A Geometric Characterization of Smooth Linearizability

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The question of how smoothly a diffeomorphism or flow can be linearized in a neighborhood of a hyperbolic fixed point has a long and venerable history, going back at least to Poincaré [P; S1]. Of particular note is the work of Sternberg [S1; S2], which has been built on by Belitskii [B1; B2], Sell [Se], and others. Sternberg proved that a transformation is smoothly conjugate to its linear part provided there is an absence of "resonance" in the eigenvalues of the linear part. Belitskii has shown nonresonance conditions to be both sufficient and necessary, in the sense that they are necessary for smooth linearizability of the entire *class* of maps having the same linear part. On the other hand, it is clear that conditions on the eigenvalues could never be necessary conditions for linearizability of any one particular map.

In this paper we present a completely different type of result, which can be seen as a generalization of a theorem of Hartman [H]. For the special case of hyperbolic attracting or repelling fixed points, we give criteria for linearizability that are geometric in nature. These are not computable like resonance conditions (although can possibly be verified as in the example below), but do precisely characterize how smoothly a given diffeomorphism can be linearized near its fixed points. Therefore, this result clearly indicates where the obstruction to linearizability manifests itself for attracting and repelling fixed points.

By the local nature of the question, we can take the fixed point to be the origin in $\mathbb{R}^n$. Therefore, let f be a local diffeomorphism fixing $0 \in \mathbb{R}^n$, which is at least C^r. Throughout this paper we assume the origin is a hyperbolic repelling fixed point. The attracting case can be treated by taking f^{-1}. Consequently, let $\{\lambda_i\}$ be the moduli of the eigenvalues of $(Df)_0$, where $1 < \lambda_1 < \cdots < \lambda_k$. To each splitting of the spectrum, $\{\lambda_1, \ldots, \lambda_i\} < \{\lambda_{i+1}, \ldots, \lambda_k\}$, there exist complementary, generalized eigenspaces for $(Df)_0$, the weak unstable and strong unstable subspaces, respectively. A general result is that f has nonlinear analogs of these subspaces, the weak and strong unstable manifolds. Locally these are embedded disks tangent to the corresponding subspaces at 0 (and so can be given as graphs over these subspaces), which

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are overflowing invariant under f. Moreover, under iteration by f, points in these invariant submanifolds exhibit the same asymptotic rates of expansion as vectors in the corresponding subspaces do under the linear map.

By the pseudostable manifold theorem [HPS], the strong unstable manifold is unique and as smooth as the diffeomorphism. The weak unstable manifolds, however, are not unique. In this case, the pseudostable manifold theorem guarantees the existence of weak unstable manifolds of at least a minimum smoothness, which depends upon the gap between the two pieces of the spectrum. (For the splitting as above, the weak unstable manifold will be at least C^s provided that $\lambda_i^s < \lambda_{i+1}$.) This does not imply, however, that there might not exist weak unstable manifolds that are more smooth than predicted by this theorem. We say that *f has a complete set of C^r weak unstable manifolds* if to every such splitting of the spectrum of $(Df)_0$ there exists at least one weak unstable manifold that is C^r. Under a linearization, a weak unstable subspace must be taken to a corresponding weak unstable manifold. Hence, if f is C^r linearizable, f must certainly have a complete set of C^r weak unstable manifolds. We prove the converse.

THEOREM 1. *Assume $0 \in \mathbb{R}^n$ is a hyperbolic repelling fixed point for a C^{r+2} $(1 \le r \le \infty)$ germ of a diffeomorphism f. Then*

f is locally C^r linearizable
$\Leftrightarrow$ f has a complete set of C^r weak unstable manifolds.

Moreover, if $r > 1$, then the given C^r conjugacy is the only conjugacy between f and its linear part that is at least $C^{1+\theta}$ $(\theta > 0)$, takes the given weak unstable manifolds to the corresponding weak unstable planes, and whose derivative at the origin is the identity.

