The Continuity of Commutators on Herz-Type Spaces

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Dedicated to Professor Guido Weiss

1. Introduction

Let $b \in \mathrm{BMO}(\mathbb{R}^n)$ and let T be a standard Calderón–Zygmund singular integral operator. The commutator $[b, T]$ generated by b and T is defined by

$$[b, T]f(x) = b(x)Tf(x) - T(bf)(x).$$

A celebrated result of Coifman, Rochberg, and Weiss [5] states that the operator $[b, T]$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Chanillo [3] considered the similar question when the Calderón–Zygmund operator is replaced by the fractional integral operator. The main purpose of this paper is to generalize these results to the case of Herz spaces. Let us first introduce some notation.

Let $B_k = \{x \in \mathbb{R}^n : |x| \le 2^k\}$ and $A_k = B_k \backslash B_{k-1}$ for $k \in \mathbb{Z}$. Let $\chi_k = \chi_{A_k}$ for $k \in \mathbb{Z}$, where χ_E is the characteristic function of the set E.

DEFINITION 1.1. Let $\alpha \in \mathbb{R}$, $0 < p \le \infty$, and $0 < q \le \infty$. The homogeneous Herz space $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ is defined by

$$\dot{K}_q^{\alpha,p}(\mathbb{R}^n) = \{ f \in L^q_{\mathrm{loc}}(\mathbb{R}^n \backslash \{0\}) : \|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} < \infty \},$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} = \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f\chi_k\|^p_{L^q(\mathbb{R}^n)} \right\}^{1/p} < \infty$$

with the usual modifications made when $p = \infty$ or $q = \infty$.

Obviously, $\dot{K}_p^{0,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ for all $0 < p \le \infty$. It is worth pointing out that the study involving these spaces has a long history. We refer the reader to [9] and [11] for details.

In Section 2 we establish the boundedness on the Herz spaces for a large class of the commutators related to linear operators. In fact, we shall prove that if $[b, T]$ is bounded on $L^q(\mathbb{R}^n)$ for some $q \in (1, \infty)$, then $[b, T]$ is bounded on $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ for any $\alpha \in (-n/q, n(1 - 1/q))$ and $p \in (0, \infty]$ only under certain very weak local conditions on the size of T. Similar conclusions are also true for the commutator related to the fractional integral operator. As some special cases, we shall prove the following theorems in Section 2.

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THEOREM 1.1. *Let* $b \in \mathrm{BMO}(\mathbb{R}^n)$ *and let* T *be a standard Calderón–Zygmund operator. If* $1 < q < \infty$, $-n/q < \alpha < n(1 - 1/q)$, *and* $0 < p \le \infty$, *then the commutator* $[b, T]$ *is bounded on Herz spaces* $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$.

THEOREM 1.2. *Let* $b \in \mathrm{BMO}(\mathbb{R}^n)$ *and let* I_l *be the usual fractional integral operator of order* l *with* $0 < l < n$, *that is,*

$$I_l f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-l}}\, dy.$$

If $1 < q_1 < n/l$, $1/q_2 = 1/q_1 - l/n$, $-n/q_1 + l < \alpha < n(1 - 1/q_1)$, *and* $0 < p_1 \le p_2 \le \infty$, *then* $[b, I_l]$ *maps* $\dot{K}_{q_1}^{\alpha,p_1}(\mathbb{R}^n)$ *continuously into* $\dot{K}_{q_2}^{\alpha,p_2}(\mathbb{R}^n)$.

By choosing $\alpha = 0$ and $p = q$ in Theorem 1.1, we obtain the result in [5]. Similarly, setting $\alpha = 0$, $p_1 = q_1$, and $p_2 = q_2$ yields the result in [3]. Actually, our results can be regarded as the localization of these previous results.

On the other hand, a well-known result of Stein [22] states that if the operator

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} K(x, y) f(y)\, dy$$

is bounded on $L^q(\mathbb{R}^n)$ with $q \in (1, \infty)$, and if $K(x, y)$ satisfies the "standard" size condition

$$|K(x, y)| \le C|x - y|^{-n}$$

for any $x, y \in \mathbb{R}^n$ and $x \ne y$, then T is bounded on $L^q_{|x|^\alpha}(\mathbb{R}^n)$ for $|x|^\alpha \in A_q$ (the weight function class of Muckenhoupt) or, equivalently, $-n < \alpha < n(q - 1)$. Note that $L^q_{|x|^\alpha}(\mathbb{R}^n) = \dot{K}_q^{\alpha/q,q}(\mathbb{R}^n)$, a special case of general homogeneous Herz spaces. As a simple corollary of our main theorems in Section 2, we generalize Stein's result to the case of the commutators $[b, I_l]$ as follows.

THEOREM 1.3. *Let* $0 \le l < n$, $1 < q_1 < n/l$, $1/q_2 = 1/q_1 - l/n$, $-n + q_1 l < \alpha < n(q_1 - 1)$, *and* $\beta = q_2\alpha/q_1$. *Let* $b \in \mathrm{BMO}(\mathbb{R}^n)$ *and let* T_l *be a linear operator. Suppose that* T_l *maps* $L^{q_1}(\mathbb{R}^n)$ *continuously into* $L^{q_2}(\mathbb{R}^n)$ *and satisfies the size condition*

$$|T_l f(x)| \le C \int_{\mathbb{R}^n} \frac{|f(y)|}{|x - y|^{n-l}}\, dy \tag{1.1}$$

for $f \in L^1(\mathbb{R}^n)$ *with compact support, and that* $x \notin \operatorname{supp} f$. *Then* $[b, T_l]$ *maps* $L^{q_1}_{|x|^\alpha}(\mathbb{R}^n)$ *continuously into* $L^{q_2}_{|x|^\beta}(\mathbb{R}^n)$.

Soria and Weiss [21] gave an elementary proof of Stein's result and some of its beautiful generalizations, which include, in addition to the case $q = 1$, more general weights and the possibility of considering maximal functions associated with a sequence of operators. We also establish some results for commutators related to linear operators that parallel results of Soria and Weiss for linear operators. To state our result, we need to introduce the weight function class $A(q_1, q_2)$.

DEFINITION 1.2 (see [20]). Let $w(x)$ be a nonnegative and locally integrable function on $\mathbb{R}^n$. We say that $w \in A(q_1, q_2)$ with $1 \le q_1 \le \infty$ and $1 \le q_2 \le \infty$ if

$$\left(|B|^{-1}\int_B w(x)^{-q_1'}\,dx\right)^{1/q_1'}\left(|B|^{-1}\int_B w(x)^{q_2}\,dx\right)^{1/q_2} \le C \qquad (1.2)$$

holds for any ball B with some positive constant C. Here $1/q_1 + 1/q_1' = 1$. The smallest constant C satisfying (1.2) will be called the constant of w in the class $A(q_1, q_2)$.

THEOREM 1.4. *Let $b \in \mathrm{BMO}(\mathbb{R}^n)$ and let $\{T_l^j\}_{j\in I}$ (I is some index set) be linear operators such that, for any $f \in L^1(\mathbb{R}^n)$ with compact support and for $x \notin \operatorname{supp} f$,*

$$|T_l^j f(x)| \le C_0 \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-l}}\,dy$$

with C_0 independent of j. Let $0 \le l < n$, $1 < q_1 < n/l$, $1/q_2 = 1/q_1 - l/n$, and $T_{l,b}^* f(x) = \sup_{j\in I} |[b, T_l^j] f(x)|$. If $T_{l,b}^*$ maps $L^{q_1}(\mathbb{R}^n)$ continuously into $L^{q_2}(\mathbb{R}^n)$, then $T_{l,b}^*$ also maps $L^{q_1}_{w^{q_1}}(\mathbb{R}^n)$ continuously into $L^{q_2}_{w^{q_2}}(\mathbb{R}^n)$ provided that $w \in A(q_1, q_2)$, and there exists a constant $C_1 > 0$ such that

$$\sup_{2^{k-2}<|x|\le 2^{k+1}} w(x) \le C_1 \inf_{2^{k-2}<|x|\le 2^{k+1}} w(x) \qquad (1.3)$$

for any $k \in \mathbb{Z}$.

We also note that although some operators (e.g., the Bochner–Riesz operators below the critical index) satisfy (1.1), they are bounded operators on $L^q(\mathbb{R}^n)$ only for some $q \in (1, \infty)$. For the commutator related to these operators, we have the following result.

THEOREM 1.5. *Let $1 \le q < \infty$, $0 < l < n/q$, $0 < p \le \infty$, and $b \in \mathrm{BMO}(\mathbb{R}^n)$. Suppose that the linear operator T_l satisfies* (1.1) *and that the commutator $[b, T_l]$ is bounded on $L^q(\mathbb{R}^n)$. Then $[b, T_l]$ maps $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ continuously into $\dot{K}_q^{\beta,p}(\mathbb{R}^n)$, provided that $l - n/q < \alpha < 0$ and $\beta = \alpha n/(n - lq)$.*

As a corollary of Theorem 1.5, we can obtain the following result for commutators related to linear operators. This result is similar to that of Lu and Soria [13] for linear operators.

COROLLARY 1.1. *Let $1 \le q < \infty$ and $0 < l < n/q$. Let $b \in \mathrm{BMO}(\mathbb{R}^n)$ and let T_l be a linear operator. Suppose that T_l satisfies* (1.1) *and that the commutator $[b, T_l]$ is bounded on $L^q(\mathbb{R}^n)$. Then $[b, T_l]$ also maps $L^q_{|x|^\alpha}(\mathbb{R}^n)$ continuously into $L^q_{|x|^\beta}(\mathbb{R}^n)$, provided that $lq - n < \alpha \le 0$ and $\beta = \alpha n/(n - lq)$.*

As a direct application of Theorem 1.5, Corollary 1.1, and the result of Hu and Lu [10], we shall obtain the boundedness of the commutator related to the Bochner–Riesz operator below the critical index on Herz spaces and weighted Lebesgue spaces. These results are new and have special meanings, since the weighted L^p-boundedness criterion for the commutators related to linear operators obtained by Alvarez et al. [2] cannot be applied to these cases. To be more precise, let $\delta > 0$. We first define the Bochner–Riesz operator of order δ for a Schwartz function f by

$$(B^\delta f)\hat{}(\xi) = (1 - |\xi|^2)_+^\delta \hat{f}(\xi),$$

where $\hat{f}$ is the Fourier transform of f. If $n = 2$, $0 < \delta < 1/2$, $4/(3+2\delta) < q < 4/(1-2\delta)$, $l = 1/2 - \delta$, $0 < p \le \infty$, $l - 2/q < \alpha < 0$, and $\beta = 2\alpha/(2 - lq)$, then Theorem 1.5 can be applied to the case of the commutator $[b, B^\delta]$. For $n \ge 3$, $(n-1)/(2n+2) < \delta < (n-1)/2$, $2n/(n+1+2\delta) < q < 2n/(n-1-2\delta)$, $l = (n-1)/2 - \delta$, $0 < p \le \infty$, $l - n/q < \alpha < 0$, and $\beta = \alpha n/(n - lq)$, Theorem 1.5 is also true for the commutator $[b, B^\delta]$. In particular, we have the following boundedness result on the weighted Lebesgue spaces with power weights.

COROLLARY 1.2. *Let $b \in \mathrm{BMO}(\mathbb{R}^n)$, and let B^δ be the Bochner–Riesz operator of order δ.*

(1) *If $0 < \delta < 1/2$, $4/(3+2\delta) < q < 4/(1-2\delta)$, $1/2 - \delta - 2/q < \alpha \le 0$, and $\beta = 4\alpha/(4-(1-2\delta)q)$, then $[b, B^\delta]$ maps $L^q_{|x|^\alpha}(\mathbb{R}^2)$ continuously into $L^q_{|x|^\beta}(\mathbb{R}^2)$.*

(2) *If $n \ge 3$, $(n-1)/(2n+2) < \delta < (n-1)/2$, $(n-1)/2 - \delta - n/q < \alpha \le 0$, and $\beta = 2n\alpha/(2n - (n-1-2\delta)q)$, then $[b, B^\delta]$ maps $L^q_{|x|^\alpha}(\mathbb{R}^n)$ continuously into $L^q_{|x|^\beta}(\mathbb{R}^n)$.*

In the proofs of our main theorems in Section 2, we borrowed many techniques developed by Soria and Weiss [21]; see also [11]. The basic philosophy is that each commutator we are considering can be decomposed into two operators: one local part, which automatically extends the boundedness properties of $[b, T]$ on Lebesgue spaces to weighted Lebesgue spaces (or Herz space); and one second part, which depends only on the "size" of the operators T and can be controlled by the commutator related to the Hardy–Littlewood maximal operator or the commutator related to the fractional maximal operator.

