\mathbb{A}(\mathbb{D}^N)$ that have been previously answered for $N = 1$.

One of the problems left open by Douglas and Paulsen [5] was that of finding projective objects in the cateogry $\mathcal{H} = \mathcal{H}(\mathbb{A})$ for any function algebra $\mathbb{A}$. An answer in the case that $\mathbb{A} = \mathbb{A}(\mathbb{D})$ was given in [2], where it was proved that unitary Hilbert modules (Hilbert modules where the operator variable is unitary) are always projective. One purpose of the present note is to obtain an extension of the result to modules over $\mathbb{A}(\mathbb{D}^N)$ (see Theorem 3.1).

A second question of importance is whether the category $\mathcal{H}$ has *enough* projectives, in the sense that every object in $\mathcal{H}$ is a quotient of some projective module. Another result of [2] shows that the category $\mathcal{C}$ of *cramped* Hilbert modules—that is, Hilbert modules similar to contractive ones—has enough projectives. Specifically, it was proved that isometric Hilbert modules are projective in $\mathcal{C}$ and that the Sz.-Nagy–Foiaş model for a completely nonunitary contractive Hilbert module over $\mathbb{A}(\mathbb{D})$ gives a projective resolution. On the other hand, Pisier's recent example ([7], see also [4]) demonstrates that the isometric Hilbert modules are not in general projective in $\mathcal{H}(\mathbb{A}(\mathbb{D}))$. In particular, the vector-valued Hardy space $\mathbb{H}^2(H)$ is not projective, though it remains unknown whether non–vector-valued $\mathbb{H}^2$ is projective in $\mathcal{H}$. The examples make it seem doubtful that $\mathcal{H}(\mathbb{A}(\mathbb{D}))$ has enough projectives, though there is not yet any proof.

For $\mathbb{A}(\mathbb{D}^N)$-modules, the situation is worse. In the cramped category $\mathcal{C}(\mathbb{A}(\mathbb{D}^N))$, the best that can be proved is that, if one of the operator variables is an isometry and all of the others are unitary, then the object is projective (Theorem

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3.2). On the other hand, a result of Cotlar and Sadosky can be used to prove that $\mathrm{Ext}^1_{\mathcal{C}}(\mathbb{H}^2(\mathbb{D}^2), \mathbb{H}^2(\mathbb{D}^2)) \neq 0$ and hence $\mathbb{H}^2(\mathbb{D}^2)$ is not projective (see Section 6).

Much of our paper [1] was devoted to characterizations of $\mathrm{Ext}^1_{\mathcal{H}}(K, \mathbb{H}^2(\mathbb{D}))$. In Sections 4 and 5, we give several criteria for the vanishing of $\mathrm{Ext}^1_{\mathcal{H}}(K, \mathbb{H}^2(\mathbb{D}^N))$ for $N \geq 2$. This happens, for example, when K is finite-dimensional, in sharp contrast to the case in which $N = 1$.

2. Lemmas on Vanishing of Cocycles

Our principal tool is Lemma 2.1 of this section. Recall from [1] that $\mathrm{Ext}^1_{\mathcal{H}}(K, H)$ is the quotient $\mathcal{A}/\mathcal{B}$, where $\mathcal{A}$ is the set of all continuous bilinear functions $\sigma: A \times K \to H$ that satisfy $\sigma(ab, k) = a\sigma(b, k) + \sigma(a, bk)$ for $a, b \in \mathbb{A}$ and $k \in K$, and where $\mathcal{B}$ is the set of all $\sigma \in \mathcal{A}$ of the form $\sigma(a, k) = aL(k) - L(ak)$, where $a \in \mathbb{A}$, $k \in K$, and $L: K \to H$ is a bounded operator.