From the uniqueness in Theorem 1 we will be able to obtain the corresponding theorem for flows.

THEOREM 2. *Assume $0 \in \mathbb{R}^n$ is a hyperbolic repelling singularity for a C^{r+2} $(1 \le r \le \infty)$ germ of a vector field X. Let φ_t be the local flow of X, and set $B := (DX)_0$. Then*

φ_t is locally C^r conjugate to the linear flow e^{Bt}
$\Leftrightarrow$ φ_t has a complete set of C^r weak unstable manifolds.

Moreover, if $r > 1$, then the given C^r conjugacy is the only conjugacy between φ_t and e^{Bt} that is at least $C^{1+\theta}$ $(\theta > 0)$, takes the given weak unstable manifolds to the corresponding weak unstable planes, and whose derivative at the origin is the identity.

Consider the following example. Let f be a local diffeomorphism of the plane with $f(x, y) = [\kappa x, \kappa^2 y] + [r^2\rho_1, y^2\rho_2]$, where $r^2 = x^2 + y^2$ and ρ_1, ρ_2: $\mathbb{R}^2 \to \mathbb{R}$ are C^∞ functions. The origin is a fixed point, and the eigenvalues of $(Df)_0$ are $\kappa < \kappa^2$. With these eigenvalues a map is not in general even C^2

linearizable [S1, p. 812]. However, for this example the x-axis is invariant, and so the system has a C^∞ weak unstable manifold (for the only possible splitting of the eigenvalues). Consequently, Theorem 1 implies that f will be C^∞ linearizable for *any* choice of the ρ_i $(i = 1, 2)$.

As already observed, it is obvious that there must exist a complete set of C^r weak unstable manifolds if there is a C^r linearization. It is somewhat surprising, as in the above example, that this should also be sufficient. A linearizing change of coordinates implies considerable geometric structure in the neighborhood of the fixed point. Not only will every invariant subspace of the derivative have a corresponding smooth invariant submanifold, but the family of planes parallel to any such subspace will be taken by the conjugacy to a smooth invariant foliation. In fact, for any hyperbolic fixed point, the existence of a smooth linearization is equivalent to the existence of these invariant foliations of the same smoothness. This result hinges on having certain crucial invariant foliations of sufficient smoothness. For these foliations, the existence/smoothness of the foliation will depend entirely on the smoothness of just one (particular) leaf. This is certainly not usual even for invariant foliations—the smoothness of the leaves (let alone one leaf) is typically quite independent of the overall smoothness of the foliation.

The uniqueness of the conjugacy says that there is considerable rigidity with regards to linearizations in the smooth category. This is totally at odds with the C^0 case, where one need only define a reasonable homeomorphism between fundamental domains (i.e., one that "glues" properly at its boundaries). Consequently, one is free to make (almost) arbitrary modifications on the interiors of the domains. For the smooth case, this theorem says there is a one-to-one correspondence (up to multiplication by a linear map) between the C^r conjugacies and complete sets of C^r weak unstable manifolds. If there exists one C^r conjugacy, then the correspondence extends to the complete sets of C^r weak unstable manifolds of the linear map as well. However, for a linear map there may be only one (smooth) weak unstable manifold for a given splitting—the linear subspace itself. For the above example, the only C^2 weak unstable manifolds of the linear part are the parabolas $y = ax^2$. Hence, there is exactly a 1-parameter family of C^2 conjugacies (which happen to be C^∞). If the eigenvalues had not been related by an integer power, the conjugacy would have been unique.

