Minimal Norm Interpolation with Nonnegative Real Part on Multiply Connected Planar Domains

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1. Introduction

Let Ω be a domain in the plane whose boundary is composed of a finite number of disjoint smooth simple closed curves. The space $H^2(\Omega)$ consists of those analytic functions f on Ω for which the subharmonic function $|f(z)|^2$ has a harmonic majorant. $K(\Omega)$ is the convex cone of those elements in $H^2(\Omega)$ whose real part is nonnegative on Ω.

In this paper we describe the projection of $H^2(\Omega)$ onto $K(\Omega)$ and also describe the unique element of $K(\Omega)$ of minimal norm satisfying a finite number of interpolation conditions:

$$\min\{\, \|f\|_{H^2(\Omega)} : f \in K(\Omega) \text{ and } f(z_j) = w_j,\ j = 1, 2, \dots, n\,\}, \tag{1.1}$$

assuming, of course, that there is at least one element of $K(\Omega)$ satisfying these conditions.

A problem similar to (1.1) was solved by Sarason [11] for the space $H^\infty(\Delta)$ where Δ is the open unit disc. He proved that the minimal norm interpolant is rational. As explained in [3], this result has importance in signal processing. The $H^2(\Omega)$ version of the problem studied here was suggested to us by J. D. Ward, to whom we express our appreciation.

The plan of the paper is this. In Sections 2 and 3 we give a description of the projection of $H^2(\Omega)$ onto $K(\Omega)$ in the case when Ω is a finitely connected domain. The special case when $\Omega = \Delta$ is examined separately in Section 2 because of its importance and simplicity. In Section 4 we show how this knowledge, combined with a result from [9], leads us to the the solution of problem (1.1) when $\Omega = \Delta$. In Section 5 we give some results similar to those of Section 3 but for finitely connected domains, where additional attention is given to the special case of an annulus.

2. The Projection onto Functions with Nonnegative Real Part: The Unit Disc

Let $\Delta = \{\, z : |z| < 1 \,\}$ be the open unit disc, T the unit circle, and $dm = \frac{1}{2\pi}\, d\theta$ Lebesgue measure on T normalized so that $\int_T dm = 1$. Also, we use $\|\cdot\|_2$ to denote the L^2 norm relative to this measure.

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THEOREM 2.1. *Let $K(\Delta) = \{g : g \in H^2(\Delta), \operatorname{Re} g(z) \geq 0,\ z \in \Delta\}$. Then the projection of $H^2(\Delta)$ onto $K(\Delta)$ is given by the formula*

$$(P_{K(\Delta)}f)(z) = i \operatorname{Im} f(0) + \frac{1}{2\pi} \int_T \frac{e^{i\theta} + z}{e^{i\theta} - z} (\operatorname{Re} f(e^{i\theta}) - \lambda)_+ \, d\theta, \quad f \in H^2(\Delta), \tag{2.1}$$

where λ is the unique nonpositive number satisfying the equation

$$\lambda(1 + m\{\operatorname{Re} f < \lambda\}) = \int_{\operatorname{Re} f < \lambda} \operatorname{Re} f \, dm. \tag{2.2}$$

Proof. For each $f \in H^2(\Delta)$, observe that equation (2.2) uniquely determines some $\lambda \leq 0$ because the function

$$u(t) = t(1 + m\{\operatorname{Re} f < t\}) - \int_{\operatorname{Re} f < t} \operatorname{Re} f \, dm = t + \int_{\operatorname{Re} f < t} (t - \operatorname{Re} f) \, dm$$

is a strictly increasing function of t, with $\lim_{t \to -\infty} u(t) = -\infty$ and $u(0) \geq 0$.

Next, we note that for any $f \in H^2(\Delta)$ we have the relationship

$$\|f\|^2_{H^2(\Delta)} = 2\|\operatorname{Re} f\|_2^2 - \operatorname{Re}(f(0))^2 \tag{2.3}$$

and consequently for $f, g \in H^2(\Delta)$ we have the formula

$$\operatorname{Re}(f, g)_{H^2(\Delta)} = 2(\operatorname{Re} f, \operatorname{Re} g)_2 - \operatorname{Re}(f(0)g(0)). \tag{2.4}$$

To establish (2.1) we must show that

$$\operatorname{Re}(f - P_{K(\Delta)}f, g)_{H^2(\Delta)} \leq 0, \quad g \in K(\Delta), \tag{2.5}$$

with equality when $g = P_{K(\Delta)}f$.

We begin by noting that equation (2.1) implies that

$$\operatorname{Re} f(e^{i\theta}) - \operatorname{Re}(P_{K(\Delta)}f)(e^{i\theta}) = \begin{cases} \lambda & \text{when } \operatorname{Re} f(e^{i\theta}) \geq \lambda, \\ \operatorname{Re} f(e^{i\theta}) & \text{when } \operatorname{Re} f(e^{i\theta}) < \lambda. \end{cases}$$

Recall that, for every $f \in H^2(\Delta)$, we have

$$\operatorname{Re} f(0) = \int_T \operatorname{Re} f \, dm.$$

Hence, using equation (2.4) we have

$$\begin{aligned}
&\operatorname{Re}(f - P_{K(\Delta)}f, g)_{H^2(\Delta)} \\
&\quad = 2\int_T (\operatorname{Re} f - \operatorname{Re} P_{K(\Delta)}f) \operatorname{Re} g \, dm - \operatorname{Re}((f(0) - (P_{K(\Delta)}f)(0)) g(0)) \\
&\quad = 2\lambda \int_{\operatorname{Re} f \geq \lambda} \operatorname{Re} g \, dm + 2\int_{\operatorname{Re} f < \lambda} \operatorname{Re} f \operatorname{Re} g \, dm \\
&\qquad - \operatorname{Re} g(0) \left\{ \lambda \int_{\operatorname{Re} f \geq \lambda} dm + \int_{\operatorname{Re} f < \lambda} \operatorname{Re} f \, dm \right\},
\end{aligned}$$

which by definition (2.2) of λ

$$= 2\lambda \int_{\operatorname{Re} f \geq \lambda} \operatorname{Re} g \, dm + 2 \int_{\operatorname{Re} f < \lambda} \operatorname{Re} f \operatorname{Re} g \, dm - 2\lambda \int_T \operatorname{Re} g \, dm$$

$$= 2 \int_{\operatorname{Re} f < \lambda} (\operatorname{Re} g)(\operatorname{Re} f - \lambda) \, dm.$$

This last quantity is nonpositive for $g \in K(\Delta)$ and zero for $g = P_{K(\Delta)} f$. □

3. The Projection onto Functions with Nonnegative Real Part: Finitely Connected Domains

In this section we extend Theorem 2.1 in two ways. First we study the case of multiply connected domains Ω, and then we consider more general inequality constraints on the real part of f. Let Ω be a bounded domain in the complex plane whose boundary Γ consists of $m + 1$ disjoint analytic simple closed curves. Fix a point $t_0 \in \Omega$ and let μ be harmonic measure on Γ corresponding to the point t_0.