LEMMA 2.1. *Let H and K be Hilbert modules over $\mathbb{A}(\mathbb{D}^N)$, let b lie in the unit ball of $\mathbb{A}(\mathbb{D}^N)$, and suppose that the operator $Uf = bf$ on K is a co-isometry. If $\sigma: \mathbb{A}(\mathbb{D}^N) \times K \to H$ represents a cocycle, then σ is equivalent to a bounded bilinear map σ^* that satisfies the following conditions:*

(i) *$\sigma^*(b, g) = 0$ for all $g \in K$ such that $U^*Ug = g$; and*
(ii) *if $c \in \mathbb{A}(\mathbb{D}^N)$ is such that multiplication by c on K doubly commutes with U, and if $K_0 \subset \{h \in K \mid \sigma(c, h) = 0\}$ is U^*-invariant, then $\sigma^*(c, h) = 0$ for $h \in K_0$.*

Proof. As $\|\sigma(b^n, U^{*n}f)\|$ is bounded ($n = 0, 1, \dots$), we can define a translation-invariant Banach limit LIM_n on H such that

$$Lf = \operatorname*{LIM}_n \sigma(b^n, U^{*n}f)$$

exists in the weak operator topology on $\mathcal{L}(K, H)$. Set

$$\sigma^*(a, f) = \sigma(a, f) + aLf - Laf$$

for $a \in \mathbb{A}(\mathbb{D}^N)$ and $f \in K$. Then σ^* is equivalent to σ.

(i) For g as in Lemma 2.1(i), we have

$$\begin{aligned}
\sigma^*(b, g) &= \sigma(b, g) - \operatorname*{LIM}_n[\sigma(b^n, U^{*n}Ug) - b\sigma(b^n, U^{*n}g)] \\
&= \sigma(b, g) - \operatorname*{LIM}_n[\sigma(b^n, U^{*n-1}g) - b\sigma(b^n, U^{*n}g)] \\
&= \sigma(b, g) - \operatorname*{LIM}_n[b\sigma(b^{n-1}, U^{*n-1}g) + \sigma(b, g) - b\sigma(b^n, U^{*n}g)] \\
&= b\operatorname*{LIM}_n[\sigma(b^n, U^{*n}g) - \sigma(b^{n-1}, U^{*n-1}g)] = 0
\end{aligned}$$

by translation-invariance.

(ii) Let $h \in K_0$. We have

$$\begin{aligned}\sigma^*(c,h) &= cLh - Lch = \operatorname*{LIM}_n[c\sigma(b^n, U^{*n}h) - \sigma(b^n, U^{*n}ch)]\\ &= \operatorname*{LIM}_n[\sigma(cb^n, U^{*n}h) - \sigma(c,h) - \sigma(cb^n, U^{*n}h) + b^n\sigma(c, U^{*n}h)]\\ &= \operatorname*{LIM}_n[b^n\sigma(c, U^{*n}h) - \sigma(c,h)] = 0\end{aligned}$$

by U^*-invariance of K_0. □

Lemma 2.1 will be used in the following form.

LEMMA 2.2. *Let H and K be Hilbert modules over $\mathbb{A}(\mathbb{D}^N)$ and suppose that:*

(a) *$b_1, b_2, \dots, b_n$ lie in the unit ball of $\mathbb{A}(\mathbb{D}^N)$ for some $n \le N$;*
(b) *$U_i: K \to K$, defined by $U_i f = b_i f$, are doubly commuting and are coisometries, $i = 1, \dots, n$;*
(c) *K_i are subspaces of K with $U_i^* U_i f = f$ for $f \in K_i$ and with K_i invariant under U_j^* $(j = i+1, i+2, \dots, n)$; and*
(d) *σ represents a cocycle in $\operatorname{Ext}^1_{\mathcal{H}}(K, H)$.*

Then σ is equivalent to σ^* satisfying

$$\sigma^*(b_i, f) = 0 \tag{2.3}$$

for $f \in K_i$ and for $i = 1, \dots, n$.

Proof. We use induction on n. If $n = 1$, apply Lemma 2.1 with $b = b_1$.

If we have obtained σ satisfying (2.3), for $i = 1, \dots, k$, we then apply Lemma 2.1 with $b = b_{k+1}$ to obtain σ^* with

$$\sigma^*(b_i, g) = 0 \quad \text{for } g \in K_i \tag{2.4}$$

for $i = k+1$. By part (ii) of Lemma 2.1 with $c = b_i$ and $K_0 = K_i$, we see that σ^* also satisfies (2.4) for each $i \le k$. □

3. Projectivity

In this section, we use Lemma 2.2 to obtain generalizations of the projectivity results of [2].