DEFINITION 1.3 (see [19; 20]). Let $b \in \mathrm{BMO}(\mathbb{R}^n)$. The commutators of the Hardy–Littlewood maximal function and the fractional Hardy–Littlewood maximal operator function are defined, respectively, by

$$M_b f(x) = \sup_{r>0} |B(x,r)|^{-1} \int_{B(x,r)} |b(x) - b(y)| \cdot |f(y)|\, dy$$

and

$$M_b^\lambda f(x) = \sup_{r>0} |B(x,r)|^{-1/\lambda'} \int_{B(x,r)} |b(x) - b(y)| \cdot |f(y)|\, dy,$$

where $B(x,r) = \{ y \in \mathbb{R}^n : |x-y| \le r \}$, $1 < \lambda \le \infty$, and $1/\lambda + 1/\lambda' = 1$.

In the beginning of Section 2, we shall establish the boundedness of $M_b f$ and $M_b^\lambda f$ on Herz spaces, which will be useful in the proofs of our main theorems in Section 2.

Note that, in Theorems 1.1 and 1.2, for the range of α we respectively have the restrictions $\alpha < n(1-1/q)$ and $\alpha < n(1-1/q_1)$. It is natural to ask what will happen if $\alpha \ge n(1-1/q)$ or $\alpha \ge n(1-1/q_1)$. The main purpose of Section 3 is to answer this question. In fact, we remark that Theorems 1.1 and 1.2 will be no longer true in these cases; see also [17] and [1]. For Theorem 1.1, we have the following counterexample. Let $T = H$ be the one-dimensional Hilbert transform, and let $b = \chi_{(0,\infty)}$. Let $n = 1$, $p = 1$, and $\alpha = 1 - 1/q$ with $1 < q < \infty$. If we take $f(x) = \chi_{(0,1/2)}(x) - \chi_{(-1/2,0)}(x)$, then $f \in \dot{K}_q^{1-1/q,1}(\mathbb{R})$ and, for $x > 1$,

$$|[\chi_{(0,\infty)}, H]f(x)| = \int_0^{1/2} \frac{dy}{x+y} \geq \frac{1}{2x+1} \geq \frac{1}{3x}.$$

Thus,

$$\|[\chi_{(0,\infty)}, H]f\|_{\dot{K}_q^{1-1/q,1}(\mathbb{R})} \geq \sum_{k=1}^{\infty} 2^{k(1-1/q)} \left(\int_{2^{k-1}}^{2^k} \frac{dx}{3x^q} \right)^{1/q} = \infty.$$

That is, $[\chi_{(0,\infty)}, H]f \notin \dot{K}_q^{1-1/q,1}(\mathbb{R})$. Therefore, Theorem 1.1 is not true for the case $\alpha \geq n(1-1/q)$. A slight modification yields a similar example to indicate that Theorem 1.2 is also false if $\alpha \geq n(1-1/q_1)$.

In recent years, a theory of the Hardy spaces associated with Herz spaces has been developed (see [4; 6; 8; 14; 15; 16]). In [15], Lu and Yang proved that the Calderón–Zygmund operators do map $H\dot{K}_q^{n(1-1/q),1}(\mathbb{R}^n)$ into $\dot{K}_q^{n(1-1/q),1}(\mathbb{R}^n)$, although they are not bounded on $\dot{K}_q^{n(1-1/q),1}(\mathbb{R}^n)$. It seems reasonable to guess that the commutator related to the Calderón–Zygmund operator maps $H\dot{K}_q^{n(1-1/q),1}(\mathbb{R}^n)$ into $\dot{K}_q^{n(1-1/q),1}(\mathbb{R}^n)$. Unfortunately, just as with the cases involving the standard Hardy space H^1 and the Lebesgue space L^1 (see [17; 18; 1]), this is not true. In fact, the function in the previous example is a central $(1-1/q, 1)$-atom supported in $(-1/2, 1/2)$ and belongs to the space $H\dot{K}_q^{1-1/q,1}(\mathbb{R})$ by the atomic decomposition for the space $H\dot{K}_q^{1-1/q,1}(\mathbb{R})$ in [15]; see also [8] and [16]. But $[\chi_{(0,\infty)}, H]f \notin \dot{K}_q^{1-1/q,1}(\mathbb{R})$. However, if we replace $H\dot{K}_q^{1-1/q,1}(\mathbb{R}^n)$ by a suitable atomic space $H\dot{K}_{q,b}^{n(1-1/q),1,0}(\mathbb{R}^n)$ defined as follows, then the commutator $[b, T]$ does map $H\dot{K}_{q,b}^{n(1-1/q),1,0}(\mathbb{R}^n)$ into $\dot{K}_q^{n(1-1/q),1}(\mathbb{R}^n)$; see Section 3 for the details and see also [18] and [1] for the case of the Hardy space H_b^1.

DEFINITION 1.4. Let $\alpha \in \mathbb{R}$, $s \in \mathbb{N} \cup \{0\}$, $1 < q \leq \infty$, $1/q + 1/q' = 1$, and $b \in L_{\mathrm{loc}}^{q'}(\mathbb{R}^n)$. A function $a(x)$ on $\mathbb{R}^n$ is a central $(\alpha, q, s;\ b)$-atom if

(i) $\operatorname{supp} a \subseteq B(0, r)$ for some $r > 0$;

(ii) $\|a\|_{L^q(\mathbb{R}^n)} \leq |B(0,r)|^{-\alpha/n}$; and

(iii) $$\int_{\mathbb{R}^n} x^\beta a(x)\, dx = 0 = \int_{\mathbb{R}^n} x^\beta a(x) b(x)\, dx \quad \text{for } |\beta| \leq s,$$

where $\beta = (\beta_1, \dots, \beta_n) \in (\mathbb{N} \cup \{0\})^n$ and $x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$.

DEFINITION 1.5. Let $\alpha \in \mathbb{R}$, $1 < q < \infty$, $1/q + 1/q' = 1$, $b \in L_{\mathrm{loc}}^{q'}(\mathbb{R}^n)$, $0 < p \leq \infty$, and $s \in \mathbb{N} \cup \{0\}$. A "temperate" distribution f is said to belong to $H\dot{K}_{q,b}^{\alpha,p,s}(\mathbb{R}^n)$ (or $HK_{q,b}^{\alpha,p,s}(\mathbb{R}^n)$) if, in the $\mathcal{S}'(\mathbb{R}^n)$ sense, it can be written as $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$ (or $f = \sum_{j=0}^{\infty} \lambda_j a_j$), where a_j is a central $(\alpha, q, s;\ b)$-atom supported on $B(0, 2^j)$, $\lambda_j \in \mathbb{C}$, and $\sum_{j=-\infty}^{\infty} |\lambda_j|^p < \infty$ (or $\sum_{j=0}^{\infty} |\lambda_j|^p < \infty$). We define on $H\dot{K}_{q,b}^{\alpha,p,s}(\mathbb{R}^n)$ (or $HK_{q,b}^{\alpha,p,s}(\mathbb{R}^n)$) the quasinorm

$$\|f\|_{H\dot{K}_{q,b}^{\alpha,p,s}(\mathbb{R}^n)} = \inf_{\sum_{j=-\infty}^{\infty} \lambda_j a_j = f} \left\{ \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^p \right)^{1/p} \right\}$$

$$\left(\text{or } \|f\|_{HK_{q,b}^{\alpha,p,s}(\mathbb{R}^n)} = \inf_{\sum_{j=0}^{\infty}\lambda_j a_j = f}\left\{\left(\sum_{j=0}^{\infty}|\lambda_j|^p\right)^{1/p}\right\}\right),$$

where the infimum is taken over all decompositions of f.

Obviously, if $1 < q < \infty$, $\alpha \geq n(1-1/q)$, $0 < p < \infty$, $s \geq [\alpha + n(1/q-1)]$, and $b \equiv 1$, then $H\dot{K}_{q,b}^{\alpha,p,s}(\mathbb{R}^n) = H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ and $HK_{q,b}^{\alpha,p,s}(\mathbb{R}^n) = HK_q^{\alpha,p}(\mathbb{R}^n)$, which are studied by Lu and Yang [16]. In particular, $HK_{q,1}^{n(1-1/q),1,0}(\mathbb{R}^n) = HA^q(\mathbb{R}^n)$ is introduced in [4] and [6]; $H\dot{K}_{q,1}^{n(1-1/q),1,0}(\mathbb{R}^n) = HK_q(\mathbb{R}^n)$ is discussed in [14]. If $0 < p \leq 1 < q < \infty$, $\alpha = n(1/p - 1/q)$, and $s \geq [n(1/p-1)]$, then $H\dot{K}_{q,1}^{\alpha,p,s}(\mathbb{R}^n)$ and $HK_{q,1}^{\alpha,p,s}(\mathbb{R}^n)$ are just the same spaces as those spaces introduced, respectively, in [8] and [15].

In Section 3, we shall establish the boundedness of commutators related to the (local) Calderón–Zygmund type operators (see [24, p. 63] for the definitions), fractional integral operators, and the Bochner–Riesz operators on the above atomic Herz-type Hardy spaces and Herz spaces.

For nonhomogeneous spaces, we have similar results to all the theorems for homogeneous spaces. To limit the length of this paper, we omit the details. In addition, throughout this paper, C, C_0, and C_1 denote constants that are independent of the main parameters involved but whose values may differ from line to line. The expression $A \sim B$ means that there are constants C and C_0 such that $C_0 \leq A/B \leq C$. For any power exponent p with $1 \leq p \leq \infty$ and nonnegative function w, we define the conjugate exponent $p' \equiv p/(p-1)$ and

$$\|f\|_{L_w^p(\mathbb{R}^n)} \equiv \left(\int_{\mathbb{R}^n}|f(x)|^p w(x)\,dx\right)^{1/p}.$$

2. Commutators on Herz Spaces

We begin with the boundedness of the commutators of the (fractional) Hardy–Littlewood maximal functions on Herz spaces, which will be useful in the proofs of our main theorems in this section and are themselves of independent interest; see [19] and [20].