For $1 \leq p < \infty$ we let $H^p(\Omega)$ consist of those analytic functions f on Ω for which the subharmonic function $|f(z)|^p$ has a harmonic majorant on Ω. If f is in $H^p(\Omega)$, then $|f|^p$ has a unique least harmonic majorant u_f on Ω. If we set

$$\|f\| = \|f\|_{H^p(\Omega)} = (u_f(t_0))^{1/p} \tag{3.1}$$

then $\|\cdot\|$ is a norm on $H^p(\Omega)$.

It is also true that each $f \in H^p(\Omega)$ has boundary values a.e. μ on Γ that lie in $L^p(\Gamma, \mu)$ and

$$\|f\|_{H^p(\Omega)} = \|f\|_{L^p(\Gamma,\mu)}. \tag{3.2}$$

$H^\infty(\Omega)$ is the space of bounded analytic functions on Ω with the sup norm.

Since μ is multiplicative on $H^2(\Omega)$, we have

$$\int_\Gamma f^2 \, d\mu = \left(\int_\Gamma f \, d\mu \right)^2, \quad f \in H^2(\Omega)$$

or

$$\int_\Gamma [(\operatorname{Re} f)^2 - (\operatorname{Im} f)^2] \, d\mu = \left(\int_\Gamma \operatorname{Re} f \, d\mu \right)^2 - \left(\int_\Gamma \operatorname{Im} f \, d\mu \right)^2.$$

This equation yields the formula

$$\int_\Gamma [(\operatorname{Re} f)^2 + (\operatorname{Im} f)^2] \, d\mu$$
$$= 2 \int_\Gamma (\operatorname{Re} f)^2 \, d\mu - \left(\int_\Gamma \operatorname{Re} f \, d\mu \right)^2 + \left(\int_\Gamma \operatorname{Im} f \, d\mu \right)^2$$

or, in other words,

$$\int_\Gamma |f|^2 \, d\mu = 2 \int_\Gamma (\operatorname{Re} f)^2 \, d\mu - \operatorname{Re}(f(t_0))^2. \tag{3.3}$$

Consequently, for every $f, g \in H^2(\Omega)$ we find that

$$\operatorname{Re}(f, g)_{H^2(\Omega)} = 2 \int_\Gamma (\operatorname{Re} f)(\operatorname{Re} g) \, d\mu - \operatorname{Re}(f(t_0) g(t_0)). \tag{3.4}$$

Let $K(\Omega)$ be the convex cone of those functions in $H^2(\Omega)$ whose real part is nonnegative on Ω. Fix $f \in H^2(\Omega)$, $f \notin K(\Omega)$, and let F be the unique element of $K(\Omega)$ that is nearest f in $H^2(\Omega)$. Then

$$\operatorname{Re}(f - F, g)_{H^2(\Omega)} \leq 0, \quad g \in K(\Omega) \tag{3.5}$$

and

$$\operatorname{Re}(f - F, F)_{H^2(\Omega)} = 0. \tag{3.6}$$

From (3.4) we may write

$$\begin{aligned}\operatorname{Re}(f - F, g)_{H^2(\Omega)} &= 2\int_\Gamma \operatorname{Re}(f - F)\operatorname{Re} g\, d\mu - \operatorname{Re}((f(t_0) - F(t_0))g(t_0)) \\ &= 2\int_\Gamma \operatorname{Re}(f - F)\operatorname{Re} g\, d\mu - \operatorname{Re}(f(t_0) - F(t_0))\operatorname{Re} g(t_0) \\ &\quad + \operatorname{Im}(f(t_0) - F(t_0))\operatorname{Im} g(t_0).\end{aligned}$$

Since $\operatorname{Im} g(t_0)$ can be any number, we evidently must have from inequality (3.5) that

$$\operatorname{Im} F(t_0) = \operatorname{Im} f(t_0).$$

Write $U := 2\operatorname{Re}(f - F)$ and $c := \operatorname{Re}(f(t_0) - F(t_0)) = \frac{1}{2}\int_\Gamma U\, d\mu$. Then (3.5) and (3.6) take the respective forms

$$\int_\Gamma (U - c)\operatorname{Re} g\, d\mu \leq 0, \quad g \in K(\Omega) \tag{3.7}$$

and

$$\int_\Gamma (U - c)\operatorname{Re} F\, d\mu = 0. \tag{3.8}$$

Note that the choice $g = 1$ in (3.7) gives $c \leq 0$. Moreover, replacing g by $\lambda h + \|h\|_\infty$, where $|\lambda| = 1$ and h is any function in $H^\infty(\Omega)$, yields the inequality

$$\left|\int_\Gamma (U - c)h\, d\mu\right| \leq |c|\|h\|_\infty. \tag{3.9}$$

That is, the function $U - c$ gives a bounded linear functional on $H^\infty(\Omega)$ of norm at most $|c|$. To make use of this observation we recall that the subspace of $L^1(\Gamma, \mu)$ of functions orthogonal to $H^\infty(\Omega)$ has a special form. Namely, $u \in L^1(\Gamma, \mu)$ is orthogonal to $H^\infty(\Omega)$ if and only if $u = g + w$, where $g \in H_0^1(\Omega) := \{v : v \in H^1(\Omega), v(t_0) = 0\}$ and w lies in an m-dimensional subspace N of $C(\Gamma)$. In fact, N is spanned by m *real-valued*, smooth functions (cf. [4, Thm. 4.2.3 and Sec. 4.5]). The functions in N that are real-valued on Γ are known as *Schottky functions*; see [5]. Moreover, we know that whenever a sequence $\{g_k\}_{k\in\mathbb{Z}_+}$ lies in $H_0^1(\Omega)$ and another $\{w_k\}_{k\in\mathbb{Z}_+}$ in N with $\{g_k + w_k\}_{k\in\mathbb{Z}_+}$ bounded in $L^1(\Gamma, \mu)$, then there is a subsequence $\{g_l + w_l\}_{l\in\mathbb{Z}_+}$, and functions $g \in H_0^1(\Omega)$ and $w \in N$ such that $(g_l + w_l)d\mu \to (g + w)d\mu$ weak-$*$ as measures on $C(\Gamma)$

