Orbits of Hyponormal Operators

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1. Introduction

We show that orbits of hyponormal operators display simple growth patterns. We then use our orbital-growth observations to prove that hyponormal operators are never supercyclic, which generalizes a result due to Hilden and Wallen [6, p. 564] and answers a question raised by Kitai [7, p. 4.5]. We also establish that every hyponormal operator is "power regular," which means that if T is a hyponormal operator on the Hilbert space H, then $\lim_n \|T^n h\|^{1/n}$ exists for every $h \in H$. That every normal operator is power regular follows from results in [2] (see also [5]).

Interest in the behavior of orbits of bounded linear operators on Hilbert space derives from the invariant subspace (subset) problem for Hilbert space operators, which is to determine whether every bounded linear operator on a separable, infinite-dimensional Hilbert space H must leave invariant some proper, nonzero, closed subspace (subset) of H. Consider, for example, the following simple proposition, well known to operator theorists.

PROPOSITION. *Suppose that the linear operator T is a contraction on the Hilbert space H such that*

(a) *there is a nonzero vector h in H whose orbit under T $(T^n h : n = 0, 1, 2, \dots)$ has limit 0, and*
(b) *there is a vector $g \in H$ whose orbit under T is bounded away from 0.*

Then there is a proper, nonzero, closed subspace M of H that is invariant for T.

Proof. Let M be the closed linear span of $\{T^n h : n = 0, 1, 2, \dots\}$. That M is invariant for T and is nonzero is clear. We now argue by contradiction that g is not in M so that M is properly contained in H.

Suppose that g is in M; thus, there must be a sequence (p_n) of polynomials such that $(p_n(T)h)$ converges to g. Let $\varepsilon > 0$ be such that $\|T^n g\| > \varepsilon$ for all nonnegative n (here, $\|\cdot\|$ denotes the norm induced by the inner product on H). Choose the positive integer j large enough so that $\|p_j(T)h - g\| < \varepsilon/2$. We then have, for every nonnegative integer k,

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$$\begin{aligned}\varepsilon/2 &> \|p_j(T)h - g\| \\ &\geq \|p_j(T)(T^k h) - T^k g\| \qquad (T \text{ is a contraction}) \\ &\geq \|T^k g\| - \|p_j(T)(T^k h)\| \\ &\geq \varepsilon - \|p_j(T)(T^k h)\|;\end{aligned}$$

letting $k \to \infty$, we obtain the contradiction $\varepsilon/2 \geq \varepsilon$. □

Our argument showing that hyponormal operators are never supercyclic involves a variation on the theme of the proof of the preceding proposition. We remark that the proposition above may be stated more generally: H may be replaced by a Banach space X and the hypothesis that T be a contraction may be replaced by "T is power bounded" (which means $\|T^n\| \leq C$ for some constant C and every positive integer n).

Recall that a bounded linear operator T on the Hilbert space H is *hyponormal* provided that

$$\|Th\| \geq \|T^*h\|$$

for every $h \in H$, where, as usual, we have used T^* to denote the Hilbert-space adjoint of T. Normal and subnormal operators are hyponormal; the backward shift plus twice the forward shift acting on ℓ^2 is an example of a hyponormal operator that is not subnormal [4, p. 145]. We show in the next section that orbits of hyponormal operators either strictly decrease in norm, increase in norm, or strictly decrease up to a point and increase thereafter. We also observe that if $T: H \to H$ is hyponormal and $h \in H$ is not in the kernel of T, then the sequence

$$\left(\frac{\|T^{n+1}h\|}{\|T^n h\|}\right)$$

is increasing.

Using our observations about orbits, we show in the third section of this paper that hyponormal operators are never supercyclic. Supercyclicity is one of three types of cyclicity studied by operator theorists; the other two are hypercyclicity and cyclicity. A bounded linear operator T on the Hilbert space H is *hypercyclic* if there is a vector h in H whose orbit under T, $\mathrm{Orb}(T, h) := \{T^n h : n = 0, 1, 2, \dots\}$, is dense in H, in which case h is called a hypercyclic vector for T. If for some $h \in H$ the set of scalar multiples of elements in $\mathrm{Orb}(T, h)$ is dense in H, then T is *supercyclic* and h is a supercyclic vector for T. Finally, if the linear span of $\mathrm{Orb}(T, h)$ is dense in H for some $h \in H$, then T is *cyclic* with cyclic vector h. Hypercyclicity and supercyclicity are infinite-dimensional phenomena; that is, if $1 < \text{dimension } H < \infty$, then $T: H \to H$ cannot be supercyclic (and hence cannot be hypercyclic) [6, p. 564]. Observe that every operator on a one-dimensional Hilbert space is supercyclic. We wish to ignore this trivial situation; hence, throughout the remainder of this paper we assume that the dimension of H exceeds 1.