THEOREM 2.1. *Let $1 < q < \infty$, $-n/q < \alpha < n(1-1/q)$, and $0 < p \leq \infty$. Let $b \in \mathrm{BMO}(\mathbb{R}^n)$ and let M_b be defined as in Definition 1.3. Then M_b is bounded on $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$.*

Proof. Set $f = \sum_{j=-\infty}^{\infty} f\chi_j \equiv \sum_{j=-\infty}^{\infty} f_j$. We write

$$\begin{aligned}\|M_b f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} &= \left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\|(M_b f)\chi_k\|_{L^q(\mathbb{R}^n)}^p\right\}^{1/p}\\ &\leq C\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-3}\|(M_b f_j)\chi_k\|_{L^q(\mathbb{R}^n)}\right)^p\right\}^{1/p}\end{aligned}$$

$$+ C\left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=k-2}^{k+2} \|(M_b f_j)\chi_k\|_{L^q(\mathbb{R}^n)} \right)^p \right\}^{1/p}$$

$$+ C\left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=k+3}^{\infty} \|(M_b f_j)\chi_k\|_{L^q(\mathbb{R}^n)} \right)^p \right\}^{1/p}$$

$$\equiv \mathrm{D}_1 + \mathrm{D}_2 + \mathrm{D}_3.$$

For D_2, by [7, Thm. 2.4, p. 1401] we know that M_b is bounded on $L^q(\mathbb{R}^n)$. Therefore,

$$\mathrm{D}_2 \le C\left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=k-2}^{k+2} \|f_j\|_{L^q(\mathbb{R}^n)} \right)^p \right\}^{1/p} \le C\|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}.$$

Let b_j denote the mean value of b on $B(0, 2^j)$. Observe that if $j \le k-3$, from the properties of $\mathrm{BMO}(\mathbb{R}^n)$ functions (see [23, Chap. 4]) it follows that

$$\begin{aligned}
\|(M_b f_j)\chi_k\|_{L^q(\mathbb{R}^n)} &\le C2^{-kn}\left[\int_{A_k} \left(\int_{A_j} |b(x) - b(y)| \cdot |f_j(y)|\, dy \right)^q dx \right]^{1/q} \\
&\le C2^{-kn}\|f_j\|_{L^1(\mathbb{R}^n)} \left(\int_{A_k} |b(x) - b_j|^q\, dx \right)^{1/q} \\
&\quad + C2^{kn(1/q-1)}\|f_j\|_{L^q(\mathbb{R}^n)} \left(\int_{A_j} |b_j - b(y)|^{q'}\, dy \right)^{1/q'} \\
&\le C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} 2^{(j-k)n(1-1/q)}(k-j)\|f_j\|_{L^q(\mathbb{R}^n)}.
\end{aligned}$$

Therefore, Hölder's inequality gives us that

$$\mathrm{D}_1 \le C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \left\{ \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-3} 2^{j\alpha}\|f_j\|_{L^q(\mathbb{R}^n)} 2^{(k-j)(\alpha - n(1-1/q))}(k-j) \right)^p \right\}^{1/p}$$

$$\le C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \begin{cases} \left[\displaystyle\sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-3} 2^{j\alpha p}\|f_j\|^p_{L^q(\mathbb{R}^n)} \right.\right. \\ \qquad \left.\left. \times\, 2^{(k-j)(\alpha-n(1-1/q))p}(k-j)^p \right) \right]^{1/p}, \quad 0 < p \le 1 \\ \left[\displaystyle\sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-3} 2^{j\alpha p}\|f_j\|^p_{L^q(\mathbb{R}^n)} 2^{(k-j)(\alpha-n(1-1/q))p/2} \right) \right. \\ \quad \left. \times \left(\displaystyle\sum_{j=-\infty}^{k-3} (k-j)^{p'} 2^{(k-j)(\alpha-n(1-1/q))p'/2} \right)^{p/p'} \right]^{1/p}, \\ \qquad\qquad 1 < p \le \infty \end{cases}$$

$$\leq C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \begin{cases} \left[\sum_{j=-\infty}^{\infty} 2^{j\alpha p} \|f_j\|_{L^q(\mathbb{R}^n)}^p \right. \\ \quad \left. \times \left(\sum_{k=j+3}^{\infty} 2^{(k-j)(\alpha - n(1-1/q))p} (k-j)^p \right) \right]^{1/p}, \\ \hfill 0 < p \leq 1 \\ \left[\sum_{j=-\infty}^{\infty} 2^{j\alpha p} \|f_j\|_{L^q(\mathbb{R}^n)}^p \left(\sum_{k=j+3}^{\infty} 2^{(k-j)(\alpha - n(1-1/q))p/2} \right) \right]^{1/p}, \\ \hfill 1 < p \leq \infty \end{cases}$$

$$\leq C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \left\{ \sum_{j=-\infty}^{\infty} 2^{j\alpha p} \|f_j\|_{L^q(\mathbb{R}^n)}^p \right\}^{1/p} = C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)},$$

since $\alpha < n(1 - 1/q)$.

For D_3, we first observe that for $j \geq k + 3$ and $x \in A_k$,

$$\begin{aligned} M_b f_j(x) &\leq C 2^{-jn} \int_{A_j} |b(x) - b(y)| \cdot |f_j(y)|\, dy \\ &\leq C 2^{-jn} |b(x) - b_k| \cdot \|f_j\|_{L^1(\mathbb{R}^n)} + C 2^{-jn} \int_{A_j} |b_k - b(y)| \cdot |f(y)|\, dy. \end{aligned}$$

Thus,

$$\begin{aligned} \|(M_b f_j)\chi_k\|_{L^q(\mathbb{R}^n)} &\leq C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} 2^{(k-j)n/q} \|f_j\|_{L^q(\mathbb{R}^n)} \\ &\quad + C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} 2^{(k-j)n/q} (j-k) \|f_j\|_{L^q(\mathbb{R}^n)} \\ &\leq C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} 2^{(k-j)n/q} (j-k) \|f_j\|_{L^q(\mathbb{R}^n)} \end{aligned}$$

and so

$$\mathrm{D}_3 \leq C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \left\{ \sum_{k=-\infty}^{\infty} \left(\sum_{j=k+3}^{\infty} 2^{j\alpha} \|f_j\|_{L^q(\mathbb{R}^n)} (j-k) 2^{-(j-k)(\alpha + n/q)} \right)^p \right\}^{1/p}$$

$$\leq C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \begin{cases} \left[\sum_{k=-\infty}^{\infty} \left(\sum_{j=k+3}^{\infty} 2^{j\alpha p} \|f_j\|_{L^q(\mathbb{R}^n)}^p \right. \right. \\ \quad \left. \left. \times (j-k)^p 2^{-(j-k)(\alpha+n/q)p} \right) \right]^{1/p}, \quad 0 < p \leq 1 \\ \left[\sum_{k=-\infty}^{\infty} \left(\sum_{j=k+3}^{\infty} 2^{j\alpha p} \|f_j\|_{L^q(\mathbb{R}^n)}^p 2^{-(j-k)(\alpha+n/q)p/2} \right) \right. \\ \quad \left. \times \left(\sum_{j=k+3}^{\infty} (j-k)^{p'} 2^{-(j-k)(\alpha+n/q)p'/2} \right)^{p/p'} \right]^{1/p}, \\ \hfill 1 < p \leq \infty \end{cases}$$

$$\leq C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\begin{cases}\Bigg[\sum_{j=-\infty}^{\infty} 2^{j\alpha p}\|f_j\|_{L^q(\mathbb{R}^n)}^p \\ \quad\times\Bigg(\sum_{k=-\infty}^{j-3}(j-k)^p 2^{-(j-k)(\alpha+n/q)p}\Bigg)\Bigg]^{1/p}, & 0<p\leq 1 \\ \Bigg[\sum_{j=-\infty}^{\infty} 2^{j\alpha p}\|f_j\|_{L^q(\mathbb{R}^n)}^p\Bigg(\sum_{k=-\infty}^{j-3} 2^{-(j-k)(\alpha+n/q)p/2}\Bigg)\Bigg]^{1/p}, \\ & 1<p\leq\infty\end{cases}$$

$$\leq C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\Bigg\{\sum_{j=-\infty}^{\infty} 2^{j\alpha p}\|f_j\|_{L^q(\mathbb{R}^n)}^p\Bigg\}^{1/p} = C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)},$$

because $\alpha > -n/q$.

This finishes the proof of Theorem 2.1. □

THEOREM 2.2. *Let* $1<\lambda<\infty$, $1<q_1<\lambda$, $1/q_2 = 1/q_1 - 1/\lambda$, $-n/q_1 + n/\lambda < \alpha < n(1-1/q_1)$, *and* $0 < p_1 \leq p_2 \leq \infty$. *Let* $b \in \mathrm{BMO}(\mathbb{R}^n)$ *and let* M_b^λ *be defined as in Definition 1.3. Then* M_b^λ *maps* $\dot K_{q_1}^{\alpha,p_1}(\mathbb{R}^n)$ *continuously into* $\dot K_{q_2}^{\alpha,p_2}(\mathbb{R}^n)$.

Proof. Note that if $p_1 < p_2$ then

$$\dot K_{q_2}^{\alpha,p_1}(\mathbb{R}^n) \subset \dot K_{q_2}^{\alpha,p_2}(\mathbb{R}^n). \tag{2.1}$$

We only need to prove Theorem 2.2 for the case $p_1 = p_2$.

As in the proof of Theorem 2.1, we can write

$$\begin{aligned}\|M_b^\lambda f\|_{\dot K_{q_2}^{\alpha,p_1}(\mathbb{R}^n)} &= \Bigg\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p_1}\|\chi_k M_b^\lambda f\|_{L^{q_2}(\mathbb{R}^n)}^{p_1}\Bigg\}^{1/p_1} \\ &\leq C\Bigg\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p_1}\Bigg(\sum_{j=-\infty}^{k-3}\|\chi_k M_b^\lambda f_j\|_{L^{q_2}(\mathbb{R}^n)}\Bigg)^{p_1}\Bigg\}^{1/p_1} \\ &\quad + C\Bigg\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p_1}\Bigg(\sum_{j=k-2}^{k+2}\|\chi_k M_b^\lambda f_j\|_{L^{q_2}(\mathbb{R}^n)}\Bigg)^{p_1}\Bigg\}^{1/p_1} \\ &\quad + C\Bigg\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p_1}\Bigg(\sum_{j=k+3}^{\infty}\|\chi_k M_b^\lambda f_j\|_{L^{q_2}(\mathbb{R}^n)}\Bigg)^{p_1}\Bigg\}^{1/p_1} \\ &\equiv \mathrm{E}_1 + \mathrm{E}_2 + \mathrm{E}_3.\end{aligned}$$

For E_2, since M_b^λ maps $L^{q_1}(\mathbb{R}^n)$ continuously into $L^{q_2}(\mathbb{R}^n)$ (see [20, Thm. 3.2, p. 544]), we obtain

$$\mathrm{E}_2 \leq C\Bigg\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p_1}\Bigg(\sum_{j=k-2}^{k+2}\|f_j\|_{L^{q_1}(\mathbb{R}^n)}\Bigg)^{p_1}\Bigg\}^{1/p_1} \leq C\|f\|_{\dot K_{q_1}^{\alpha,p_1}(\mathbb{R}^n)}.$$

Let b_j denote the mean value of b on $B(0,2^j)$. For E_1, note that if $j \leq k-3$ then

$$\begin{aligned}\|\chi_k M_b^\lambda f_j\|_{L^{q_2}(\mathbb{R}^n)} &\le C2^{-kn/\lambda'}\left[\int_{A_k}\left(\int_{A_j}|b(x)-b(y)|\cdot|f_j(y)|\,dy\right)^q dx\right]^{1/q}\\ &\le C2^{-kn/\lambda'}\|f_j\|_{L^1(\mathbb{R}^n)}\left(\int_{A_k}|b(x)-b_j|^{q_2}\,dx\right)^{1/q_2}\\ &\quad + C2^{-kn/\lambda'+kn/q_2}\|f_j\|_{L^{q_1}(\mathbb{R}^n)}\left(\int_{A_j}|b_j-b(y)|^{q_1'}\,dy\right)^{1/q_1'}\\ &\le C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}(k-j)2^{(j-k)n(1-1/q_1)}\|f_j\|_{L^{q_1}(\mathbb{R}^n)}.\end{aligned}$$

Therefore, by the fact that $\alpha < n(1-1/q_1)$ and an argument similar to the estimate for D_1, we obtain

$$\begin{aligned}\mathrm{E}_1 &\le C\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-3}2^{j\alpha}\|f_j\|_{L^{q_1}(\mathbb{R}^n)}2^{(j-k)(n(1-1/q_1)-\alpha)}(k-j)\right)^{p_1}\right\}^{1/p_1}\\ &\le C\left\{\sum_{j=-\infty}^{\infty}2^{j\alpha p_1}\|f_j\|_{L^{q_1}(\mathbb{R}^n)}^{p_1}\right\}^{1/p_1} = C\|f\|_{\dot K_{q_1}^{\alpha,p_1}(\mathbb{R}^n)}.\end{aligned}$$

For E_3, observe that for $j \ge k+3$ we have

$$\begin{aligned}\|\chi_k M_b^\lambda f_j\|_{L^{q_2}(\mathbb{R}^n)} &\le C2^{-jn/\lambda'}\left[\int_{A_k}\left(\int_{A_j}|b(x)-b(y)|\cdot|f(y)|\,dy\right)^q dx\right]^{1/q}\\ &\le C2^{-jn/\lambda'}\|f_j\|_{L^1(\mathbb{R}^n)}\left(\int_{A_k}|b(x)-b_k|^{q_2}\,dx\right)^{1/q_2}\\ &\quad + C2^{-jn/\lambda'+kn/q_2}\|f_j\|_{L^{q_1}(\mathbb{R}^n)}\left(\int_{A_j}|b_k-b(y)|^{q_1'}\,dy\right)^{1/q_1'}\\ &\le C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}(j-k)2^{(k-j)n/q_2}\|f_j\|_{L^{q_1}(\mathbb{R}^n)}.\end{aligned}$$

As with the estimate for D_3, we similarly obtain

$$\begin{aligned}\mathrm{E}_3 &\le C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{j=k+3}^{\infty}2^{j\alpha}\|f_j\|_{L^{q_1}(\mathbb{R}^n)}2^{(k-j)(\alpha+n/q_2)}(j-k)\right)^{p_1}\right\}^{1/p_1}\\ &\le C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\left\{\sum_{j=-\infty}^{\infty}2^{j\alpha p_1}\|f_j\|_{L^{q_1}(\mathbb{R}^n)}^{p_1}\right\}^{1/p_1} = C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_{\dot K_{q_1}^{\alpha,p_1}(\mathbb{R}^n)},\end{aligned}$$

since $\alpha > -n/q_2 = -n/q_1 + n/\lambda$.