Using these facts, we conclude that there is a $g \in H_0^1(\Omega)$ and a $w \in N$ such that

$$\begin{aligned}\int_\Gamma |U-c+g+w|\,d\mu \\ &= \inf\left\{\int_\Gamma |U-c-k|\,d\mu : k\in L^1(\Gamma,\mu),\ \int_\Gamma kv\,d\mu = 0,\ v\in H^\infty(\Omega)\right\} \\ &= \sup\left\{\left|\int_\Gamma (U-c)h\,d\mu\right| : h\in H^\infty(\Omega),\ \|h\|_\infty\le 1\right\} \le |c| = -c.\end{aligned}$$

Hence

$$|c| \ge \int_\Gamma |U-c+g+w|\,d\mu \ge \int_\Gamma (-U+c-g-w)\,d\mu = -c = |c|$$

and we conclude that

$$U-c+g+w\le 0 \quad \text{a.e. } \mu \text{ on } \Gamma.$$

In particular, $U-c+g+w$ is real a.e. μ on Γ so that

$$U-c+g+w \equiv U-c+\operatorname{Re} g+\operatorname{Re} w \quad \text{a.e. } \mu \text{ on } \Gamma;$$

hence

$$\operatorname{Im} g = -\operatorname{Im} w \quad \text{a.e. } \mu \text{ on } \Gamma.$$

But $\operatorname{Im} w\in N$, since N has a basis of real functions. Thus, $\operatorname{Im} g\in N$ and so is orthogonal to $H^1(\Omega)$:

$$0 = \int_\Gamma (\operatorname{Im} g)g\,d\mu = \int_\Gamma (\operatorname{Re} g)(\operatorname{Im} g)\,d\mu + i\int_\Gamma (\operatorname{Im} g)^2\,d\mu.$$

This implies that $\operatorname{Im} g = 0$ and so g is constant. Since $g(t_0)=0$, we have $g=0$.

So far we have proved that

$$U-c+w\le 0 \quad \text{a.e. } \mu \text{ on } \Gamma, \tag{3.10}$$

which implies that

$$\operatorname{Re} F \le \operatorname{Re} f - \tfrac{1}{2}c + \tfrac{1}{2}w. \tag{3.11}$$

However, from (3.6) we also have

$$\begin{aligned}0 &= \operatorname{Re}(f-F,F)_{H^2(\Omega)} \\ &= \int_\Gamma (U-c+w)\operatorname{Re} F\,d\mu \le 0,\end{aligned}$$

which implies that

$$(U-c+w)\operatorname{Re} F = 0 \quad \text{a.e. } \mu \text{ on } \Gamma. \tag{3.12}$$

Therefore, a.e. μ whenever $\zeta\in\Gamma$ is such that $\operatorname{Re} F(\zeta)>0$ it follows that $2\operatorname{Re}(f-F)(\zeta)-c+w=0$. Equivalently, $\operatorname{Re} F(\zeta)>0$ implies that $\operatorname{Re} F(\zeta) = \operatorname{Re} f(\zeta)-\frac{c}{2}+\frac{1}{2}w(\zeta)$. Also, from (3.11) we see that $\operatorname{Re} F(\zeta)\le 0$ when $\operatorname{Re} f(\zeta)-\frac{c}{2}+\frac{1}{2}w(\zeta)\le 0$ a.e. μ on Γ. Consequently, we have established that

$$\operatorname{Re} F = (\operatorname{Re} f - \tfrac{1}{2}c + \tfrac{1}{2}w)_+ \quad \text{a.e. } \mu \text{ on } \Gamma. \qquad \square$$

We summarize this information in the next theorem.

THEOREM 3.1. *Suppose the boundary of Ω consists of a finite number of disjoint smooth simple closed curves. Let $K(\Omega)$ be the convex cone of functions in $H^2(\Omega)$ with positive real part on Ω. If $f \in H^2(\Omega)$ but is not in $K(\Omega)$, and if F is the best approximation to f in $H^2(\Omega)$ from $K(\Omega)$, then there is a real number c and a function $w \in N$ such that*

$$\operatorname{Re} F = (\operatorname{Re} f - c - w)_+ \quad \text{a.e. } \mu \text{ on } \Gamma. \tag{3.13}$$

As an example of this result, we consider the case where Ω is the annulus

$$\{z : R^{-1} < |z| < R\},$$

where R is some real constant greater than unity. For the point $t_0 = 1$, the harmonic measure $d\mu$ is given by the formula

$$d\mu = \begin{cases} \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \frac{R^n}{R^{2n}+1} e^{-in\theta}\, d\theta, & z = Re^{i\theta}, \\ \frac{1}{2\pi} \sum_{-\infty}^{\infty} \frac{R^n}{R^{2n}+1} e^{-in\theta}\, d\theta, & z = R^{-1}e^{i\theta}, \end{cases}$$

and N is the one-dimensional space spanned by the function defined by the equation

$$v\, d\mu = \begin{cases} d\theta & \text{if } z = Re^{i\theta}, \\ -d\theta & \text{if } z = R^{-1}e^{i\theta}; \end{cases}$$

see [1]. Hence the projection onto $K(\Omega)$ is given by

$$\operatorname{Re} F = (\operatorname{Re} f + av + b)_+,$$

where a and b are scalars such that

$$\int_\Gamma (\operatorname{Re} F) v\, d\mu = 0$$

and

$$2 \int_\Gamma (\operatorname{Re} F)(\operatorname{Re}(f - F))\, d\mu + (\operatorname{Re} F(1))^2 = 0.$$

As a corollary of this result, we note the following fact.