It is easy to check that hypercyclic and supercyclic operators must have dense range and that the closure of the range of a cyclic operator has codimension at most 1. Twice the backward shift on ℓ^2 is an example of a hypercyclic operator [10]. The backward shift itself is thus supercyclic, but it is not hypercyclic because all of its orbits are bounded. Finally, the forward shift is an example of a cyclic operator that is not supercyclic (because, for example, its range is not dense in ℓ^2).

The notion of supercyclicity was introduced by Hilden and Wallen in [6], which contains a proof (p. 564) that normal operators are never supercyclic. Kitai [7] shows that hyponormals are never hypercyclic and raises the question [7, p. 4.5] of whether a hyponormal operator can be supercyclic, which we settle in the negative (Theorem 3.1 below).

In the final section of this paper we show that every hyponormal operator is power regular. Following Atzmon [2], we call a bounded linear operator T on a complex Hilbert (or Banach) space X *power regular* provided $\lim_{n\to\infty}\|T^n x\|^{1/n}$ exists for every $x \in X$. That normal operators are power regular follows from results in [2]. The backward shift on ℓ^2 is an example of an operator that is not power regular (see [2, p. 3107]). The notion of power regularity is also connected with the invariant-subspace problem. For example, in Theorem 4.4 we show that if T is a hyponormal on H and $\lim_{n\to\infty}\|T^n h\|^{1/n} < \|T\|$, then h is not cyclic for T; that is, the closed invariant subspace generated by h will be properly contained in H.

It is not known whether hyponormal operators must have nontrivial invariant subspaces; that subnormal operators do have nontrivial invariant subspaces was established in the late 1970s by S. Brown [3].

2. Elementary Orbital Estimates

Our first proposition is simple and well known. It appears, for example, as the first line of the proof of Proposition 4.7 in [4, Chap. 3].

PROPOSITION 2.1. *Suppose that the operator T mapping the Hilbert space H to itself is hyponormal. Then, for every $f \in H$ and nonnegative integer n,*

$$\|T^n f\|^2 \le \|T^{n+1} f\|\,\|T^{n-1} f\|. \tag{2.2}$$

Proof. Let $f \in H$ be arbitrary and let $n \ge 1$. We have

$$\begin{aligned}\|T^n f\|^2 &= \langle T^n f, T^n f\rangle \\ &= \langle T^*T^n f, T^{n-1} f\rangle \\ &\le \|T^*T^n f\|\,\|T^{n-1} f\| \\ &\le \|T^{n+1} f\|\,\|T^{n-1} f\|,\end{aligned}$$

where we have used hyponormality to obtain the final inequality. □

Observe that the preceding proposition shows that if Tf is nonzero for a hyponormal T, then all elements in $\mathrm{Orb}(T, f)$ must be nonzero. Hence, if Tf is nonzero, Proposition 2.1 shows that

$$\text{the sequence } \left(\frac{\|T^{n+1}f\|}{\|T^n f\|}\right) \text{ is increasing.} \tag{2.3}$$

Because this sequence is clearly bounded above by $\|T\|$, we have the following.

COROLLARY 2.4. *Suppose that T is hyponormal on the Hilbert space H and that f is not in the kernel of T. Then the sequence*

$$\left(\frac{\|T^{n+1}f\|}{\|T^n f\|}\right) \tag{2.5}$$

converges.

We show in Section 4 that whenever f is cyclic for T, the sequence (2.5) converges to $\|T\|$. (More generally, we show that the sequence will converge to the norm of the restriction of T to the invariant subspace for T generated by f.)