This finishes the proof of Theorem 2.2. □

LEMMA 2.1 (see [9]). *Let $\alpha \in \mathbb{R}$ and $1 \le p, q < \infty$. Then $f \in \dot K_q^{\alpha,p}(\mathbb{R}^n)$ if and only if*

$$\left|\int_{\mathbb{R}^n} f(x)g(x)\,dx\right| < \infty$$

for every $g \in \dot{K}_{q'}^{-\alpha,p'}(\mathbb{R}^n)$ *and, in this case,*

$$\|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} = \sup\left\{\left|\int_{\mathbb{R}^n} f(x)g(x)\,dx\right| : \|g\|_{\dot{K}_{q'}^{-\alpha,p'}(\mathbb{R}^n)} \leq 1\right\}.$$

Now let us state one of our main theorems in this section as follows.

THEOREM 2.3. *Let* $b \in \mathrm{BMO}(\mathbb{R}^n)$ *and let* T *be a linear operator. Suppose that the commutator* $[b, T]$ *is bounded on* $L^q(\mathbb{R}^n)$ *for some* $q \in (1, \infty)$. *Suppose that* T *satisfies the local size condition*

$$|Tf(x)| \leq C|x|^{-n} \int |f(y)|\,dy \tag{2.2}$$

for $f \in L^1(\mathbb{R}^n)$, $\operatorname{supp} f \subseteq A_k$, *and* $|x| \geq 2^{k+1}$ *with* $k \in \mathbb{Z}$, *and the condition*

$$|Tf(x)| \leq C2^{-kn}\|f\|_{L^1(\mathbb{R}^n)} \tag{2.3}$$

for $f \in L^1(\mathbb{R}^n)$, $\operatorname{supp} f \subseteq A_k$ *and* $|x| \leq 2^{k-2}$ *with* $k \in \mathbb{Z}$. *Then* $[b, T]$ *is also bounded on the Herz space* $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$, *provided that* $-n/q < \alpha < n(1 - 1/q)$ *and* $0 < p \leq \infty$.

Proof. Write $f = \sum_{j=-\infty}^{\infty} f\chi_j \equiv \sum_{j=-\infty}^{\infty} f_j$. We then have

$$\begin{aligned}
\|[b, T]f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} &= \left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\|\chi_k[b, T]f\|_{L^q(\mathbb{R}^n)}^p\right\}^{1/p} \\
&\leq C\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\left\|\chi_k[b, T]\left(\sum_{j=-\infty}^{k-3} f_j\right)\right\|_{L^q(\mathbb{R}^n)}\right)^p\right\}^{1/p} \\
&\quad + C\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\left\|\chi_k[b, T]\left(\sum_{j=k-2}^{k+2} f_j\right)\right\|_{L^q(\mathbb{R}^n)}\right)^p\right\}^{1/p} \\
&\quad + C\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\left\|\chi_k[b, T]\left(\sum_{j=k+3}^{\infty} f_j\right)\right\|_{L^q(\mathbb{R}^n)}\right)^p\right\}^{1/p} \\
&\equiv \mathrm{F}_1 + \mathrm{F}_2 + \mathrm{F}_3.
\end{aligned}$$

Since $[b, T]$ is bounded on $L^q(\mathbb{R}^n)$, we have

$$\mathrm{F}_2 \leq C\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=k-2}^{k+2} \|f_j\|_{L^q(\mathbb{R}^n)}\right)^p\right\}^{1/p} \leq C\|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}.$$

For F_1, note that if $x \in A_k$ and $j \leq k - 3$ then

$$\left|[b, T]\left(\sum_{j=-\infty}^{k-3} f_j\right)(x)\right| \leq C|x|^{-n}\int |b(x)-b(y)|\cdot\left|\sum_{j=-\infty}^{k-3} f_j(y)\right| dy \leq CM_b f(x).$$

Therefore, by Theorem 2.1,

$$F_1 \le C\|M_b f\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)} \le C\|f\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)}.$$

On F_3, note that if $x \in A_k$, $y \in A_j$, and $j \ge k+3$, then $2|x| < |y|$. Thus,

$$\begin{aligned}\left|[b,T]\Big(\sum_{j=k+3}^{\infty} f_j\Big)(x)\right| &\le C\int_{2|x|<|y|} |b(x)-b(y)|\cdot|y|^{-n}\cdot|f(y)|\,dy\\ &\equiv CT_b^1(|f|)(x).\end{aligned}$$

Set $1 < p < \infty$ and

$$T_b^0 g(x) = |x|^{-n}\int_{|y|<2|x|} |b(x)-b(y)|g(y)\,dy.$$

Therefore, by Lemma 2.1,

$$\begin{aligned}\mathrm{F}_3 &\le C\|T_b^1(|f|)\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)} = C \sup_{\|g\|_{\dot K_{q'}^{-\alpha,p'}(\mathbb{R}^n)}\le 1} |(T_b^1(|f|), g)|\\ &= C \sup_{\|g\|_{\dot K_{q'}^{-\alpha,p'}(\mathbb{R}^n)}\le 1} |(|f|, T_b^0 g)|\\ &\le C\|f\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)} \sup_{\|g\|_{\dot K_{q'}^{-\alpha,p'}(\mathbb{R}^n)}\le 1} \|M_b g\|_{\dot K_{q'}^{-\alpha,p'}(\mathbb{R}^n)}.\end{aligned}$$

Since $1 < q' < \infty$, $-n/q' < -\alpha < n(1-1/q')$, and $0 < p' < \infty$, by Theorem 2.1 we have

$$\|M_b g\|_{\dot K_{q'}^{-\alpha,p'}(\mathbb{R}^n)} \le C\|g\|_{\dot K_{q'}^{-\alpha,p'}(\mathbb{R}^n)}.$$

Thus,

$$\mathrm{F}_3 \le C\|f\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)}.$$

This finishes the estimate of F_3 for $1 < p < \infty$.

Now let $0 < p \le 1$. By (2.3), we first deduced that, when $j \ge k+3$,

$$\begin{aligned}&\|\chi_k[b,T]f_j\|_{L^q(\mathbb{R}^n)}\\ &\quad\le C\Big(\int_{A_k} |b(x)-b_k|^q\cdot|Tf_j(x)|^q\,dx\Big)^{1/q}\\ &\qquad + C\Big(\int_{A_k} \big|T\big((b-b_k)f_j\big)(x)\big|^q\,dx\Big)^{1/q}\\ &\quad\le C2^{(k-j)n/q}\|f_j\|_{L^q(\mathbb{R}^n)}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\\ &\qquad + C2^{(k-j)n/q}\|f_j\|_{L^q(\mathbb{R}^n)}\Big(|B(0,2^j)|^{-1}\int_{B(0,2^j)} |b(y)-b_k|^{q'}\,dy\Big)^{1/q'}\\ &\quad\le C2^{(k-j)n/q}(j-k)\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_j\|_{L^q(\mathbb{R}^n)}.\end{aligned}$$

Similar to the estimate for D_3, it follows from this that

$$F_3 \le C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{j=k+3}^{\infty} 2^{j\alpha}\|f_j\|_{L^q(\mathbb{R}^n)}(j-k)2^{-(j-k)(\alpha+n/q)}\right)^p\right\}^{1/p}$$
$$\le C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\|f_k\|^p_{L^q(\mathbb{R}^n)}\right\}^{1/p} = C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)},$$

since $\alpha > -n/q$.

The estimate for F_3 with the case $p = \infty$ is simple. We leave it to the reader.

This finishes the proof of Theorem 2.3. □

Corollary 2.1. *Let $b \in \mathrm{BMO}(\mathbb{R}^n)$ and let T be a linear operator. Suppose that the commutator $[b, T]$ is bounded on $L^q(\mathbb{R}^n)$ for some $q \in (1, \infty)$, and that*

$$|Tf(x)| \le C\int \frac{|f(y)|}{|x-y|^n}\,dy \tag{2.4}$$

for $f \in L^1(\mathbb{R}^n)$ with compact support and $x \notin \operatorname{supp} f$. Then $[b, T]$ is bounded on Herz spaces $\dot K_q^{\alpha,p}(\mathbb{R}^n)$, provided that $-n/q < \alpha < n(1-1/q)$ and $0 < p \le \infty$.

We remark that (2.4) is satisfied by many operators in harmonic analysis, such as Calderón–Zygmund operators, C. Fefferman's singular multiplier, Ricci–Stein's oscillatory singular integral, the Bochner–Riesz operators at the critical index, and so on (see [21], [9], or [11]). Therefore, Theorem 1.1 is just a special case of Corollary 2.1.

Corollary 2.2. *Let $1 < q < \infty$ and $-n < \alpha < n(q-1)$. Let $b \in \mathrm{BMO}(\mathbb{R}^n)$ and let T be a linear operator. Then the commutator $[b, T]$ is bounded on $L^q_{|x|^\alpha}(\mathbb{R}^n)$ if $[b, T]$ is bounded on $L^q(\mathbb{R}^n)$ and if T satisfies one of the following conditions:*

(a) (2.2) *and* (2.3) *in Theorem 2.1; or*
(b) (2.4) *in Corollary 2.1.*

Our Corollary 2.2 actually is a generalization of Stein's result in [22] to linear commutators; see also [21, Thm. 4, p. 193].

If we take a look at the proof of Theorem 2.3, we easily see that the following result holds for the maximal commutators.

Theorem 2.4. *Let $1 < q < \infty$, $-n/q < \alpha < n(1-1/q)$, and $0 < p \le \infty$. Given a $\mathrm{BMO}(\mathbb{R}^n)$ function b and linear operators $\{T_j\}_{j\in I}$ (I is some index set), the maximal commutator $T_b^* f(x) = \sup_{j\in I}|[b, T_j]f(x)|$ is bounded on $\dot K_q^{\alpha,p}(\mathbb{R}^n)$ if $T_b^* f$ is bounded on $L^q(\mathbb{R}^n)$ and if the T_j satisfy one of the following conditions.*

(i) *For $f \in L^1(\mathbb{R}^n)$, $\operatorname{supp} f \subseteq A_k$, and $|x| \ge 2^{k+1}$ with $k \in \mathbb{Z}$, the T_j uniformly satisfy*

$$|T_j f(x)| \le C_0|x|^{-n}\|f\|_{L^1(\mathbb{R}^n)}; \tag{2.5}$$

for $f \in L^1(\mathbb{R}^n)$, $\operatorname{supp} f \subseteq A_k$, and $|x| \le 2^{k-2}$ with $k \in \mathbb{Z}$, they uniformly satisfy

$$|T_j f(x)| \leq C_0 2^{-kn} \|f\|_{L^1(\mathbb{R}^n)}. \tag{2.6}$$

Here C_0 is independent of j.

(ii) *For $f \in L^1(\mathbb{R}^n)$ with compact support and $x \notin \operatorname{supp} f$,*

$$|T_j f(x)| \leq C_0 \int \frac{|f(y)|}{|x-y|^n}\, dy, \tag{2.7}$$

with C_0 independent of j.