THEOREM 3.1. *If K is a Hilbert module over $\mathbb{A}(\mathbb{D}^N)$ such that $U_1, U_2, \dots, U_N$: $K \to K$ defined by $U_i f = z_i f$ are unitary, then K is projective in the category of Hilbert modules over $\mathbb{A}(\mathbb{D}^N)$.*

Proof. The hypotheses of Lemma 2.2 are satisfied with $b_i = z_i$ $(i = 1, \dots, N)$ and with $K_1 = K_2 = \cdots = K_n = \mathbb{H}^2(\mathbb{D}^N)$. Thus any σ representing a cocycle in $\operatorname{Ext}^1_{\mathcal{H}}(K, H)$ is equivalent to 0 for any H. □

The other projectivity result of [2] assumes that multiplication by z on K is isometric and yields projectivity in the category $\mathcal{C}$ of Hilbert modules similar to contractive ones, the so-called *cramped* Hilbert modules. The generalization that follows is disappointing in that $N-1$ of the U_i must be assumed to be unitary. However, it appears to be the best possible result.

THEOREM 3.2. *If K is a Hilbert module over $\mathbb{A}(\mathbb{D}^N)$, if $U_i\colon K \to K$ are given by $U_i f = z_i f$ $(i = 1, \dots, N)$, and if U_1 is an isometry and $U_2, \dots, U_N$ are unitary, then K is projective in the category $\mathcal{C}$ of cramped Hilbert modules over $\mathbb{A}(\mathbb{D}^N)$.*

Proof. Let H be a cramped Hilbert module over $\mathbb{A}(\mathbb{D}^N)$ and let σ represent a cocycle in $\mathrm{Ext}^1_{\mathcal{C}}(K, H)$. View H and K as cramped Hilbert modules over $\mathbb{A}(\mathbb{D})$ by restricting the action of $\mathbb{A}(\mathbb{D}^N)$ to the first variable. By the projectivity of isometric Hilbert modules in the one-variable cramped category, there exists $L \in \mathcal{L}(K, H)$ with

$$\sigma(z_1, f) = Lz_1 f - z_1 Lf, \quad f \in K.$$

Let

$$\sigma'(a, f) = \sigma(a, f) + aLf - Laf \quad \text{for } f \in K,\ a \in \mathbb{A}(\mathbb{D}^N).$$

Then σ' represents a cocycle in $\mathrm{Ext}_{\mathcal{C}}(K, H)$ equivalent to σ, and satisfying $\sigma'(z_1, f) = 0$ for $f \in K$.

Now apply Lemma 2.2 with σ' replacing σ, with $b_i = z_{i+1}$, $i = 1, \dots N-1$ $(= n)$, and with $K_i = \mathbb{H}^2(\mathbb{D}^N)$, $i = 1, \dots, N-1$. The proof of Lemma 2.2 shows that the resulting σ^* satisfies $\sigma^*(z_1, f) = 0$ for $f \in K$. Indeed, at each step in the induction, σ satisfies $\sigma(z_1, f) = 0$ for $f \in K = K_0$, and Lemma 2.1 implies that $\sigma^*(z_1, f) = 0$ for $f \in K$. □

If the hypothesis of Theorem 3.2 is replaced by the assumption that $U_1, \dots, U_k$ are isometries and $U_{k+1}, \dots, U_N$ are unitary for some k $(1 < k \le N)$, then K need not be projective in the cramped category over $\mathbb{A}(\mathbb{D}^N)$. Indeed, for the case $k = N$, we shall show in Section 6 that $\mathrm{Ext}^1_{\mathcal{C}}(\mathbb{H}^2(\mathbb{D}^N), \mathbb{H}^2(\mathbb{D}^N)) \neq 0$ for $N > 1$. Hence a counterexample to a generalization of the last theorem is obtained as follows.

For $1 < k < N$, let $K = \mathbb{H}^2(\mathbb{D}^k)$, as a Hilbert module over $\mathbb{A}(\mathbb{D}^N)$, where we define multiplication by the first k coordinate functions according to the usual action of $\mathbb{A}(\mathbb{D}^N)$ on $\mathbb{H}^2(\mathbb{D}^k)$:

$$U_1 f = z_1 f, \dots, U_k f = z_k f, \quad f \in K.$$

For $j = k+1, \dots, N$, multiplication by the remaining coordinate functions is defined by

$$U_j f = f, \quad j = k+1, \dots, N.$$

Then $U_1, \dots, U_k$ are isometric, $U_{k+1}, \dots, U_n$ are unitary, and

$$\mathrm{Ext}^1_{\mathcal{C}}(K, K) = \mathrm{Ext}^1_{\mathcal{C}}(\mathbb{H}^2(\mathbb{D}^k), \mathbb{H}^2(\mathbb{D}^k)) \neq 0$$

by Theorem 6.1 below.