PROPOSITION 2.6. *Suppose that $f \in H$ is such that $\|Tf\| \geq \|f\|$ for the hyponormal operator $T: H \to H$. Then $(\|T^n f\|)$ is an increasing sequence.*

Proof. For each nonnegative integer n, set $a_n = \|T^n f\|$. If $f = 0$, then the sequence (a_n) is constant (each term zero) so that the proposition holds. Suppose that f is nonzero. Then, applying the hypothesis of the proposition, we have $a_1 \geq a_0 > 0$. Now, observe that if $a_n \geq a_{n-1} > 0$ for some $n \geq 1$ then

$$\begin{aligned} a_{n+1} - a_n &\geq \frac{(a_n)^2}{a_{n-1}} - a_n \qquad \text{by (2.2)} \\ &= \frac{a_n(a_n - a_{n-1})}{a_{n-1}} \\ &\geq 0. \end{aligned}$$

Thus, by induction, the proposition holds. □

Note that Proposition 2.6 shows the following: An orbit of a hyponormal operator either strictly decreases in norm, increases in norm, or strictly decreases up to a point and increases thereafter. Clearly, then, no hyponormal operator can have a dense orbit; that is, no hyponormal can be hypercyclic. Thus Proposition 2.6 yields a different proof of Kitai's result [7, Cor. 4.5] (as well as a different proof of the weaker result [9, Thm. 2.10]).

We remark that all three possible growth patterns for orbits of a hyponormal operator may occur in the collection of orbits of a single such operator. Suppose that T is hyponormal and has eigenvalues λ_1 and λ_2 such that $|\lambda_1| < 1$ and $|\lambda_2| > 1$. Let e_1 and e_2 be norm-1 eigenvectors for T corresponding respectively to λ_1 and λ_2 (it is not difficult to verify that e_1 and e_2 must be orthogonal; see e.g. [4, Prop. 4.4, p. 140]). Then the orbit of e_1 strictly decreases in norm, the orbit of e_2 (strictly) increases, and—by choosing the positive number β large enough so that $\beta^2 + 1 > \beta^2|\lambda_1|^2 + |\lambda_2|^2$—one may easily verify that the orbital norms of $\beta e_1 + e_2$ strictly decrease up to a point and then increase (to infinity) thereafter.

3. Hyponormal Operators Are Never Supercyclic

Recall that an operator T on a complex Hilbert space H is *supercyclic* if there is a vector $h \in H$ such that

$$\{cT^n h : c \in \mathbf{C} \text{ and } n = 0, 1, 2, \dots\}$$

is dense in H. As noted in Section 1, it is easy to verify that a supercyclic operator must have dense range. Because all powers of a dense-range operator will map dense sets to dense sets, if h is a supercyclic vector for T and c is a nonzero scalar, then $cT^n h$ will also be supercyclic for T for every nonnegative integer n. Thus, the collection of supercyclic vectors for a given supercyclic operator T on H will always be dense in H.

THEOREM 3.1. *Suppose that T is hyponormal on the Hilbert space H. Then T cannot be supercyclic.*

Proof. Suppose that T is supercyclic. Set $A = T/\|T\|$ so that A is hyponormal, supercyclic, and has norm 1. We claim that A cannot be an isometry. If it were, it would have to be onto because supercyclic operators have dense range; thus "isometry" here means "unitary" and the Hilden–Wallen result [6, p. 564] shows that A cannot be supercyclic. (For a different proof that isometries cannot be supercyclic—one that applies in a Banach-space setting—see [1].) Thus we may assume that there is a vector g in H such that $\|Ag\| < \|g\|$. Let α be a scalar of modulus > 1 such that

$$\|\alpha A g\| < \|g\|.$$

Now set $S = \alpha A$ so that S is supercyclic, hyponormal, has norm > 1, and satisfies $\|Sg\| < \|g\|$.