COROLLARY 2.3. *Let $1 < q < \infty$ and $-n < \alpha < n(q-1)$. Assume that b, $\{T_j\}_{j\in I}$, and T_b^* are defined as in Theorem 2.4. Then the maximal commutator T_b^* is bounded on $L^q_{|x|^\alpha}(\mathbb{R}^n)$ if T_b^* is bounded on $L^q(\mathbb{R}^n)$ and if the T_j satisfy one of* (i) *and* (ii) *in Theorem 2.4.*

Obviously, the commutators of the truncated operators and the commutators of the Bochner–Riesz operators satisfy Corollary 2.3.

For Corollary 2.3, we have the following generalization (see also [21, Thm. 1, p. 189]).

THEOREM 2.5. *Let $b \in \mathrm{BMO}(\mathbb{R}^n)$ and let the linear operators $\{T_j\}_{j\in I}$ (I is some index set) satisfy* (2.7). *Define $T_b^* f$ as in Theorem 2.4. If $T_b^* f$ is bounded on $L^q(\mathbb{R}^n)$ for some $q \in (1, \infty)$ then $T_b^* f$ is also bounded on $L^q_w(\mathbb{R}^n)$, where $w \in A_q$ and w satisfies* (1.3).

Proof. As in the proof of [21, Thm. 1], we write $f_{k,0}(x) = f(x)\chi_{\{2^{k-2}<|x|\leq 2^{k+1}\}}(x)$ and $f_{k,1}(x) = f(x) - f_{k,0}(x)$. Then

$$\begin{aligned} T_b^* f(x) &= \sum_{k\in\mathbb{Z}} T_b^* f(x)\chi_k(x) \leq \sum_{k\in\mathbb{Z}} T_b^*(f_{k,0})(x)\chi_k(x) + \sum_{k\in\mathbb{Z}} T_b^*(f_{k,1})(x)\chi_k(x) \\ &\equiv T_0 f(x) + T_1 f(x). \end{aligned}$$

If we set $m_k = \inf\{ w(x) : 2^{k-2} < |x| \leq 2^{k+1} \}$, then $w(x) \sim m_k$ for every x satisfying $2^{k-2} < |x| \leq 2^{k+1}$. Therefore, by the $L^q(\mathbb{R}^n)$-boundedness of $T_b^* f$, we have

$$\begin{aligned} \|T_0 f\|^q_{L^q_w(\mathbb{R}^n)} &= \sum_{k\in\mathbb{Z}} \|\chi_k T_b^*(f_{k,0})\|^q_{L^q_w(\mathbb{R}^n)} \leq C \sum_{k\in\mathbb{Z}} m_k \|T_b^*(f_{k,0})\|^q_{L^q(\mathbb{R}^n)} \\ &\leq C \sum_{k\in\mathbb{Z}} m_k \|f_{k,0}\|^q_{L^q(\mathbb{R}^n)} \leq C \sum_{k\in\mathbb{Z}} \|f_{k,0}\|^q_{L^q_w(\mathbb{R}^n)} \leq C\|f\|^q_{L^q_w(\mathbb{R}^n)}. \end{aligned}$$

To prove Theorem 2.5, it suffices to prove

$$\|T_1 f\|_{L^q_w(\mathbb{R}^n)} \leq C\|f\|_{L^q_w(\mathbb{R}^n)}.$$

Note that

$$\begin{aligned} T_1 f(x) &\leq C_0 \sum_{k=-\infty}^{\infty} \left(\int_{\mathbb{R}^n} \frac{|b(x)-b(y)| \cdot |f_{b,1}(y)|}{|x-y|^n}\, dy \right) \chi_k(x) \\ &\leq C_0 \int_{\mathbb{R}^n} \frac{|b(x)-b(y)| \cdot |f(y)|}{|x|^n + |y|^n}\, dy \end{aligned}$$

$$\begin{aligned}&\le C_0|x|^{-n}\int_{|y|\le|x|}|b(x)-b(y)|\cdot|f(y)|\,dy\\&\quad+C_0\int_{|x|\le|y|}|b(x)-b(y)|\cdot|f(y)|\cdot|y|^{-n}\,dy\\&\equiv \mathrm{L}_1f(x)+\mathrm{L}_2f(x).\end{aligned}$$

Now, $\mathrm{L}_1f(x)\le CM_bf(x)$. Since $w\in A_q(\mathbb{R}^n)$, by [7, Thm. 2.4], we have

$$\|\mathrm{L}_1f\|_{L^q_w(\mathbb{R}^n)}\le C\|M_bf\|_{L^q_w(\mathbb{R}^n)}\le C\|f\|_{L^q_w(\mathbb{R}^n)}.$$

By duality, L_2f satisfies the same conclusion.

This finishes the proof of Theorem 2.5. □

Obviously, Theorem 2.5 is just the case with $l=0$ of Theorem 1.4.

Now let us turn to the case with fractional commutators.

THEOREM 2.6. *Let $b\in\mathrm{BMO}(\mathbb{R}^n)$ and $0<l<n$. Suppose that the linear operator T_l satisfies*

$$|T_lf(x)|\le C|x|^{-(n-l)}\|f\|_{L^1(\mathbb{R}^n)} \tag{2.8}$$

when $f\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $\operatorname{supp} f\subseteq A_k$, and $|x|\ge 2^{k+1}$ with $k\in\mathbb{Z}$, and satisfies

$$|T_lf(x)|\le C2^{-k(n-l)}\|f\|_{L^1(\mathbb{R}^n)} \tag{2.9}$$

when $f\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $\operatorname{supp} f\subseteq A_k$, and $|x|\le 2^{k-2}$ with $k\in\mathbb{Z}$. Assume also that $1<q_1<n/l$, $1/q_2=1/q_1-l/n$, $-n/q_1+l<\alpha<n(1-1/q_1)$, $0<p_1\le p_2\le\infty$, and that $[b,T_l]$ maps $L^{q_1}(\mathbb{R}^n)$ continuously into $L^{q_2}(\mathbb{R}^n)$. Then $[b,T_l]$ maps $\dot K^{\alpha,p_1}_{q_1}(\mathbb{R}^n)$ continuously into $\dot K^{\alpha,p_2}_{q_2}(\mathbb{R}^n)$.

Proof. In view of (2.1), we need only show Theorem 2.6 in the case $p_1=p_2$. Let $f=\sum_{j=-\infty}^{\infty}f\chi_j\equiv\sum_{j=-\infty}^{\infty}f_j$. Write

$$\begin{aligned}\|[b,T_l]f\|_{\dot K^{\alpha,p_1}_{q_2}(\mathbb{R}^n)}&=\left\{\sum_{k=-\infty}^{\infty}2^{k\alpha p_1}\|\chi_k[b,T_l]f\|^{p_1}_{L^{q_2}(\mathbb{R}^n)}\right\}^{1/p_1}\\&\le C\left\{\sum_{k=-\infty}^{\infty}2^{k\alpha p_1}\left(\left\|\chi_k[b,T_l]\left(\sum_{j=-\infty}^{k-3}f_j\right)\right\|_{L^{q_2}(\mathbb{R}^n)}\right)^{p_1}\right\}^{1/p_1}\\&\quad+C\left\{\sum_{k=-\infty}^{\infty}2^{k\alpha p_1}\left(\sum_{j=k-2}^{k+2}\|\chi_k[b,T_l]f_j\|_{L^{q_2}(\mathbb{R}^n)}\right)^{p_1}\right\}^{1/p_1}\\&\quad+C\left\{\sum_{k=-\infty}^{\infty}2^{k\alpha p_1}\left(\left\|\chi_k[b,T_l]\left(\sum_{j=k+3}^{\infty}f_j\right)\right\|_{L^{q_2}(\mathbb{R}^n)}\right)^{p_1}\right\}^{1/p_1}\\&\equiv \mathrm{G}_1+\mathrm{G}_2+\mathrm{G}_3.\end{aligned}$$

For G_2, using that $[b,T_l]$ maps $L^{q_1}(\mathbb{R}^n)$ continuously into $L^{q_2}(\mathbb{R}^n)$, we obtain

$$\mathrm{G}_2\le C\left\{\sum_{k=-\infty}^{\infty}2^{k\alpha p_1}\left(\sum_{j=k-2}^{k+2}\|f_j\|_{L^{q_1}(\mathbb{R}^n)}\right)^{p_1}\right\}^{1/p_1}\le C\|f\|_{\dot K^{\alpha,p_1}_{q_1}(\mathbb{R}^n)}.$$

To estimate G_1, observe that when $x \in A_k$, $y \in A_j$, and $j \le k-3$, we then have $2|y| < |x|$ and therefore

$$\left|[b, T_l]\left(\sum_{j=-\infty}^{k-3} f_j\right)(x)\right| \le C|x|^{-(n-l)} \int_{2|y|<|x|} |b(x)-b(y)| \left|\sum_{j=-\infty}^{k-3} f_j(y)\right| dy \le CM_b^{n/l} f(x),$$

by (2.8).

By Theorem 2.2 we now have that

$$G_1 \le C\|M_b^{n/l} f\|_{\dot{K}_{q_2}^{\alpha,p_1}(\mathbb{R}^n)} \le C\|f\|_{\dot{K}_{q_1}^{\alpha,p_1}(\mathbb{R}^n)}.$$

Now we turn our attention to G_3. Let $1 < p_1 < \infty$. Note that, when $x \in A_k$, $y \in A_j$, and $j \ge k+3$, we have $|x| \le 2|y|$ and

$$\left|[b, T_l]\left(\sum_{j=k+3}^{\infty} f_j\right)(x)\right| \le C \int_{|x|\le 2|y|} \frac{|b(x)-b(y)|}{|y|^{n-l}} |f|\, dy \equiv CT_{b,l}^1(|f|)(x).$$

Set

$$T_{b,l}^0 g(x) \equiv C|x|^{-(n-l)} \int_{|y|\le 2|x|} |b(x)-b(y)| g(y)\, dy.$$

By Lemma 2.1 we obtain that, for $1 < p_1 < \infty$,

$$\begin{aligned} G_3 &\le C\|T_{b,l}^1(|f|)\|_{\dot{K}_{q_2}^{\alpha,p_1}(\mathbb{R}^n)} = C \sup_{\|g\|_{\dot{K}_{q_2'}^{-\alpha,p_1'}(\mathbb{R}^n)}\le 1} |(T_{b,l}^1(|f|), g)| \\ &= C \sup_{\|g\|_{\dot{K}_{q_2'}^{-\alpha,p_1'}(\mathbb{R}^n)}\le 1} |(|f|, T_{b,l}^0 g)| \\ &\le C\|f\|_{\dot{K}_{q_1}^{\alpha,p_1}(\mathbb{R}^n)} \sup_{\|g\|_{\dot{K}_{q_2'}^{-\alpha,p_1'}(\mathbb{R}^n)}\le 1} \|M_b^{n/l} g\|_{\dot{K}_{q_1'}^{-\alpha,p_1'}(\mathbb{R}^n)} \le C\|f\|_{\dot{K}_{q_1}^{\alpha,p_1}(\mathbb{R}^n)}, \end{aligned}$$

where in the last inequality we have used Theorem 2.2 for $M_b^{n/l}$.

In the following let $0 < p_1 \le 1$ and, for $j \in \mathbb{Z}$, let b_j be the mean value of b on $B(0, 2^j)$. When $j \ge k+3$, (2.9) states that

$$\begin{aligned} \|\chi_k [b, T_l] f_j\|_{L^{q_2}(\mathbb{R}^n)} &\le \|\chi_k (b-b_k) T_l f_j\|_{L^{q_2}(\mathbb{R}^n)} + \|\chi_k T_l((b-b_k) f_j)\|_{L^{q_2}(\mathbb{R}^n)} \\ &\le C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} (j-k) 2^{(k-j)n/q_2} \|f_j\|_{L^{q_1}(\mathbb{R}^n)}. \end{aligned}$$

Thus, by a similar estimate to E_3, we obtain

$$\begin{aligned} G_3 &\le C\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{j=k+3}^{\infty} 2^{j\alpha}\|f_j\|_{L^{q_1}(\mathbb{R}^n)} 2^{-(j-k)(n/q_2+\alpha)}(j-k)\right)^{p_1}\right\}^{1/p_1} \\ &\le C\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p_1}\|f_k\|_{L^{q_1}(\mathbb{R}^n)}^{p_1}\right\}^{1/p_1} = C\|f\|_{\dot{K}_{q_1}^{\alpha,p_1}(\mathbb{R}^n)}, \end{aligned}$$

since $\alpha > -n/q_2$.