PROPOSITION 3.1. *Let Ω be finitely connected as above. If $f \in H^2(\Omega)$ but $f \notin K(\Omega)$, and if F is the best approximation to f from $K(\Omega)$, then* Re F *must vanish on some set of positive measure on* Γ.

Proof. Suppose to the contrary that Re $F > 0$ a.e. μ on Γ. It therefore follows by Theorem 3.1 that

$$\operatorname{Re} F = \operatorname{Re} f - \tfrac{1}{2}c + \tfrac{1}{2}w \quad \text{a.e. } \mu \text{ on } \Gamma.$$

This implies that $w \in \operatorname{Re} H^2(\Omega)$. But w is also orthogonal to $\operatorname{Re} H^2(\Omega)$ and so $w = 0$, that is, $\operatorname{Re}(f - F) = \frac{1}{2}c$. Hence,

$$\begin{aligned} 0 &= \operatorname{Re}(f - F, F)_{H^2(\Omega)} \\ &= 2 \int_\Gamma \operatorname{Re}(f - F) \operatorname{Re} F\, d\mu - \operatorname{Re}((f(t_0) - F(t_0)) F(t_0)) \\ &= 2 \int_\Gamma \frac{1}{2} c \operatorname{Re} F\, d\mu - \frac{1}{2} c \operatorname{Re} F(t_0) \\ &= \frac{1}{2} c \operatorname{Re} F(t_0). \end{aligned}$$

Since $\operatorname{Re} F(t_0) > 0$, we must have $c = 0$; that is, $\operatorname{Re} F = \operatorname{Re} f$. This implies that $f \in K(\Omega)$, contradicting our original hypothesis. □

We conclude this section with an extension of Theorem 3.1. To begin, we let u_1 and u_2 be two real valued functions in $L^2_{\mathbb{R}}(\Gamma, \mu)$ such that $u_1 < u_2$ a.e. μ on Γ. With these functions we consider the convex subset of $H^2(\Omega)$ defined by

$$K(\Omega; u_1, u_2) := \{ h \in H^2(\Omega) : u_1 \le \operatorname{Re} h \le u_2 \text{ a.e. } \mu \text{ on } \Gamma \}.$$

Under the condition that $K(\Omega; u_1, u_2) \neq \phi$ we shall describe the orthogonal projection of $H^2(\Omega)$ onto $K(\Omega; u_1, u_2)$. Toward this end, we introduce the following truncation operator $T_{u_1,u_2}: L^2_{\mathbb{R}}(\Gamma; \mu) \to L^2_{\mathbb{R}}(\Gamma, \mu)$ defined for $g \in L^2_{\mathbb{R}}(\Gamma; \mu)$ by

$$T_{u_1,u_2} g = \begin{cases} u_1 & \text{if } g \le u_1, \\ g & \text{if } u_1 \le g \le u_2, \\ u_2 & \text{if } g \ge u_2. \end{cases}$$

Of course, if $u_1 = 0$ and $u_2 = \infty$ then $T_{u_1,u_2} g = g_+$, since g is finite a.e. μ on Γ.

THEOREM 3.2. *Suppose u_1 and u_2 are two real-valued functions in $L^2(\Gamma, \mu)$. Let the boundary of Ω consist of a finite number of disjoint smooth simple closed curves. Let $K(\Omega; u_1, u_2)$ be the convex subset of functions in $H^2(\Omega)$ whose real part lies in $[u_1, u_2]$ on Γ. We suppose that $K(\Omega; u_1, u_2)$ is nonempty. If $f \in H^2(\Omega)$ but not in $K(\Omega; u_1, u_2)$ and if F is the best approximation to f from $K(\Omega; u_1, u_2)$, then there is a real number c and a function $w \in N$ such that*

$$\operatorname{Re} F = T_{u_1,u_2}(\operatorname{Re} f - c - w) \quad \text{a.e. } \mu \text{ on } \Gamma. \tag{3.14}$$

Proof. Recall that F is defined by the variational inequalities

$$\operatorname{Re}(f - F, g - F)_{H^2(\Omega)} \le 0, \quad g \in K(\Omega; u_1, u_2), \tag{3.15}$$

from which it follows from (3.4) that

$$0 \ge \int_\Gamma U \operatorname{Re}(g - F)\, d\mu - c \operatorname{Re}(g - F)(t_0) + \operatorname{Im}(f - F)(t_0) \operatorname{Im}(g - F)(t_0),$$

where $U = 2 \operatorname{Re}(f - F)$ and

$$c = \operatorname{Re}(f - F)(t_0) = \frac{1}{2} \int_\Gamma U\, d\mu.$$

As in Theorem 3.1, we conclude that $\operatorname{Im}(f - F)(t_0) = 0$, since $\operatorname{Im} g(t_0)$ can be any number. This gives us the variational inequality

$$0 \ge \int_\Gamma (U - c) \operatorname{Re}(g - F)\, d\mu, \quad g \in K(\Omega; u_1, u_2) \tag{3.16}$$

which we will proceed to solve.

We introduce subsets of Γ:

$$A^j = \{ \zeta : \zeta \in \Gamma, \operatorname{Re} F(\zeta) = u_j \}, \quad j = 1, 2,$$

and also $A = \Gamma \backslash (A^1 \cup A^2)$.