Because $\|S\| > 1$ and the set of supercyclic vectors for S is dense in H, there is a supercyclic $f \in H$ such that $\|Sf\| > \|f\|$. Because f is supercyclic, there is a subsequence (n_j) of the sequence of nonnegative integers and a sequence (c_j) of scalars such that

$$(c_j S^{n_j} f) \to g.$$

By continuity, we have

$$(c_j S^{n_j+1} f) \to Sg.$$

However, because S is hyponormal and $\|Sf\| > \|f\|$, Proposition 2.6 tells us $\|S^{n_j+1} f\| \geq \|S^{n_j} f\|$ for every j; thus

$$\begin{aligned}\|Sg\| &= \lim_j \|c_j S^{n_j+1} f\| \\ &\geq \lim_j \|c_j S^{n_j} f\| \\ &= \|g\|,\end{aligned}$$

a contradiction. □

Observe that the proof of Theorem 3.1 establishes the following.

COROLLARY 3.2. *Suppose that T is a bounded linear operator on the Hilbert space H such that either*

(a) *there is an open set of vectors with orbits under T increasing in norm and there is a vector g in H with $\|Tg\| < \|g\|$, or*
(b) *there is an open set of vectors with orbits decreasing in norm and there is a vector $f \in H$ with $\|Tf\| > \|f\|$;*

then T is not supercyclic.

4. Hyponormal Operators Are Power Regular

Recall that $T: H \to H$ is power regular provided that $\lim\|T^n h\|^{1/n}$ exists for every $h \in H$. That every hyponormal operator is power regular follows quickly from Corollary 2.4.

THEOREM 4.1. *Every hyponormal operator is power regular.*

Proof. Suppose that T is hyponormal on H and that $h \in H$. We need to show that the sequence

$$(\|T^n h\|^{1/n}) \tag{4.2}$$

converges. If h is in the kernel of T then clearly (4.2) converges to 0. Hence we assume that $Th \neq 0$ and set $a_n = \|T^n h\|$ for every nonnegative integer n. By Corollary 2.4, the sequence (a_n/a_{n-1}) converges to a positive real number s. We have

$$\begin{aligned}\lim_n(\|T^n h\|^{1/n}) &= \lim_n (a_n)^{1/n}\\ &= \lim_n a_0^{1/n}\left(\prod_{k=1}^{n}\frac{a_k}{a_{k-1}}\right)^{1/n}\\ &= s.\end{aligned}$$

□

The proof of the preceding theorem shows that if h is not in the kernel of the hyponormal operator T, then

$$\lim_n \|T^n h\|^{1/n} = \lim_n \frac{\|T^{n+1}h\|}{\|T^n h\|}. \tag{4.3}$$

We now identify the common value of these two limits as the norm of the restriction of T to the closed invariant subspace for T generated by h.

THEOREM 4.4. *Suppose that T is hyponormal on H, that $h \in H$, and that M is the closure of $\{p(T)h : p$ is a polynomial$\}$. Then*

$$\lim_{n\to\infty}\|T^n h\|^{1/n} = \|T|_M\|, \tag{4.5}$$

where $T|_M$ denotes the restriction of T to M.

Proof. If either $h = 0$ or $\|T|_M\| = 0$, the equality (4.5) clearly holds. Thus we assume that both $h \neq 0$ and $\|T|_M\| \neq 0$. Let $S = T|_M$ and let ε be a positive number less than $\|S\|$. Since h is cyclic for S, there is a polynomial p such that the point $y := p(S)h/\|p(S)h\|$ on the unit sphere of M satisfies

$$\|Sy\| > \|S\| - \varepsilon.$$

Because the restriction of a hyponormal operator to one of its invariant subspaces is still hyponormal (see e.g. [4, Prop. 4.4, p. 140]), $S\colon M \to M$ is hyponormal and we may apply (4.3) and (2.3) to obtain

$$\begin{aligned}\lim_n \|S^n y\|^{1/n} &= \lim_n \frac{\|S^{n+1}y\|}{\|S^n y\|} \\ &\geq \frac{\|Sy\|}{\|y\|} = \|Sy\| > \|S\| - \varepsilon.\end{aligned}$$

Now observe that, for each nonnegative n,

$$\begin{aligned}\|S^n y\|^{1/n} &= \left\| \frac{p(S)S^n h}{\|p(S)h\|} \right\|^{1/n} \\ &\leq \|p(S)\|^{1/n} \|p(S)h\|^{-1/n} \|S^n h\|^{1/n}.\end{aligned}$$