We leave the estimate for G_3 with the case $p_1 = \infty$ to the reader.
This finishes the proof of Theorem 2.6. □

The following corollary is obvious.

COROLLARY 2.4. *Let b, l, q_1, q_2, α, p_1, and p_2 be as in Theorem 2.6. If the linear operator T_l satisfies*

$$|T_l f(x)| \leq C \int \frac{|f(y)|}{|x-y|^{n-l}}\, dy \tag{2.10}$$

for $f \in L^1(\mathbb{R}^n)$ with compact support and $x \notin \operatorname{supp} f$, and if $[b, T_l]$ maps $L^{q_1}(\mathbb{R}^n)$ continuously into $L^{q_2}(\mathbb{R}^n)$, then $[b, T_l]$ maps $\dot{K}_{q_1}^{\alpha,p_1}(\mathbb{R}^n)$ continuously into $\dot{K}_{q_2}^{\alpha,p_2}(\mathbb{R}^n)$.

Obviously, Theorem 1.4 is just a special case of Corollary 2.4.

Note that $L^{q_1}_{|x|^\alpha}(\mathbb{R}^n) = \dot{K}_{q_1}^{\alpha/q_1,q_1}(\mathbb{R}^n)$ and $L^{q_2}_{|x|^\beta}(\mathbb{R}^n) = \dot{K}_{q_2}^{\beta/q_2,q_2}(\mathbb{R}^n)$. From this follows our next corollary.

COROLLARY 2.5. *Let $0 < l < n$, $1 < q_1 < \infty$, $1/q_2 = 1/q_1 - l/n$, $-n + q_1 l < \alpha < n(q_1 - 1)$, and $\beta = q_2\alpha/q_1$. Given a $\mathrm{BMO}(\mathbb{R}^n)$ function b and a linear function T_l, the linear commutator $[b, T_l]$ maps $L^{q_1}_{|x|^\alpha}(\mathbb{R}^n)$ continuously into $L^{q_2}_{|x|^\beta}(\mathbb{R}^n)$ if T_l satisfies one of the following:*

(a) *T_l satisfies* (2.8) *and* (2.9) *in Theorem 2.6, and $[b, T_l]$ maps $L^{q_1}(\mathbb{R}^n)$ continuously into $L^{q_2}(\mathbb{R}^n)$;*
(b) *T_l satisfies* (2.10) *in Corollary 2.4, and $[b, T_l]$ maps $L^{q_1}(\mathbb{R}^n)$ continuously into $L^{q_2}(\mathbb{R}^n)$; or*
(c) *T_l is the standard fractional integral I_l of order l.*

Obviously, Theorem 1.3 is just some special cases of Corollaries 2.2 and 2.5. Moreover, Corollary 2.5 is a generalization of the result in [13] to the linear commutators.

By a careful look at the proof of Theorem 2.6, we easily see that the following theorem holds for the maximal fractional commutator.

THEOREM 2.7. *Let $b \in \mathrm{BMO}(\mathbb{R}^n)$, and let l, q_1, q_2, p_1, p_2, and α be as in Theorem 2.6. Suppose $\{T_l^j\}_{j\in I}$ (I is some index set) are linear operators. Then $T_{l,b}^* f(x) = \sup_{j\in I} |[b, T_l^j] f(x)|$ maps $\dot{K}_{q_1}^{\alpha,p_1}(\mathbb{R}^n)$ continuously into $\dot{K}_{q_2}^{\alpha,p_2}(\mathbb{R}^n)$ if $T_{l,b}^*$ maps $L^{q_1}(\mathbb{R}^n)$ continuously into $L^{q_2}(\mathbb{R}^n)$, and $\{T_l^j\}_{j\in I}$ satisfies one of the following conditions.*

(a) *For $f \in L^1(\mathbb{R}^n)$, $\operatorname{supp} f \subseteq A_k$, and $|x| \geq 2^{k+1}$ with $k \in \mathbb{Z}$,*

$$|T_l^j f(x)| \leq C_0 |x|^{-(n-l)} \|f\|_{L^1(\mathbb{R}^n)}; \tag{2.11}$$

for $f \in L^1(\mathbb{R}^n)$, $\operatorname{supp} f \subseteq A_k$, and $|x| \leq 2^{k-2}$ with $k \in \mathbb{Z}$,

$$|T_l^j f(x)| \leq C_0 2^{-k(n-l)} \|f\|_{L^1(\mathbb{R}^n)}. \tag{2.12}$$

(b) *For* $f \in L^1(\mathbb{R}^n)$ *with compact support and* $x \notin \operatorname{supp} f$,

$$|T_l^j f(x)| \le C_0 \int \frac{|f(y)|}{|x-y|^{n-l}}\, dy. \tag{2.13}$$

Here C_0 *is independent of* j.

COROLLARY 2.6. *Let* b, $\{T_l^j\}_{j\in I}$, *and* $T_{l,b}^*$ *be as in Theorem 2.7. Let* $0 < l < n$, $1 < q_1 < \infty$, $1/q_2 = 1/q_1 - l/n$, $-nq_1/q_2 < \alpha < n(q_1-1)$, *and* $\beta = q_2\alpha/q_1$. *Then the maximal fractional commutator* $T_{l,b}^*$ *maps* $L^{q_1}_{|x|^\alpha}(\mathbb{R}^n)$ *continuously into* $L^{q_2}_{|x|^\beta}(\mathbb{R}^n)$ *if* $T_{l,b}^*$ *maps* $L^{q_1}(\mathbb{R}^n)$ *continuously into* $L^{q_2}(\mathbb{R}^n)$ *and if the* T_j $(j \in I)$ *satisfy one of* (a) *and* (b) *in Theorem 2.7.*

The generalization of Corollary 2.6 is just the case with $0 < l < n$ of Theorem 1.4. Let us prove it now.

Proof of Theorem 1.4 with $0 < l < n$. As in the proof of Theorem 2.5 (see also [21, Thm. 1]), we write $f_{k,0}(x) = f(x)\chi_{\{2^{k-2}<|x|\le 2^{k+1}\}}(x)$ and $f_{k,1}(x) = f(x) - f_{k,0}(x)$. Then

$$\begin{aligned} T_{l,b}^* f(x) &= \sum_{k=-\infty}^{\infty} T_{l,b}^* f(x)\chi_k(x) \\ &\le \sum_{k=-\infty}^{\infty} T_{l,b}^*(f_{k,0})(x)\chi_k(x) + \sum_{k=-\infty}^{\infty} T_{l,b}^*(f_{k,1})(x)\chi_k(x) \\ &\equiv T_0^l f(x) + T_1^l f(x). \end{aligned}$$

If we set $m_k = \inf\{ w(x) : 2^{k-2} < |x| \le 2^{k+1} \}$, then $w(x) \sim m_k$ for $2^{k-2} < |x| \le 2^{k+1}$ by (1.3). Therefore, by the fact that $T_{l,b}^* f$ maps $L^{q_1}(\mathbb{R}^n)$ continuously into $L^{q_2}(\mathbb{R}^n)$, we obtain

$$\begin{aligned} \|T_0^l f\|^{q_2}_{L^{q_2}_{w^{q_2}}(\mathbb{R}^n)} &= \sum_{k=-\infty}^{\infty} \|T_{l,b}^*(f_{k,0})\chi_k\|^{q_2}_{L^{q_2}_{w^{q_2}}(\mathbb{R}^n)} \sim \sum_{k=-\infty}^{\infty} (m_k)^{q_2} \|T_{l,b}^*(f_{k,0})\chi_k\|^{q_2}_{L^{q_2}(\mathbb{R}^n)} \\ &\le C \sum_{k=-\infty}^{\infty} (m_k)^{q_2} \|f_{k,0}\|^{q_2}_{L^{q_1}(\mathbb{R}^n)} \sim C \sum_{k=-\infty}^{\infty} \|f_{k,0}\|^{q_2}_{L^{q_1}_{w^{q_1}}(\mathbb{R}^n)} \\ &\le C\left(\sum_{k=-\infty}^{\infty} \int_{2^{k-2}<|x|\le 2^{k+1}} |f(x)|^{q_1} w(x)^{q_1}\, dx\right)^{q_2/q_1} \\ &\le C\|f\|^{q_2}_{L^{q_1}_{w^{q_1}}(\mathbb{R}^n)}. \end{aligned}$$

To prove the theorem, it suffices to verify that T_1^l maps $L^{q_1}_{w^{q_1}}(\mathbb{R}^n)$ continuously into $L^{q_2}_{w^{q_2}}(\mathbb{R}^n)$. In fact,

$$\begin{aligned} T_1^l f(x) &\le C_0 \sum_{k=-\infty}^{\infty} \left(\int_{\mathbb{R}^n} \frac{|b(x)-b(y)| \cdot |f_{k,1}(y)|}{|x-y|^{n-l}}\, dy \right) \chi_k(x) \\ &\le C|x|^{-(n-l)} \int_{|y|\le|x|} |b(x)-b(y)| \cdot |f(y)|\, dy \end{aligned}$$

$$+C\int_{|x|\le|y|}|b(x)-b(y)|\cdot|f(y)|\cdot|y|^{-(n-l)}\,dy$$
$$\equiv L_l^1 f(x)+L_l^2 f(x).$$

Now, $L_l^1 f(x)\le CM_b^{n/l}f(x)$. Since $w\in A(q_1,q_2)$, by [20, Thm. 3.2, p. 544], we obtain

$$\|L_l^1 f\|_{L^{q_2}_{w^{q_2}}(\mathbb{R}^n)}\le C\|M_b^{n/l}f\|_{L^{q_2}_{w^{q_2}}(\mathbb{R}^n)}\le C\|f\|_{L^{q_1}_{w^{q_1}}(\mathbb{R}^n)}.$$

For $L_l^2 f$, note that $w\in A(q_1,q_2)$. It follows from Definition 1.2 that $w^{-1}\in A(q_2',q_1')$. Thus, by duality,

$$\|L_l^2 f\|_{L^{q_2}_{w^{q_2}}(\mathbb{R}^n)}=\sup_{\|g\|_{L^{q_2'}_{w^{-q_2'}}(\mathbb{R}^n)}\le 1}|(L_l^2 f,g)|=\sup_{\|g\|_{L^{q_2'}_{w^{-q_2'}}(\mathbb{R}^n)}\le 1}|(f,L_l^1 g)|$$
$$\le C\|f\|_{L^{q_1}_{w^{q_1}}(\mathbb{R}^n)}\sup_{\|g\|_{L^{q_2'}_{w^{-q_2'}}(\mathbb{R}^n)}\le 1}\|M_b^{n/l}g\|_{L^{q_1'}_{w^{-q_1'}}(\mathbb{R}^n)}\le C\|f\|_{L^{q_1}_{w^{q_1}}(\mathbb{R}^n)},$$

where we used that $1/q_1'=1/q_2'-l/n$ and Theorem 3.2 in [20].

This finishes the proof of Theorem 1.4 for the case $0<l<n$. □

To end this section, we now prove Theorem 1.5.