4. Vanishing of $\mathrm{Ext}^1_{\mathcal{H}}(K, \mathbb{H}^2(\mathbb{D}^N))$

The main topic of [1] was the group $\mathrm{Ext}^1_{\mathcal{H}}(K, \mathbb{H}^2(\mathbb{D}^N))$, for $N = 1$ and for a Hilbert module K over $\mathbb{A}(\mathbb{D})$. In the present paper, we obtain versions of some of the theorems of [1] for $\mathbb{H}^2(\mathbb{D}^N)$, $N > 1$. Some theorems are less precise than in [1]. For example, we have not obtained a characterization of coboundaries analogous to [1, Prop. 3.3.1]. Other theorems are radically different in the polydisk.

The main tool in [1] was that, if σ represents a cocycle in $\mathrm{Ext}^1_{\mathcal{H}}(K, \mathbb{H}^2(\mathbb{D}^1))$, then σ can be taken to have the form

$$\sigma(z, f) = \langle f, k_0 \rangle$$

for some $k_0 \in K$. The analog to this in several variables is the following.

THEOREM 4.1. *If* σ *represents a cocycle in* $\mathrm{Ext}^1_{\mathcal{H}}(K, \mathbb{H}^2(\mathbb{D}^N))$, *then* σ *is equivalent to*

$$\sigma^*: \mathbb{A} \times K \to \mathbb{H}^2(\mathbb{D}^N),$$

with the property that

$$\sigma^*(z_j, f) \in \mathbb{H}^2(\mathbb{D}^N) \ominus z_j\mathbb{H}^2(\mathbb{D}^N) \quad \text{for } f \in K$$

with $j = 1, 2, \ldots, N$.

Proof. For convenience we consider adjoints. Suppose that $\overline{\mathbb{H}^2(\mathbb{D}^N)}$ denotes the Hilbert space $\mathbb{H}^2(\mathbb{D}^N)$ with the actions

$$z_j \cdot f = T^*_{z_j} f, \quad f \in \mathbb{H}^2(\mathbb{D}^N),$$

where T_{z_j} is the usual multiplication by z_j.

Let σ represent a cocycle in $\mathrm{Ext}^1_{\mathcal{H}}(\overline{\mathbb{H}^2(\mathbb{D}^N)}, H)$. We claim that σ is equivalent to a (bounded, bilinear) σ^* such that

$$\ker \sigma^*(z_j, \cdot) \supset z_j\mathbb{H}^2(\mathbb{D}^N), \quad j = 1, \ldots, N. \tag{4.2}$$

Choose $n = N$ and $b_j = z_j$ in Lemma 2.2. Then U_j is the adjoint of T_{z_j}, and so U_j^* is an isometry and $U_j^* U_j f = f$ for all $f \in z_j\mathbb{H}^2(\mathbb{D}^N)$. From Lemma 2.2, we conclude that σ is equivalent to σ^* satisfying (4.2). □

COROLLARY 4.3. *Suppose that* $\zeta_1, \ldots, \zeta_N \in \bar{\mathbb{D}}$, $n_1, \ldots, n_N \in \mathbb{Z}_+$, *and* $f \in K$ *such that*

$$(z_j - \zeta_j)^{n_j} f = 0, \quad j = 1, \ldots, N.$$

If σ *is the bilinear function* σ^* *obtained in Theorem 4.1, then* $\sigma(a, f) = 0$ *for all* $a \in \mathbb{A}(\mathbb{D}^N)$.