Our first goal is to show that there is a $w \in N$ such that

$$U - c = w \quad \text{a.e. } \mu \text{ on } A.$$

When A has measure zero there is nothing to prove. If A has positive measure then we decompose it by introducing the subsets

$$A_\delta := \{\, \zeta : \zeta \in A,\ u_2(\zeta) - u_1(\zeta) \geq \delta \,\},$$

so that

$$A = \bigcup \{\, A_\delta : \delta > 0 \,\}.$$

For a fixed $\delta > 0$ such that $\mu(A_\delta) > 0$, we choose $v \in L^\infty_{\mathbb{R}}(A_\delta, \mu)$ such that

$$\int_{A_\delta} vu \, d\mu = 0, \qquad u \in N,$$

and define $g \in H^2(\Omega)$ by setting

$$\operatorname{Re} g := \operatorname{Re} F + \varepsilon v \chi_{A_\delta}.$$

Then there exists an $\varepsilon_0 > 0$ such that $g \in K(\Omega; u_1, u_2)$ for all ε, $|\varepsilon| < \varepsilon_0$. Substituting this g into the variational inequality (3.16), we conclude that

$$\int_{A_\delta} (U - c) v \, d\mu = 0.$$

From this observation we conclude that there exists a $w \in N$ such that $U - c = w$ on the set A_δ. Since the sets A_δ ($\delta > 0$) are nested in δ, it is easy to see that w is independent of δ and so

$$U(\zeta) - w(\zeta) - c = 0, \qquad \zeta \in A.$$

Next, we consider this function off the set A and define for $i = 1, 2$ the sets

$$E_i = \{\, \zeta : \zeta \in A^i,\ \{U(\zeta) - w(\zeta) - c\}(-1)^{i-1} > 0 \,\}.$$

Let $v_1 \in L^\infty_{\mathbb{R}}(\Gamma, \mu)$ be chosen to be supported on A_δ, so that the function

$$v := \chi_{E_1} - \chi_{E_2} + v_1$$

is in $\operatorname{Re} H^2(\Omega)$. Such a v_1 exists since $\mu(A_\delta) > 0$ and N has finite dimension. As before, define $g \in H^2(\Omega)$ by the equation

$$\operatorname{Re} g = \operatorname{Re} F + \varepsilon v.$$

When ε is a sufficiently small positive number we conclude that $g \in K(\Omega; u_1, u_2)$. Again, substituting this function into the variational inequality (3.16) gives us

$$\begin{aligned} 0 &\geq \int_\Gamma (U - c) v \, d\mu = \int_\Gamma (U - c - w) v \, d\mu \\ &= \int_{E_1} (U - c - w) \, d\mu + \int_{E_2} (U - c - w)(-1) \, d\mu \\ &\quad + \int_A (U - c - w) v \, d\mu. \end{aligned}$$

Suppose that one or both of E_1 and E_2 have positive measure. The last integral above is zero, since $U - c - w$ vanishes on A and one or both of the others are positive. This contradiction implies that meas $E_1 =$ meas $E_2 = 0$. Hence,

$$U - c - w \begin{cases} \leq 0 & \text{where } \operatorname{Re} F = u_1, \\ \geq 0 & \text{where } \operatorname{Re} F = u_2, \\ = 0 & \text{where } u_1 < \operatorname{Re} F < u_2, \end{cases}$$

or (equivalently)

$$\operatorname{Re} f - \tfrac{1}{2}c - \tfrac{1}{2}w \begin{cases} \leq u_1 & \text{where } \operatorname{Re} F = u_1, \\ \geq u_2 & \text{where } \operatorname{Re} F = u_2, \\ = \operatorname{Re} F & \text{where } u_1 < \operatorname{Re} F < u_2. \end{cases}$$

That is,

$$\operatorname{Re} F = T_{u_1,u_2}(\operatorname{Re} f - \tfrac{1}{2}c - \tfrac{1}{2}w). \qquad \square$$

For the special case of the unit disc, $w = 0$ since $N = \{0\}$; also, $c/2 = \lambda$, where λ is the unique root of the equation

$$\lambda + \int_T (\operatorname{Re} f - \lambda - u_1)_+ \, dm = \int_T (\operatorname{Re} f - \lambda - u_2)_+ \, dm. \tag{3.17}$$

To see this, we compute

$$\begin{aligned} c = \int_T \operatorname{Re}(f - F)\, dm &= \int_{A_1} (\operatorname{Re} f - u_1)\, dm + \int_{A_2} (\operatorname{Re} f - u_2)\, dm + \frac{c}{2} \int_A dm \\ &= \int_{A_1} \Big(\operatorname{Re} f - u_1 - \frac{c}{2}\Big)\, dm + \int_{A_2} \Big(\operatorname{Re} f - u_2 - \frac{c}{2}\Big)\, dm + \frac{c}{2}. \end{aligned}$$

Recalling that the integrand is (respectively) negative on A_1, positive on A_2, and zero on A, we conclude that indeed (3.17) holds.

4. Minimal Interpolation with Nonnegative Real Part: The Unit Disc

Given distinct points $z_1, z_2, \dots, z_n \in \Delta$ and complex numbers $w_1, w_2, \dots, w_n$, we wish to characterize the unique g in $H^2(\Delta)$ of minimal norm satisfying the conditions

$$g \in K(\Delta), \quad g(z_j) = w_j, \quad j = 1, 2, \dots, n. \tag{4.1}$$

Extremal problems of this general type have been studied before (see e.g. [2; 6; 8; 10]). First, we recall the following well-known necessary and sufficient conditions on $z_1, z_2, \dots, z_n$ and $w_1, w_2, \dots, w_n$, which ensure the existence of a $g \in H^2(\Delta)$ satisfying (4.1).

LEMMA 4.1. *Assume $w_1, w_2, \dots, w_n$ are complex numbers that are not all the same. There exists a function in $H^2(\Delta)$ satisfying* (4.1) *with* $\operatorname{Re} g(\zeta) > 0$ ($\zeta \in \Delta$) *if and only if the matrix*

$$A = \left(\frac{w_j + \overline{w_k}}{1 - z_j \overline{z_k}} \right)_{j,k=1}^{n}$$

is strictly positive definite.