Proof of Theorem 1.5. Let $f(x)=\sum_{j=-\infty}^{\infty}f(x)\chi_j(x)\equiv\sum_{j=-\infty}^{\infty}f_j(x)$ and $m=n/(n-lq)$. Then $m>1$. Write

$$\|[b,T_l]f\|_{\dot K_q^{\beta,p}(\mathbb{R}^n)}\le C\left\{\sum_{k=-\infty}^{\infty}2^{k\beta p}\left(\sum_{j\le m(k+1)}\|\chi_k[b,T_l]f_j\|_{L^q(\mathbb{R}^n)}\right)^p\right\}^{1/p}$$
$$+C\left\{\sum_{k=-\infty}^{\infty}2^{k\beta p}\left(\sum_{j>m(k+1)}\|\chi_k[b,T_l]f_j\|_{L^q(\mathbb{R}^n)}\right)^p\right\}^{1/p}$$
$$\equiv \mathrm{H}_1+\mathrm{H}_2.$$

For H_1, by the same argument as for D_1 in the proof of Theorem 2.1 and the hypothesis that $[b,T_l]$ is bounded on $L^q(\mathbb{R}^n)$, we obtain

$$\mathrm{H}_1\le C\left\{\sum_{k=-\infty}^{\infty}2^{k\beta p}\left(\sum_{j\le m(k+1)}\|f_j\|_{L^q(\mathbb{R}^n)}\right)^p\right\}^{1/p}=C\|f\|_{\dot K_q^{\alpha,p}(\mathbb{R}^n)}.$$

For H_2, note that $j>m(k+1)$. Denote by b_{mk} the mean value of b on the ball $B(0,2^{mk})$. For $x\in A_k$, we have

$$|[b,T_l]f_j(x)|\le|b(x)-b_{mk}|\cdot|T_lf_j(x)|+|T_l((b-b_{mk})f_j)(x)|$$
$$\le C|b(x)-b_{mk}|\int_{A_j}\frac{|f(y)|}{|x-y|^{n-l}}\,dy$$
$$+C\int_{A_j}\frac{|b(y)-b_{mk}|\cdot|f(y)|}{|x-y|^{n-l}}\,dy$$

$$\leq C2^{j(l-n/q)}|b(x)-b_{mk}|\cdot\|f_j\|_{L^q(\mathbb{R}^n)}$$
$$+C2^{j(l-n/q)}(j-mk)\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_j\|_{L^q(\mathbb{R}^n)},$$

by (1.1). Thus,

$$\|\chi_k[b,T_l]f_j\|_{L^q(\mathbb{R}^n)}\leq C2^{kn/q+j(l-n/q)}(j-mk)\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f_j\|_{L^q(\mathbb{R}^n)}.$$

Therefore, by the same argument as for D_3 in the proof of Theorem 2.1, we obtain

$$\begin{aligned}\mathrm{H}_2 &\leq C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\left\{\sum_{k=-\infty}^{\infty}2^{k(\beta+n/q)p}\right.\\ &\qquad\left.\times\left(\sum_{j>m(k+1)}\|f_j\|_{L^q(\mathbb{R}^n)}2^{j(l-n/q)}(j-mk)\right)^p\right\}^{1/p}\\ &\leq C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}.\end{aligned}$$

This finishes the proof of Theorem 1.5. □

3. Commutators on Herz-Type Hardy Spaces

In this section we will consider the endpoint cases of our Theorems in Section 2. As we pointed out in Section 1, Theorem 1.2 is no longer true if $\alpha\geq n(1-1/q)$. Instead of this, we have the following.

THEOREM 3.1. *Let $b\in\mathrm{BMO}(\mathbb{R}^n)$ and let T be a standard Calderón–Zygmund operator, that is,*

$$Tf(x)=\mathrm{p.v.}\int_{\mathbb{R}^n}k(x,y)f(y)\,dy,$$

with the kernel $k(x,y)$ satisfying

$$|k(x,y)-k(x,z)|\leq C\frac{|y-z|^{\varepsilon}}{|x-z|^{n+\varepsilon}} \tag{3.1}$$

for some $\varepsilon\in(0,1]$ and $2|y-z|<|x-z|$. If $1<q<\infty$, $n(1-1/q)\leq\alpha<n(1-1/q)+\varepsilon$, and $0<p\leq\infty$, then the commutator $[b,T]$ maps $H\dot{K}_{q,b}^{\alpha,p,0}(\mathbb{R}^n)$ continuously into $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$.

Proof. We first restrict $0<p\leq 1$. Let a_j be a central $(\alpha,q,0;\,b)$-atom supported in $B(0,2^j)$. In this case, it suffices to show $\|[b,T]a_j\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}\leq C$ with C independent of j. Write

$$\begin{aligned}\|[b,T]a_j\|^p_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}&=\sum_{k=-\infty}^{j+2}2^{k\alpha p}\|\chi_k[b,T]a_j\|^p_{L^q(\mathbb{R}^n)}\\ &\quad+\sum_{k=j+3}^{\infty}2^{k\alpha p}\|\chi_k[b,T]a_j\|^p_{L^q(\mathbb{R}^n)}\\ &\equiv \mathrm{J}_1+\mathrm{J}_2.\end{aligned}$$

For J_1, using the $L^q(\mathbb{R}^n)$-boundedness of the commutator $[b, T]$ (see [5]), we obtain

$$J_1 \le C\|a_j\|^p_{L^q(\mathbb{R}^n)}\left(\sum_{k=-\infty}^{j+2} 2^{k\alpha p}\right) \le C,$$

with C independent of j.

To estimate J_2, we first compute $\|\chi_k[b, T]a_j\|_{L^q(\mathbb{R}^n)}$ for $k \ge j+3$. In fact, let b_j be the mean value of b on the ball $B(0, 2^j)$. Note that if $x \in A_k$, $y \in B(0, 2^j)$, and $k \ge j+3$, then $2|y| < |x|$. By the vanishing moments of a_j and Hölder's inequality, we obtain

$$\begin{aligned}
&\|\chi_k[b, T]a_j\|_{L^q(\mathbb{R}^n)}\\
&\quad\le \|(b-b_j)\chi_k Ta_j\|_{L^q(\mathbb{R}^n)} + \|T((b-b_j)a_j)\chi_k\|_{L^q(\mathbb{R}^n)}\\
&\quad\le \left(\int_{A_k} |b(x)-b_j|^q\left|\int (k(x,y)-k(x,0))a_j(y)\,dy\right|^q dx\right)^{1/q}\\
&\qquad+\left\{\int_{A_k}\left(\int |k(x,y)-k(x,0)|\cdot|b(y)-b_j|\cdot|a_j(y)|\,dy\right)^q dx\right\}^{1/q}\\
&\quad\le C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}(k-j)2^{j[\varepsilon+n(1-1/q)-\alpha]-k[n(1-1/q)+\varepsilon]}.
\end{aligned} \tag{3.2}$$

By substituting this estimate into J_2, we obtain

$$J_2 \le C\|b\|^p_{\mathrm{BMO}(\mathbb{R}^n)}\sum_{k=j+3}^{\infty}(k-j)^p 2^{(k-j)[\alpha-n(1-1/q)-\varepsilon]} \le C\|b\|^p_{\mathrm{BMO}(\mathbb{R}^n)},$$

since $\alpha < n(1-1/q)+\varepsilon$. This finishes the proof of Theorem 3.1 for the case of $0 < p \le 1$.

Now let $1 < p \le \infty$ and $f = \sum_{j=-\infty}^{\infty}\lambda_j a_j$, where a_j is a central $(\alpha, q, 0;\ b)$-atom supported on $B(0, 2^j)$. By (3.2), Minkowski's inequality, and Hölder's inequality, we have

$$\begin{aligned}
\|[b,T]f\|_{\dot K^{\alpha,p}_q(\mathbb{R}^n)} &\le \left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-3}|\lambda_j|\cdot\|\chi_k[b,T]a_j\|_{L^q(\mathbb{R}^n)}\right)^p\right\}^{1/p}\\
&\quad+\left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=k-2}^{\infty}|\lambda_j|\cdot\|\chi_k[b,T]a_j\|_{L^q(\mathbb{R}^n)}\right)^p\right\}^{1/p}\\
&\le C\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-3}|\lambda_j|(k-j)2^{(k-j)(\alpha-n(1-1/q)-\varepsilon)}\right)^p\right\}^{1/p}\\
&\quad+C\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{j=k-2}^{\infty}|\lambda_j|2^{(k-j)\alpha}\right)^p\right\}^{1/p}\\
&\le C\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-3}|\lambda_j|^p 2^{(k-j)[\alpha-n(1-1/q)-\varepsilon]p/2}\right)\right.\\
&\qquad\left.\times\left(\sum_{j=-\infty}^{k-3}(k-j)^{p'}2^{(k-j)[\alpha-n(1-1/q)-\varepsilon]p'/2}\right)^{p/p'}\right\}^{1/p}
\end{aligned}$$

$$+C\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{j=k-2}^{\infty}|\lambda_j|^p 2^{(k-j)\alpha p/2}\right)\right.$$
$$\left.\times\left(\sum_{j=k-2}^{\infty}2^{(k-j)\alpha p'/2}\right)^{p/p'}\right\}^{1/p}$$
$$\leq C\left\{\sum_{j=-\infty}^{\infty}|\lambda_j|^p\right\}^{1/p},$$

since $\alpha < n(1-1/q)+\varepsilon$ and $\alpha > 0$.

This finishes the proof of Theorem 3.1. □

REMARK 3.1. From the proof of Theorem 3.1, we easily see that if we replace (3.1) by the local condition

$$|k(x,y)-k(x,0)| \leq C\frac{|y|^\varepsilon}{|x|^{n+\varepsilon}} \tag{3.3}$$

for $\varepsilon \in (0,1]$ and $2|y| < |x|$, then the proof of Theorem 3.1 still works. In particular, local Calderón–Zygmund type operators $\mathrm{LCZO}_q(\mathbb{R}^n)$ (see [24, p. 63] for the definition) satisfy (3.3).

REMARK 3.2. Assuming more regularity on the kernel $k(x,y)$, we can extend Theorem 3.1 to a larger range of α and s; see also Theorem 3.2.

For the endpoint cases of Theorem 1.2, we have a similar conclusion to one of Theorem 1.1.

THEOREM 3.2. *Let* $1 < q_1 < \infty$, $\alpha \geq n(1-1/q_1)$, $0 < l < n$, $1/q_2 = 1/q_1 - l/n$, $0 < p_1 \leq p_2 \leq \infty$, *and*

$$I_l f(x) = \text{p.v.}\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-l}}\,dy.$$

If $b \in \mathrm{BMO}(\mathbb{R}^n)$ *and* $s+1 > \alpha + n(1/q_1 - 1)$, *then* $[b, I_l]$ *maps* $H\dot{K}_{q_1,b}^{\alpha,p_1,s}(\mathbb{R}^n)$ *continuously into* $\dot{K}_{q_2}^{\alpha,p_2}(\mathbb{R}^n)$.

Proof. Note that (2.1) holds (see the proof of Theorem 2.2). Without loss of generality, we may assume that $p_1 = p_2$. We only prove the theorem for $0 < p_1 \leq 1$; for $1 < p_1 \leq \infty$, the proof is similar to that of Theorem 3.1. Given our restriction on p_1, it suffices to prove that if a_j is a central $(\alpha, q, s;\ b)$-atom supported on $B(0, 2^j)$ then

$$\|[b, I_l]a_j\|_{\dot{K}_{q_2}^{\alpha,p_1}(\mathbb{R}^n)} \leq C,$$

with C independent of j. Toward this end, write

$$\|[b, I_l]a_j\|^{p_1}_{\dot{K}_{q_2}^{\alpha,p_1}(\mathbb{R}^n)} = \sum_{k=-\infty}^{j+2}2^{k\alpha p_1}\|\chi_k[b, I_l]a_j\|^{p_1}_{L^{q_2}(\mathbb{R}^n)}$$
$$+\sum_{k=j+3}^{\infty}2^{k\alpha p_1}\|\chi_k[b, I_l]a_j\|^{p_1}_{L^{q_2}(\mathbb{R}^n)}$$
$$\equiv \mathrm{Q}_1 + \mathrm{Q}_2.$$

For Q_1, using that $[b, I_l]$ maps $L^{q_1}(\mathbb{R}^n)$ continuously into $L^{q_2}(\mathbb{R}^n)$ (see [3, p. 8]), we have

$$Q_1 \le C \sum_{k=-\infty}^{j+2} 2^{k\alpha p_1} \|a_j\|_{L^{q_1}(\mathbb{R}^n)}^{p_1} \le C \sum_{k=-\infty}^{j+2} 2^{(k-j)\alpha p_1} \le C$$

with C independent of j, since $\alpha \ge n(1 - 1/q_1) > 0$.