Hence,

$$\lim_n \|S^n h\|^{1/n} \geq \lim_n \|S^n y\|^{1/n} > \|S\| - \varepsilon.$$

Since $\varepsilon \in (0, \|S\|)$ is arbitrary and $\lim_n \|S^n h\|^{1/n}$ is obviously less than or equal to $\|S\|$, we have $\lim_n \|S^n h\|^{1/n} = \|S\|$; that is, $\lim_n \|T^n h\|^{1/n} = \|T|_M\|$, as desired. □

For a bounded linear operator T on H and an $h \in H$, we let $r_h(T)$ denote the spectral radius of the restriction of T to the closure of $\{\, p(T)h : p \text{ is a polynomial} \,\}$. Because the norm of a hyponormal operator equals its spectral radius [11], Theorem 4.4 shows that if $T\colon H \to H$ is hyponormal and if $h \in H$, then $(\|T^n h\|^{1/n})$ converges to $r_h(T)$. This behavior is consistent with that of the power regular operators discussed by Atzmon in [2]: For T belonging to any of the classes of power regular operators described in [2], $\lim_n \|T^n h\|^{1/n}$ always equals $r_h(T)$. Atzmon asks [2, Prob. 2] whether every power regular T has this property.

Observe that Theorem 4.4 shows that if h is cyclic for $T\colon H \to H$ then $\lim_n \|T^n h\|^{1/n} = \|T\|$; thus, whenever $\lim_n \|T^n h\|^{1/n}$ is strictly less than $\|T\|$ for some nonzero h, T will have a nontrivial, closed invariant subspace. The next proposition improves the preceding observation. Recall that a closed subspace of H is said to be *hyperinvariant* for T provided that it is invariant under every operator in the commutant $\{T\}'$ of T.

PROPOSITION 4.6. *Suppose that T is a hyponormal operator on H and that*

$$\lim_n \|T^n h\|^{1/n} < \|T\|$$

for some nonzero $h \in H$. Then T has a nontrivial hyperinvariant subspace.

Proof. Let M be the closure of $\{ Ah : A \in \{T\}' \}$. Clearly, M is hyperinvariant for T and is nonzero. Suppose $M = H$. Then one can choose $A \in \{T\}'$ to play the same role that $p(S)$ played in the proof of Theorem 4.4 and easily check that the proof remains valid with this substitution, yielding $\lim_n \|T^n h\|^{1/n} = \|T\|$. □

REMARKS. (1) As the reader may have noticed, the only property of hyponormal operators that we have used to obtain our results is the property presented in Proposition 2.1: If $T: H \to H$ is hyponormal, then

$$\|T^n f\|^2 \leq \|T^{n+1} f\| \|T^{n-1} f\| \quad (f \in H;\ n = 1, 2, 3, \ldots). \tag{4.7}$$

In other words, all the results following Proposition 2.1 remain valid if the hypothesis that T be hyponormal is replaced with the hypothesis that T satisfy (4.7). The following two observations show that the class of operators satisfying (4.7) is larger than the class of hyponormal operators: If T satisfies (4.7), then so does T^k for every positive integer k; a power of a hyponormal operator need not be hyponormal (see e.g. [4, p. 145]).

(2) After receiving a preprint of this paper, Aharon Atzmon kindly pointed out to the author that one may obtain Theorems 4.1 and 4.4 by combining [2, Thm. 3.1], which states that any decomposable operator is power regular, with [8, Thm. 4.3, p. 78], which states that every hyponormal operator is the restriction of a generalized scalar operator to one of its invariant subspaces. Because generalized scalar operators are decomposable and restrictions of power regular operators to their invariant subspaces clearly remain power regular, hyponormal operators are power regular.

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ADDED IN PROOF. The author has recently learned of connections between the results in Section 4 of this paper and results due to J. Stampfli that appear in *A local spectral theory for operators V: Spectral subspaces for hyponormal operators* (Trans. Amer. Math. Soc. 217 (1976), 285–296). Stampfli, using the methods of local spectral theory, shows that the hyponormal operator $T: H \to H$ has a nontrivial invariant subspace whenever the spectrum of T differs from the local spectrum of T at some nonzero element $h \in H$. Thus, the existence of a nontrivial invariant subspace for an operator t satisfying the hypotheses of Proposition 4.6 follows from Stampfli's work.

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