Proof. We compute

$$\begin{aligned}
&\sigma((z_1 - \zeta_1)^{n_1}(z_2 - \zeta_2)^{n_2}, f) \\
&\quad = (z_1 - \zeta_1)^{n_1}\sigma((z_2 - \zeta_2)^{n_2}, f) + \sigma((z_1 - \zeta_1)^{n_1}, (z_2 - \zeta_2)^{n_2} f) \\
&\quad = (z_2 - \zeta_2)^{n_2}\sigma((z_1 - \zeta_1)^{n_1}, f) + \sigma((z_2 - \zeta_2)^{n_2}, (z_1 - \zeta_1)^{n_1} f)
\end{aligned}$$

so that

$$(z_1 - \zeta_1)^{n_1}\sigma((z_2 - \zeta_2)^{n_2}, f) = (z_2 - \zeta_2)^{n_2}\sigma((z_1 - \zeta_1)^{n_1}, f). \tag{4.4}$$

Using

$$\sigma((z_1-\zeta_1)^{n_1},f)=\sum_{j=0}^{n_1-1}(z_1-\zeta_1)^j\sigma(z_1-\zeta_1,(z_1-\zeta_1)^{n_1-j-1}f)$$
$$=\sum_{j=0}^{n_1-1}(z_1-\zeta_1)^j\sigma(z_1,(z_1-\zeta_1)^{n_1-j-1}f) \tag{4.5}$$

we conclude that $\sigma((z_1-\zeta_1)^{n_1},f)$ is a polynomial in z_1 (with coefficients which are $\mathbb{H}^2(\mathbb{D}^{N-1})$-functions of $z_2,\dots,z_N$) of degree at most n_1-1. Therefore, the right side of (4.4) is a polynomial in z_1 of degree at most n_1-1. Fixing $z_2,\dots,z_N$ in $\mathbb{D}$ in (4.4) we get that $\sigma((z_2-\zeta_2)^{n_2},f)$ is a rational function (of z_1) of the form

$$\sigma((z_2-\zeta_2)^{n_2},f)=p(z_1)(z_1-\zeta_1)^{-n_1},$$

where p is a polynomial of degree at most n_1-1. Since $\sigma((z_2-\zeta_2)^{n-2},f)$ lies in $\mathbb{H}^2(\mathbb{D})$, for $z_2,\dots,z_N$ fixed in $\mathbb{D}$ we must have

$$\sigma((z_2-\zeta_2)^{n_2},f)=\sigma((z_1-\zeta_1)^{n_1},f)=0.$$

This implies that the coefficient of each power of $z_1-\zeta_1$ must vanish in (4.5); in particular, $\sigma(z_1,f)=0$.

Similarly, $\sigma(z_2,f)=\sigma(z_3,f)=\cdots=\sigma(z_N,f)=0$. □

For Hilbert modules over $\mathbb{A}(\mathbb{D})$, we proved in [1] that if (multiplication by z on) K is isometric, then

$$\operatorname{Ext}^1_{\mathcal{H}}(K,\mathbb{H}^2)=0$$

and that, in several cases (including $\dim K<\infty$), we have

$$\operatorname{Ext}^1_{\mathcal{H}}(K,\mathbb{H}^2)=\{k_0\in K: z^nk_0\to 0\}.$$

Three more corollaries of Theorem 4.1 show that the situation is quite different in more than one variable.

COROLLARY 4.6. *If the linear span of the joint (generalized) eigenvectors of multiplication by* $z_1,\dots,z_N$ *is dense in* K, *then* $\operatorname{Ext}^1_{\mathcal{H}}(K,\mathbb{H}^2(\mathbb{D}^N))=0$ *for* $N\ge 2$.

COROLLARY 4.7. *If* K *is finite-dimensional, then* $\operatorname{Ext}^1_{\mathcal{H}}(K,\mathbb{H}^2(\mathbb{D}^N))=0$ *for* $N\ge 2$.

COROLLARY 4.8. $\operatorname{Ext}^1_{\mathcal{H}}(\overline{\mathbb{H}^2(\mathbb{D}^N)},\mathbb{H}^2(\mathbb{D}^N))=0$ *for* $N\ge 2$.

5. More on Vanishing of $\operatorname{Ext}^1_{\mathcal{H}}(K,H)$

In this section we develop a different criterion for the vanishing of $\operatorname{Ext}^1_{\mathcal{H}}(K,H)$. The condition depends upon the action of just one element of $\mathbb{A}(\mathbb{D}^N)$ on K.

LEMMA 5.1. *Let* H *and* K *be Hilbert modules over* $\mathbb{A}(\mathbb{D}^N)$ $(N\ge 1)$, *and suppose that*

(i) *b lies in the unit ball of* $\mathbb{A}(\mathbb{D}^N)$,
(ii) $U\colon K\to K$, *defined by* $Uf=bf$, *is unitary, and*

(iii) *the only operator* $T \in \mathcal{L}(K, H)$ *satisfying*

$$bTf - Tbf = 0$$

is the zero operator.