For Q_2, we first estimate $|[b, I_l]a_j(x)|$ when $x \in A_k$ and $k \ge j+3$. In fact, if we denote by b_j the mean value of b on the ball $B(0, 2^j)$, we then have

$$\begin{aligned}
|[b, I_l]a_j(x)| &= \left| \int_{\mathbb{R}^n} \frac{b(x) - b(y)}{|x-y|^{n-l}} a_j(y)\, dy \right| \\
&\le |b(x) - b_j| \cdot \left| \int_{\mathbb{R}^n} \frac{a_j(y)}{|x-y|^{n-l}}\, dy \right| + \left| \int_{\mathbb{R}^n} \frac{b_j - b(y)}{|x-y|^{n-l}} a_j(y)\, dy \right|.
\end{aligned}$$

Using the s-order vanishing moments of a_j and the s-order Taylor expansion of $|x-y|^{-n+l}$ at x, we obtain

$$\begin{aligned}
|[b, I_l]a_j(x)| \le{}& C|b(x) - b_j| \int_{\mathbb{R}^n} \frac{|y|^{s+1}}{|x - \theta_1 y|^{n-l+s+1}} |a_j(y)|\, dy \\
&+ C \int_{\mathbb{R}^n} \frac{|y|^{s+1} |b(y) - b_j| \cdot |a_j(y)|}{|x - \theta_2 y|^{n-l+s+1}}\, dy,
\end{aligned}$$

where $\theta_1, \theta_2 \in (0,1)$. Note that $|x - \theta_i y| \ge |x| - |y| > |x|/2$ for $i = 1$ or 2. We deduce that

$$\begin{aligned}
|[b, I_l]a_j(x)| &\le C|b(x) - b_j| \frac{2^{j(s+1)+jn(1-1/q_1)-j\alpha}}{|x|^{n-l+s+1}} \\
&\quad + C \frac{2^{j(s+1)-j\alpha}}{|x|^{n-l+s+1}} \left(\int_{B(0,2^j)} |b_j - b(y)|^{q_1'}\, dy \right)^{1/q_1'} \\
&\le C|b(x) - b_j| \frac{2^{j[s+1+n(1-1/q_1)-\alpha]}}{|x|^{n-l+s+1}} \\
&\quad + C \frac{2^{j[s+1+n(1-1/q_1)-\alpha]}}{|x|^{n-l+s+1}} \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}.
\end{aligned}$$

Therefore, for $k \ge j+3$, we have

$$\begin{aligned}
\|\chi_k [b, I_l] a_j\|_{L^{q_2}(\mathbb{R}^n)} &\le C \frac{2^{j[s+1+n(1-1/q_1)-\alpha]}}{2^{k(n-l+s+1)}} \left(\int_{A_k} |b(x) - b_j|^{q_2}\, dx \right)^{1/q_2} \\
&\quad + C \|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \frac{2^{j[s+1+n(1-1/q_1)-\alpha]}}{2^{k(n-l+s+1-n/q_2)}} \\
&\le C \|b\|_{\mathrm{BMO}(\mathbb{R}^n)} (k-j) 2^{(j-k)[s+1+n(1-1/q_1)-\alpha]-k\alpha},
\end{aligned}$$

where we used that $1/q_2 = 1/q_1 - l/n$.

Thus, noting that $s+1 > \alpha + n(1/q_1 - 1)$, we deduce

$$Q_2 \le C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \sum_{k=j+3}^{\infty} 2^{(j-k)[s+1+n(1-1/q_1)-\alpha]}(k-j) \le C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}$$

with C independent of j.

This finishes the proof of Theorem 3.2. □

Let $b \in \mathrm{BMO}(\mathbb{R}^n)$. We define the maximal operator $B^{\delta}_{*,b}$ associated with the commutator of the Bochner–Riesz operator by

$$B^{\delta}_{*,b} f(x) = \sup_{r>0} |[b, B^{\delta}_r] f(x)|,$$

where $(B^{\delta}_r f)\hat{}(\xi) = (1 - r^2|\xi|^2)^{\delta}_{+} \hat{f}(\xi)$. We also set

$$B^{\delta}_{*} f(x) = \sup_{r>0} |B^{\delta}_r f(x)|.$$

THEOREM 3.3. *Let* $1 < q < \infty$, $\alpha \ge n(1-1/q)$, $\delta > \alpha + n/q - (n+1)/2$, *and* $0 < p \le \infty$. *If* $b \in \mathrm{BMO}(\mathbb{R}^n)$ *and* $s+1 > \alpha + n(1/q - 1)$, *then* $B^{\delta}_{*,b}$ *maps* $H\dot{K}^{\alpha,p,s}_{q,b}(\mathbb{R}^n)$ *continuously into* $\dot{K}^{\alpha,p}_{q}(\mathbb{R}^n)$.

Proof. Let us first suppose that $0 < p \le 1$. In this case, it is enough to show that for central $(\alpha, q, s;\ b)$-atom a_j supported in $B(0, 2^j)$,

$$\|B^{\delta}_{*,b} a_j\|_{\dot{K}^{\alpha,p}_{q}(\mathbb{R}^n)} \le C$$

with C independent of j. To do so, we write

$$\begin{aligned}\|B^{\delta}_{*,b} a_j\|^p_{\dot{K}^{\alpha,p}_{q}(\mathbb{R}^n)} &= \sum_{k=-\infty}^{j+1} 2^{k\alpha p}\|\chi_k B^{\delta}_{*,b} a_j\|^p_{L^q(\mathbb{R}^n)} + \sum_{k=j+2}^{\infty} 2^{k\alpha p}\|\chi_k B^{\delta}_{*,b} a_j\|^p_{L^q(\mathbb{R}^n)}\\ &\equiv \mathrm{N}_1 + \mathrm{N}_2.\end{aligned}$$

Theorem 2.13 in [2] now tells us that when $\alpha \ge n(1-1/q)$ and $\delta > \alpha + n/q - (n+1)/2 \ge (n+1)/2$,

$$\|B^{\delta}_{*,b} a_j\|_{L^q(\mathbb{R}^n)} \le C\|a_j\|_{L^q(\mathbb{R}^n)}.$$

Therefore,

$$\mathrm{N}_1 \le C \sum_{k=-\infty}^{j+1} 2^{k\alpha p}\|a_j\|^p_{L^q(\mathbb{R}^n)} \le C \sum_{k=-\infty}^{j+1} 2^{(k-j)\alpha p} \le C$$

with C independent of j.

Now let $k \ge j+2$ and let b_j be the mean value of b on the ball $B(0, 2^j)$. Then

$$|B^{\delta}_{*,b} a_j(x)| \le |b(x) - b_j| \cdot |B^{\delta}_{*} a_j(x)| + B^{\delta}_{*}((b - b_j)a_j)(x).$$

For $x \in A_k$, we claim

$$B^{\delta}_{*} a_j(x) \le C \frac{2^{j[\delta_0 - (n-1)/2 - \alpha + n(1-1/q)]}}{|x|^{\delta_0 + (n+1)/2}} \tag{3.4}$$

and

$$B_*^\delta((b-b_j)a_j)(x) \le C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \frac{2^{j[\delta_0-(n-1)/2-\alpha+n(1-1/q)]}}{|x|^{\delta_0+(n+1)/2}}, \tag{3.5}$$

where $\max\{(n-1)/2, \alpha+n/q-(n+1)/2\} < \delta_0 < \min\{\delta, (n+1)/2+s\}$. The proof of (3.4) is similar to that of (3.5). We prove only (3.5). Toward this end, we choose $q_1, q_2 \in (1, \infty)$ such that $1/q + 1/q_1 + 1/q_2 = 1$. Note that

$$\{(b-b_j)a_j\} * B_r^\delta(x) = r^{-n} \int_{B(0,2^j)} a_j(y)(b(y)-b_j)B^\delta((x-y)/r)\,dy.$$

We consider the following two cases.

Case I: $0 < r \le 2^j$. In this case, note that $|B^\delta(z)| \le C(1+|z|)^{-(\delta+(n+1)/2)}$ and $|x| > 2|y|$. From Hölder's inequality, it follows that

$$\begin{aligned}
|\{a_j(b-b_j)\} * B_r^\delta(x)| &\le r^{-n}\|a_j\|_{L^q(\mathbb{R}^n)}\|(b-b_j)\chi_{B(0,2^j)}\|_{L^{q_1}(\mathbb{R}^n)} \\
&\quad \times \left(\int_{B(0,2^j)} \left|B^\delta\left(\frac{x-y}{r}\right)\right|^{q_2} dy\right)^{1/q_2} \\
&\le Cr^{-n}2^{-j\alpha+jn/q_1}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \\
&\quad \times \left(\int_{B(0,2^j)} \frac{1}{(1+|x-y|/r)^{[\delta+(n+1)/2]q_2}}\,dy\right)^{1/q_2} \\
&\le C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \frac{2^{j(\delta_0-(n-1)/2-\alpha+n(1-1/q))}}{|x|^{\delta_0+(n+1)/2}}
\end{aligned}$$

with C independent of j and r.

Case II: $r > 2^j$. In this case, by the vanishing moments of a_j and s-order Taylor expansion of $B^\delta[(x-y)/r]$ at x/r, we obtain

$$\begin{aligned}
&|\{a_j(b-b_j)\} * B_r^\delta(x)| \\
&\quad \le Cr^{-n-s-1}\int_{|y|\le 2^j} |b(y)-b_j(y)| \cdot |a_j(y)| \cdot \left|\nabla^\beta B^\delta\left(\frac{x-\theta y}{r}\right)\right| \cdot |y|^{s+1}\,dy,
\end{aligned}$$

where $\beta = (\beta_1, \beta_2, \dots, \beta_n)$, $\beta_1+\beta_2+\cdots+\beta_n = s+1$, $\beta_i \in \mathbb{N}\cup\{0\}$, $\theta \in (0,1)$, and

$$\nabla^\beta = \left(\frac{\partial}{\partial x_1}\right)^{\beta_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\beta_n}.$$

Using $|\nabla^\beta B^\delta(z)| \le C(1+|z|)^{-(\delta+(n+1)/2)}$ for any $\beta \in (\mathbb{N}\cup\{0\})^n$ and Hölder's inequality, we obtain

$$\begin{aligned}
|\{a_j(b-b_j)\} * B_r^\delta(x)| &\le Cr^{-n-s-1}2^{j(s+1)}\|a_j\|_{L^q(\mathbb{R}^n)}\|(b-b_j)\chi_{B(0,2^j)}\|_{L^{q_1}(\mathbb{R}^n)} \\
&\quad \times \left(\int_{|y|\le 2^j} \frac{dy}{(1+|x-\theta y|/r)^{(\delta+(n+1)/2)q_2}}\right)^{1/q_2} \\
&\le C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \frac{2^{j(\delta_0-(n-1)/2-\alpha+n(1-1/q))}}{|x|^{\delta_0+(n+1)/2}}
\end{aligned}$$

with C independent of j and r.

Combining Case I with Case II, we obtain (3.5).

Therefore, for $k \geq j+3$, we have

$$\|\chi_k B^{\delta}_{*,b} a_j\|_{L^q(\mathbb{R}^n)} \leq C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}(k-j)\frac{2^{j(\delta_0-(n-1)/2-\alpha+n(1-1/q))}}{2^{k(\delta_0+(n+1)/2-n/q)}}. \tag{3.6}$$

Thus,

$$\mathrm{N}_2 \leq C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\sum_{k=j+2}^{\infty}(k-j)2^{(j-k)(\delta_0+(n+1)/2-\alpha-n/q)} \leq C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}$$

with C independent of j.

This finishes the proof of the theorem for the case $0 < p \leq 1$. Using (3.6) and Minkowski's inequality, we can finish the proof of the theorem for the case $1 < p \leq \infty$ in a similar way to the proof of Theorem 3.1. We omit the details here.

This finishes the proof of Theorem 3.3. □

REMARK 3.3. In Theorem 3.3, if $\alpha = n(1/p - 1/q)$ and $0 < p \leq 1$ then $\alpha + n/q - (n+1)/2 = n/p - (n+1)/2$, which is called the critical index (see [12, Chap. 3]). In general, $\delta > \alpha + n/q - (n+1)/2$ cannot be removed.

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