Proof. Let

$$\zeta_j = \frac{1 - w_j}{1 + w_j}, \quad j = 1, 2, \ldots, n,$$

and observe that a function $g \in H^2(\Delta)$ satisfies (4.1) if and only if the function $f = (1 - g)/(1 + g)$ has $H^\infty(\Delta)$ norm at most 1 and satisfies the interpolation conditions $f(z_j) = \zeta_j$, $j = 1, 2, \ldots, n$. By the Pick–Nevanlinna theorem, such a function exists if and only if the matrix

$$B = \left(\frac{1 - \zeta_j \overline{\zeta_k}}{1 - z_j \overline{z_k}} \right)_{j,k=1}^{n}$$

is nonnegative definite; B is singular if and only if the interpolant is unique, and if this is so then it is a Blaschke product of degree $n - 1$ or less. If f is a Blaschke product and $g = (1 - f)(1 + f)$, then g has poles on the boundary of Δ and so cannot be in $H^2(\Delta)$. Thus B must be strictly positive definite. In this case, we may find an f in $H^2(\Delta)$ with norm less than 1 satisfying the interpolation conditions $f(z_j) = \zeta_j$, $j = 1, 2, \ldots, n$. The corresponding g satisfies (4.1) and is also bounded and therefore in $H^2(\Delta)$. To finish the proof note that $B = DA\bar{D}^{-1}$, where

$$D = \sqrt{2}\, \text{diag}\left(\frac{1}{1 + w_1}, \frac{1}{1 + w_2}, \ldots, \frac{1}{1 + w_n} \right). \qquad \square$$

Returning to our variational problem (4.1), we wish to exclude from consideration a simple special case of this problem that appears as an exception to certain arguments that we shall present later; specifically, as noted in Lemma 4.2. The case we have in mind occurs when all the values $w_1, w_2, \ldots, w_n$ are the *same* purely imaginary number. That is, there is a real constant c such that $w_j = ic$, $j = 1, 2, \ldots, n$. In this special case we see that

$$\begin{aligned} &\min\{ \|g\|_{H^2(\Delta)}^2 : g \in K(\Delta),\ g(z_j) = w_j,\ j = 1, 2, \ldots, n \} \\ &\quad = \min\{ \|ic + f\|_{H^2(\Delta)}^2 : f \in K(\Delta),\ f(z_j) = 0,\ j = 1, 2, \ldots, n \} \\ &\quad = c^2, \end{aligned}$$

and so the optimal g is merely $g = ic$. Now that this case has been treated, we can proceed with our analysis of the general case.

The next lemma is essential for the characterization of the function in $H^2(\Delta)$ of least norm satisfying (4.1).

LEMMA 4.2. *Let $h_1, h_2, \ldots, h_n$ be linearly independent rational functions that are analytic in some neighborhood of Δ. Suppose $w_1, w_2, \ldots, w_n$ are complex numbers that are not all equal to the same purely imaginary constant. Also, suppose there exists a $g \in K(\Delta)$ such that $(g, h_j)_{H^2(\Delta)} = w_j$, $j = 1, \ldots, n$. Then,* *whenever* $\mu_1, \mu_2, \ldots, \mu_n$ *are complex numbers such that* $\sum_{j=1}^{n} |\mu_j|^2 = 1$ *and* $\operatorname{Re}(\sum_{j=1}^{n} \bar{\mu}_j w_j) \geq 0$, *it follows that* $P_{K(\Delta)}(h) \neq 0$ *where* $h = \sum_{j=1}^{n} \mu_j h_j$.

Proof. Suppose to the contrary that $P_{K(\Delta)}(h) = 0$. Then for any $z \in \Delta$ we have

$$0 = i \operatorname{Im} h(0) + \int_T \frac{e^{i\theta} + z}{e^{i\theta} - z} (\operatorname{Re} h(e^{i\theta}) - \lambda)_+ \, dm\,(\theta).$$

Consequently, we conclude that $\operatorname{Im} h(0) = 0$ and $\operatorname{Re} h(e^{i\theta}) \leq \lambda$ a.e. on T. Thus from (2.2) with $f = h$ we have that

$$\lambda = \tfrac{1}{2} \operatorname{Re}(h(0)) = \tfrac{1}{2} h(0).$$

Now let $g \in H^2$, $\operatorname{Re} g \geq 0$, $(g, h_j)_{H^2(\Delta)} = w_j$, $j = 1, \ldots, n$. Then

$$\begin{aligned} 0 &\leq \operatorname{Re} \sum_{j=1}^{n} \bar{\mu}_j w_j \\ &= \operatorname{Re}(g, h)_{H^2(\Delta)} \\ &= 2 \int_T (\operatorname{Re} h) \operatorname{Re} g \, dm - \operatorname{Re}(h(0) g(0)) \\ &= 2 \int_T (\operatorname{Re} h - \lambda) \operatorname{Re} g \, dm \\ &\leq 0. \end{aligned}$$

Hence, $\operatorname{Re}(h - \lambda) \operatorname{Re} g = 0$ a.e. on T. First, let us rule out the possibility that that $\operatorname{Re} g = 0$ a.e. on T. If indeed that were the case then it would follow that $g = i\eta$ everywhere on Δ for some real constant η. This contradicts our hypothesis about the data $w_1, w_2, \ldots, w_n$.

Thus there exists a subset E of T having positive measure such that $\operatorname{Re}(h - \lambda) = 0$ on E. The function h is rational; let $h_1(z) = \bar{h}(1/\bar{z})$. Then h_1 is also rational and $2 \operatorname{Re} h(e^{i\theta}) = h(e^{i\theta}) + h_1(e^{i\theta})$. Hence the rational function $h + h_1 - 2\lambda$ vanishes on a set E in T of positive measure and so is identically zero. Therefore, we conclude that $\operatorname{Re} h - \lambda = 0$ on all of T. As remarked above, this means that $h = \lambda + i\eta$ for some real constant η on Δ. Since we have already observed that $\operatorname{Im} h(0) = 0$, we conclude that in fact $h = \lambda$ everywhere on Δ. In particular we have $2\lambda = \operatorname{Re} h(0) = \lambda$, which means that $\lambda = 0$. This conclusion again contradicts our hypothesis. □

We are now ready to prove the characterization theorem. Recall that, for any points $z_1, z_2, \ldots, z_n$ in the disc Δ, we have

$$(g, F_j)_{H^2(\Delta)} = g(z_j), \quad j = 1, 2, \ldots, n, \quad h \in H^2(\Delta),$$

where

$$F_j(z) = \frac{1}{1 - \bar{z}_j z}, \quad z \in \Delta, \quad j = 1, 2, \ldots, n.$$

Note that if the points $z_1, z_2, \ldots, z_n$ are distinct then the functions $h_j = F_j$ $(j = 1, 2, \ldots, n)$ satisfy the hypothesis of Lemma 4.2. For this case, we set $h = F_\mu = \sum \mu_j F_j$.