Then $\mathrm{Ext}^1_{\mathcal{H}}(K, H) = 0$.

Proof. Let $\sigma\colon \mathbb{A} \times K \to H$ represent a cocycle in $\mathrm{Ext}^1_{\mathcal{H}}(K, H)$. Since $\|b\|_\infty \le 1$, we have that

$$\|\sigma(b^n, U^{*n}f)\| \le c\|f\|, \quad f \in K.$$

Let LIM_n be a translation-invariant Banach limit, and define L as the limit

$$Lf = \mathrm{LIM}_n\, \sigma(b^n, U^{*n}f), \quad f \in K$$

in the weak operator topology on $\mathcal{L}(K, H)$. We claim that

$$\sigma(a, f) = Laf - aLf, \quad f \in K \tag{5.2}$$

for every $a \in \mathbb{A}(\mathbb{D}^N)$.

First of all, (5.2) holds for $a = b$. Indeed,

$$\begin{aligned}(Lb - bL)f &= \mathrm{LIM}_n[\sigma(b^n, U^{*n-1}f) - b\sigma(b^n, U^{*n}f)]\\ &= \mathrm{LIM}_n\{\sigma(b, U^{n-1}U^{*n-1}f) + b[\sigma(b^{n-1}, U^{*n-1}f) - \sigma(b^n, U^{*n}f)]\}\\ &= \sigma(b, f).\end{aligned}$$

Now, in order to verify (5.2) for general $a \in \mathbb{A}(\mathbb{D}^N)$, we set

$$Tf = Laf - aLf - \sigma(a, f)$$

and prove that T is the zero operator by (iii). We have

$$\begin{aligned}Tbf - bTf &= (La - aL)bf - \sigma(a, bf) - b(La - aL)f + b\sigma(a, f)\\ &= Labf - aLbf - bLaf + abLf + b\sigma(a, f) - \sigma(a, bf)\\ &= (Lb - bL)af - a(Lb - bL)f + b\sigma(a, f) - \sigma(a, bf)\\ &= \sigma(b, af) - a\sigma(b, f) + b\sigma(a, f) - \sigma(a, bf),\end{aligned}$$

since (5.2) holds with a replaced by b. This yields

$$\begin{aligned}Tbf - bTf &= \sigma(b, af) + b\sigma(a, f) - [\sigma(a, bf) + a\sigma(b, f)]\\ &= \sigma(ab, f) - \sigma(ab, f) = 0,\end{aligned}$$

which proves the lemma. □

The following three corollaries demonstrate uses of Lemma 5.1

COROLLARY 5.3. *Let* $Uf = z_1 f$ *on* K *and* $Vf = z_1 f$ *on* H *satisfy*

(1) $U = I$ *and*

(2) $\ker(V - I) = 0$.

Then $\mathrm{Ext}^1_{\mathcal{H}}(K, H) = 0$.

COROLLARY 5.4. *Suppose that $Uf = z_1 f$ is unitary on K and that $Vf = z_1 f$ on H is a unilateral shift. Then $\mathrm{Ext}^1_{\mathcal{H}}(K, H) = 0$. In particular,*

$$\mathrm{Ext}^1_{\mathcal{H}}(K, \mathbb{H}^2(\mathbb{D}^N)) = 0.$$

Proof. In this case, $VT = TU$ implies $V^n T U^{*n} = T$ and $V^n T U^{*n}$ tends to 0 in the weak operator topology. □

COROLLARY 5.5. *If $Uf = bf$ is unitary on K for some nonconstant b in the unit ball of $\mathbb{A}(\mathbb{D}^N)$, then*

$$\mathrm{Ext}^1_{\mathcal{H}}(K, \mathbb{H}^2_{\mathcal{H}}(\mathbb{D}^N)) = 0.$$

6. On $\mathrm{Ext}^1(\mathbb{H}^2(\mathbb{D}^N), \mathbb{H}^2(\mathbb{D}^N))$

A recent result of Cotlar and Sadosky [3] can be used to prove the following theorem.