A final remark is that this result does not generalize to hyperbolic fixed points of mixed type. The strongest possible assumptions in character with the hypotheses of Theorem 1 would be that there exist a smooth invariant submanifold for every invariant subspace of the derivative (not just those associated to pseudohyperbolic splittings). However, a saddle point of a C^∞ diffeomorphism of the plane trivially satisfies these assumptions yet cannot in general be even C^2 linearized [St]. However, the inability to extend Theorem 1 to fixed points of mixed type is in itself interesting. Since under the above assumptions the diffeomorphism can be linearized when restricted to both the stable and unstable manifolds, additional obstructions must manifest themselves "between" these invariant manifolds. This emphasizes that for fixed points of mixed type there are obstructions, unlike those of the attracting and repelling cases, that are derived from the interaction between the stable and unstable directions.

In principle, the proof of Theorem 1 falls into three parts: the existence of strong unstable foliations; the existence of weak unstable foliations; and linearizing functions in the special case where the eigenvalues of the derivative have a common modulus. A *strong unstable foliation* is an invariant foliation on a neighborhood of the fixed point that has the strong unstable manifold as a leaf (and hence is generally "parallel" to the strong unstable subspace). A *weak unstable foliation* is defined analogously. These three parts are the content of the following propositions.

PROPOSITION 3 [M]. *Let $f: \mathbb{R}^n \to \mathbb{R}^n$ be a C^{r+2} local diffeomorphism with the origin as a hyperbolic repelling fixed point. For any strong/weak unstable splitting of the spectrum of $(Df)_0$, there exists a unique C^{r+1} strong unstable foliation.*

PROPOSITION 4a. *Let $f: \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k \times \mathbb{R}^{n-k}$ be a C^{r+1} function ($r \geq 1$) fixing the origin and having the form $f(x, y) = (f_1(x), f_2(x, y))$. Suppose that all eigenvalues of $(D_2 f_2)_0$ have a common modulus $\beta > 1$, and that $\beta > |\mathrm{sp}[(D_1 f_1)_0]| > 1$. If f has a C^r weak unstable manifold, then there exists a C^r weak unstable foliation that contains this weak unstable manifold as a leaf.*

PROPOSITION 4b. *Let $f: \mathbb{R}^n \to \mathbb{R}^n$ be a C^r ($r > 1$) local diffeomorphism with the origin as a hyperbolic repelling fixed point. If all the eigenvalues of $(Df)_0$ have a common modulus, then f is C^r linearizable.*

If $f(x, y)$ has an invariant foliation, then the foliation chart "triangularizes" f: If the chart takes the leaves to the vertical planes, then in these coordinates f leaves the vertical planes invariant, and so must have the form $(f_1(x), f_2(x, y))$. Therefore, if f has two transverse, complementary-dimension, invariant foliations (e.g., a weak and a strong unstable foliation), then relative to a common foliation chart f is "diagonal", $f(x, y) = (f_1(x), f_2(y))$. If this is performed for each of the generalized eigenspaces corresponding to the eigenvalues of a common modulus, then the resulting components can be linearized independent of each other by Proposition 4b.

In practice, it is more convenient to combine the last two propositions and find the weak unstable foliation simultaneously with linearizing the strong unstable direction. Consequently, Propositions 4a and 4b are replaced by Proposition 4 below.

REMARK. Although 4a follows directly from the statement of 4, 4b does not. While true, we will not give an explicit proof of 4b. However, one can obtain 4b from the proof of 4 by observing that in the extreme case, where the weak unstable plane is 0-dimensional, the extra level of differentiability required of f in 4 is unnecessary.

PROPOSITION 4. *Let $U \subset \mathbb{R}^l \times \mathbb{R}^{n-l}$ be a neighborhood of the origin, f: $U \to \mathbb{R}^l \times \mathbb{R}^{n-l}$ a C^{r+1} function ($r \geq 1$) fixing the origin and having the form $f(x, y) = (f_1(x), f_2(x, y))$. Assume $(Df)_0$ preserves the splitting, and set $A := (D_1 f_1)_0$ and $B := (D_2 f_2)_0$. Suppose that all eigenvalues of B have a common modulus $\beta > 1$, and that $\beta > |\mathrm{sp}(A)| > 1$. If f has a C^r weak unstable manifold, then f is locally C^r conjugate to $(f_1(x), By)$.*

This conjugacy has the form $G = \mathrm{id} + (0, g)$, takes the given weak unstable manifold to the weak unstable subspace, and has derivative equal to the identity at the origin. Moreover, this is the unique conjugacy with these properties. Specifically, if G' is any C^1 semiconjugacy with these properties and $(D_2 G')$ is θ-Hölder for some $\theta > 0$, then $G' = G$ (and so is in fact a C^r conjugacy).