THEOREM 4.1. *Let $z_1, z_2, \dots, z_n$ be distinct points in Δ, and let $w_1, w_2, \dots w_n$ be points not all of which equal the same purely imaginary constant. Suppose there exists a $g_0 \in K(\Delta)$ such that $g_0(z_i) = w_i$, $i = 1, 2, \dots, n$. Then the minimization problem*

$$\min\{ \|g\|_{H^2(\Delta)} : g \in K(\Delta),\ g(z_j) = w_j,\ j = 1, 2, \dots, n\}$$

has a unique solution of the form

$$G(z) = i \operatorname{Im} F_\mu(0) + \frac{1}{2\pi} \int_T \frac{e^{i\theta} + z}{e^{i\theta} - z} (\operatorname{Re} F_\mu(e^{i\theta}) - \lambda)_+ \, d\theta, \tag{4.2}$$

where

$$\lambda(1 + m\{\operatorname{Re} F_\mu < \lambda\}) = \int_{\operatorname{Re} F_\mu < \lambda} \operatorname{Re} F_\mu$$

and

$$G(z_j) = w_j, \quad j = 1, \dots, n. \tag{4.3}$$

Proof. The essential point in the proof is to show that the equation (4.3) has a solution of the form (4.2). It then follows that G given in (4.2) is the unique solution to the minimum problem. To see this, recall that for any complex numbers $\mu_1, \dots, \mu_n$ the function G defined in (4.2) satisfies

$$\operatorname{Re}(F_\mu - G, g)_{H^2(\Delta)} \le 0$$

for all $g \in K(\Delta)$, with equality for $g = G$. Hence, if $g(z_i) = w_i$ $(i = 1, 2, \dots, n)$ then we have

$$\|G\|^2_{H^2(\Delta)} = \operatorname{Re}(F_\mu, G)_{H^2(\Delta)} = \operatorname{Re}(F_\mu, g) \le \operatorname{Re}(G, g) \le \|G\| \|g\|.$$

Thus we must show that there exist complex constants $\mu_1, \dots, \mu_n$ such that G satisfies the interpolation conditions (4.3). For this problem, we follow [9] and consider the variational problem

$$\inf_{\mu_1, \dots, \mu_n \in \mathbb{C}} \left(\frac{1}{2} \| P_{K(\Delta)}(F_\mu) \|^2 - \operatorname{Re}\left(\sum_{i=1}^n F_\mu(z_i) w_i \right) \right). \tag{4.4}$$

From general principles given in [9], we can demonstrate that any solution of (4.4) satisfies

$$P_{K(\Delta)}(F_\mu)(z_j) = w_j, \quad j = 1, 2, \dots, n. \tag{4.5}$$

This follows from the fact that the gradient of $\frac{1}{2} \| P_{K(\Delta)}(F_\mu) \|^2$ is

$$\begin{aligned} &\operatorname{Re}((P_{K(\Delta)}(F_\mu), F_1)_{H^2(\Delta)}, (P_{K(\Delta)}(F_\mu), F_2)_{H^2(\Delta)}, \dots, (P_{K(\Delta)}(F_\mu), F_n)_{H^2(\Delta)}) \\ &\quad = \operatorname{Re}(P_{K(\Delta)}(F_\mu)(z_1), P_{K(\Delta)}(F_\mu)(z_2), \dots, P_{K(\Delta)}(F_\mu)(z_n)). \end{aligned}$$

To establish that (4.4) has a solution, we use Lemma 4.2 in the following way. Let τ^k be any minimizing sequence of (4.4). If $\sum_{i=1}^n |\tau_i^k|^2 \to \infty$ then we can find a subsequence of $\mu^k = \tau^k / (\sum_1^n |\tau_i^k|^2)^{1/2}$ that converges to a vector $(\mu_1, \dots, \mu_n)$ satisfying $\sum_{j=1}^n |\mu_j|^2 = 1$, $\operatorname{Re} \sum_{j=1}^n \bar{\mu}_j w_j \ge 0$, and $P_{K(\Delta)}(F_\mu) = 0$. This contradicts Lemma 4.2, and so τ^k must be bounded in norm and (4.4) has a solution as asserted. □

5. Minimal Interpolation with Nonnegative Real Part: Domains of Finite Connectivity

This section presents our solution to the minimum problem

$$\min\{\ \|g\|_{H^2(\Omega)} : g \in K(\Omega),\ g(z_j) = w_j,\ j = 1, 2, \ldots, n\},$$

where Ω is a domain of connectivity $m + 1$, as described earlier; $z_1, z_2, \ldots, z_n$ are distinct points in Ω, and $w_1, w_2, \ldots, w_n$ are prescribed complex numbers.

We first need a version of Lemma 4.2 that applies to the domain Ω. For this purpose, let a be any point of Ω and let K_a be the reproducing kernel in $H^2(\Omega)$ for a. That is, $K_a \in H^2(\Omega)$ and

$$\int_\Gamma f\bar{K}_a\, d\mu = f(a), \qquad f \in H^2(\Omega). \tag{5.1}$$

We also have

$$\int_\Gamma fK_a\, d\mu = f(t_0)K_a(t_0), \qquad f \in H^2(\Omega). \tag{5.2}$$

Combining these equations yields the formula

$$\int_\Gamma f(z)(z-a)\{\bar{K}_a(z) \pm K_a(z)\}\, d\mu(z) = 0, \tag{5.3}$$

which is valid for all $f \in H^2(\Omega)$ that vanish at t_0.