THEOREM 6.1. *Let $N > 1$. Then*

$$\mathrm{Ext}^1_{\mathcal{C}}(\mathbb{H}^2(\mathbb{D}^N), \mathbb{H}^2(\mathbb{D}^N)) \neq 0 \quad \textit{and} \quad \mathrm{Ext}^1_{\mathcal{H}}(\mathbb{H}^2(\mathbb{D}^N), \mathbb{H}^2(\mathbb{D}^N)) \neq 0. \tag{6.2}$$

In particular, $\mathbb{H}^2(\mathbb{D}^N)$ is not projective or injective, even in the cramped category.

This theorem is in contrast to the case where $N = 1$. In [1], the vanishing of $\mathrm{Ext}_{\mathcal{H}}(\mathbb{H}^2(\mathbb{D}), \mathbb{H}^2(\mathbb{D}))$ was proved; indeed, it was shown that

$$\mathrm{Ext}^1_{\mathcal{H}}(K, \mathbb{H}^2(\mathbb{D})) = 0$$

for any isometric Hilbert module K; see also [6, Cor. 2]. Projectivity of $\mathbb{H}^2(\mathbb{D})$ in the cramped category was proved in [2, Thm. 3.1].

Proof of Theorem 6.1. The exact sequence

$$0 \to \mathbb{H}^2(\mathbb{D}^N) \to \mathbb{L}^2(\mathbb{T}^N) \to \mathbb{L}^2(\mathbb{T}^N)/\mathbb{H}^2(\mathbb{D}^N) \to 0,$$

where the first map is inclusion and the second is the quotient map, gives rise to the long exact sequence

$$\begin{aligned} \mathrm{Hom}(\mathbb{H}^2, \mathbb{H}^2) \to \mathrm{Hom}(\mathbb{H}^2, \mathbb{L}^2) \to \mathrm{Hom}(\mathbb{H}^2, \mathbb{L}^2/\mathbb{H}^2) \to \mathrm{Ext}^1(\mathbb{H}^2, \mathbb{H}^2) \\ \to \mathrm{Ext}^1(\mathbb{H}^2, \mathbb{L}^2) \to \mathrm{Ext}^1(\mathbb{H}^2, \mathbb{L}^2/\mathbb{H}^2). \end{aligned}$$

(The proof in [2, Prop. 2.1.5] using pushouts works for modules over the polydisc algebra.) We have $\mathrm{Hom}(\mathbb{H}^2, \mathbb{L}^2) = L^\infty$; that is, every Hilbert-module map of $\mathbb{H}^2$ into $\mathbb{L}^2$ is multiplication by an L^∞ function. In addition, $\mathrm{Hom}(\mathbb{H}^2, \mathbb{L}^2/\mathbb{H}^2)$ is isomorphic to the space of Hankel operators

$$H_\varphi \colon f \to (I - P)\varphi f, \quad f \in \mathbb{H}^2. \tag{6.3}$$

The precise set of Hankel operators intended here is those with symbol φ such that (6.3) defines, for polynomials $f \in \mathbb{H}^2$, an operator H_φ with a bounded extension from $\mathbb{H}^2$ to $\mathbb{L}^2 \ominus \mathbb{H}^2$.

Now suppose $\mathrm{Ext}^1_{\mathcal{C}}(\mathbb{H}^2, \mathbb{H}^2) = 0$. The long exact sequence shows that the map from $\mathrm{Hom}(\mathbb{H}^2, \mathbb{L}^2)$ to $\mathrm{Hom}(\mathbb{H}^2, \mathbb{L}^2/\mathbb{H}^2)$ must be onto. That is, every bounded H_φ must be expressible in the form H_ψ for some $\psi \in L^\infty$. But by [3, Thm. 2.1] there exist bounded H_φ that are not of this form, for any $\psi \in L^\infty$, if $N > 1$. This contradicts the assumption that

$$\mathrm{Ext}^1_{\mathcal{C}}(\mathbb{H}^2(\mathbb{D}^N), \mathbb{H}^2(\mathbb{D}^N)) = 0.$$

The nonvanishing of $\mathrm{Ext}^1_{\mathcal{H}}(\mathbb{H}^2(\mathbb{D}^N), \mathbb{H}^2(\mathbb{D}^N))$ follows from the fact that a short exact sequence which fails to split in $\mathcal{C}$ will also not split in $\mathcal{H}$.

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