LEMMA 5.1. *Let $z_1, z_2, \ldots, z_n$ be distinct points in Ω, and let $w_1, w_2, \ldots, w_n$ be complex numbers that are not all equal to the same purely imaginary constant. Also, suppose there exists a $g \in K(\Omega)$ such that $g(z_j) = w_j$, $j = 1, 2, \ldots, n$. Then, whenever $\mu_1, \mu_2, \ldots, \mu_n$ are complex numbers such that $\sum_{j=1}^n |\mu_j|^2 = 1$ and $\mathrm{Re}(\sum_{j=1}^n \bar{\mu}_j w_j) \geq 0$, it follows that $P_{K(\Omega)}(f) \neq 0$ where $f = \sum_{j=1}^n \mu_j K_{z_j}$.*

Proof. To begin, we point out that it follows from (5.3) and Theorem 4.48 of [4, p. 92] (the F. and M. Riesz theorem for multiply connected domains) that there are functions G and H in $H^2(\Omega)$ such that

$$2\,\mathrm{Re}\, K_a(z) = \bar{K}_a(z) + K_a(z) = \frac{G(z)}{(z-a)P(z)} \quad \text{a.e. } \mu \text{ on } \Gamma$$

and

$$-2i\,\mathrm{Im}\, K_a(z) = \bar{K}_a(z) - K_a(z) = \frac{H(z)}{(z-a)P(z)} \quad \text{a.e. } \mu \text{ on } \Gamma,$$

where P is a monic polynomial of degree m whose zeros are precisely the points at which the complex derivative of the function $g(z; t_0) + ih(z; z_0)$ is zero. Here $g(z; t_0)$ is Green's function for the point t_0 and $h(z; t_0)$ is the harmonic conjugate of $g(z; t_0)$. This implies that both $\mathrm{Re}\, K_a$ and $\mathrm{Im}\, K_a$ have a meromorphic extension from Γ to Ω with $m + 1$ poles. Since both of these functions are real on Γ, they can be extended analytically across Γ. We then apply this result to our situation and conclude that the function $\mathrm{Re}(\sum_{j=1}^n \mu_j K_{z_j})$ has a meromorphic extension to a neighborhood of $\Omega \cup \Gamma$ with a finite number of poles on Ω and no poles on Γ.

Now suppose that $P_{K(\Omega)} f = 0$, where $\sum_{j=1}^{n} |\mu_j|^2 = 1$. Then, according to Theorem 3.1, there is a real scalar $c = \operatorname{Re}(f(t_0) - P_{K(\Omega)} f(t_0)) = \operatorname{Re} f(t_0)$ and a function $w \in N$ with

$$\operatorname{Re} f - \frac{c}{2} + \frac{w}{2} \leq 0 \quad \text{a.e. } \mu \text{ on } \Gamma. \tag{5.4}$$

Also, we know from the proof of Theorem 3.1 that $\operatorname{Im} f(t_0) = \operatorname{Im} P_K(\Omega) f(t_0) = 0$. Using the fact that

$$\operatorname{Re}\left(\sum_{j=1}^{n} w_j \bar{\mu}_j\right) \geq 0$$

and the function $g \in K(\Omega)$ such that $g(z_j) = w_j$ $(j = 1, 2, \ldots, n)$, by (5.4) we have that

$$\begin{aligned} 0 \leq \operatorname{Re} \sum_{j=1}^{n} w_j \bar{\mu}_j &= \operatorname{Re}\left(\int_\Gamma g \bar{f} \, d\mu\right) \\ &= 2 \int_\Gamma (\operatorname{Re} g)(\operatorname{Re} f) \, d\mu - \operatorname{Re}(g(t_0) f(t_0)) \\ &= \int_\Gamma \operatorname{Re} g(2 \operatorname{Re} f - c + w) \, d\mu \leq 0. \end{aligned}$$

We conclude that $(\operatorname{Re} f - c/2 + w/2) \operatorname{Re} g = 0$ a.e. μ on Γ. As in the proof of Lemma 4.2, we rule out the possibility that $\operatorname{Re} g = 0$ a.e. μ on Γ. Hence there is a subset E of Γ having positive measure such that $2 \operatorname{Re} f - c + w$ vanishes on E. However, Theorem 4.4.8 of [4, p. 92] implies that w is meromorphic on a neighborhood of $\Omega \cup \Gamma$ with m poles in Ω and no poles in Γ. This fact, together with the above remark about $\operatorname{Re} f$, implies that $2 \operatorname{Re} f - c + w$ is meromorphic on a neighborhood of $\Omega \cup \Gamma$ and hence must be identically zero on $\Gamma \cup \Omega$ (excluding the poles). Consequently, since $w \perp \operatorname{Re} H^2(\Omega)$, it follows that $2 \operatorname{Re} f = c$ and so $f = \frac{c}{2} + i\eta$ on $\Omega \cup \Gamma$. In particular, $c = \operatorname{Re} f(t_0) = c/2$ and so $c = 0$, which implies $\eta = 0$ as well. Finally, since the points $z_1, z_2, \ldots, z_n$ are distinct, we conclude that $\mu_1, \mu_2, \ldots, \mu_n$ are also all zero. □

This result leads to the following fact, which extends Theorem 4.1 to multiply connected domains.

THEOREM 5.1. *Suppose the boundary of Ω consists of a finite number of disjoint smooth closed curves. Let $K(\Omega)$ be the convex cone of functions in $H^2(\Omega)$ with positive real part on Ω. Suppose $z_1, z_2, \ldots, z_n$ are any distinct points in Ω and let $w_1, w_2, \ldots, w_n$ be points not all of which equal the same purely imaginary constant. Suppose there exists a $g_0 \in K(\Omega)$ such that $g_0(z_i) = w_i$, $i = 1, 2, \ldots, n$. Then the minimization problem*

$$\min\{ \|g\|_{H^2(\Omega)} : g \in K(\Omega),\ g(z_j) = w_j,\ j = 1, 2, \ldots, n \}$$

has a unique solution G of the form

$$\operatorname{Re} G = (\operatorname{Re} F_\mu - c - w)_+ \quad \text{a.e. } \mu \text{ on } \Gamma,$$

$$c \in R, \quad w \in N, \quad \mu = (\mu_1, \mu_2, \ldots, \mu_n) \in C^n, \quad F_\mu = \sum_{j=1}^{n} \mu_j K_{z_j},$$

and

$$G(z_j) = w_j, \quad j = 1, 2, \ldots, n.$$

Proof. The proof follows the method used for the proof of Theorem 4.1 and therefore relies on Lemma 5.1. □

ADDENDUM. Some preliminary results on the problem discussed here appeared in an internal IBM research report: "Minimal norm interpolation in H^2 with nonnegative real part" (RC 12351, December 1986). Shortly after we decided to return to the interesting problem studied here, we obtained a copy of "Control-oriented H_2 optimal problems with passivity constraints" by John E. McCarthy and Clas A. Jacobson, where the problem we solve in Theorem 4.1 is also considered. Their paper is motivated by considerations arising in certain problems in control synthesis and system identification.

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