All of the work in proving Theorem 1 is contained in Proposition 4, which we leave for last. Proposition 3 is a weaker version of the result in [M]. However, in its stated form it is a simple application of the C^r section theorem [HPS]. We give a sketch of this method of proof below.

We make a final remark on Theorem 1 regarding the assumption that f be C^{r+2}. This is most likely not sharp. In particular, if Propositions 3 and 4 were combined, it is conceivable that this assumption could be weakened (at the expense of drastically increasing the complexity of the proof). However, the theorem would definitely be false if this were replaced by the assumption that f be only C^r. If f is C^r linearizable, then clearly it has C^r strong unstable foliations. However, in [M] it is shown that although a strong unstable foliation for a C^r function (with a hyperbolic repelling fixed point) is $C^{r-1+\epsilon}$ (some $\epsilon > 0$), it is *not* in general C^r.

Proof of Proposition 3 (sketch). Choose coordinates near the origin so that $(Df)_0$ is block diagonal; $(Df)_0 = A \oplus B \in L(\mathbb{R}^l \times \mathbb{R}^{n-l})$. Let $\mathbb{R}^l$ be the weak unstable plane and $\mathbb{R}^{n-l}$ the strong unstable plane. By choosing an appropriate adapted norm we can assume $\|A\| =: \alpha < \beta := m(B) = \|B^{-1}\|^{-1}$. Let Λ be a small closed neighborhood of the origin on which f is overflowing; $f(\Lambda) \supset \Lambda$. If Λ is sufficiently small, then (Df) contracts the family of vertical cones $= \Lambda \times \{T \in L(\mathbb{R}^{n-l}, \mathbb{R}^l) : \|T\| \leq 1\}$ by $\alpha/\beta < 1$. Therefore, since f is purely expanding, f^{-1} is purely contracting, and by the C^r section theorem there is a unique (Df)-invariant vertical plane field that is as smooth as $(Df) = C^{r+1}$ (since f is C^{r+2}). Consequently, if this plane field is integrable, then there is a unique C^{r+1} strong unstable foliation.

On the other hand, we can apply the pseudostable manifold theorem to Λ itself. (This is more typically applied to invariant sets, but for finding strong unstable manifolds, overflowing invariant is sufficient.) From this we can conclude that through each point of Λ there exists a unique strong unstable manifold that is C^{r+2}. Since these are unique, whenever two manifolds intersect they agree. Hence they form a unique, overflowing invariant C^{r+2} strong unstable lamination. However, if this lamination is invariant under f, its tangent plane field is invariant under (Df), so this is the unique plane field found above. Since the lamination obviously integrates its tangent plane field, the plane field is integrable and the lamination is a C^{r+1} foliation.

Proof of Theorem 1. As above, let $\{\lambda_i\}$ be the moduli of the eigenvalues of $(Df)_0$, where $1<\lambda_1<\cdots<\lambda_m$. Choose coordinates $(x_1,\ldots,x_m)$ such that $(Df)_0$ is block diagonal and λ_i is the modulus of the eigenvalues of block i, B_i. By Proposition 3, there exist a total of $m-1$ C^{r+1} strong unstable foliations of decreasing dimension.

Let $s_1,\ldots,s_{m-1}$ satisfy $\lambda_1<s_1<\lambda_2<\cdots<s_{m-1}<\lambda_m$. By the pseudostable manifold theorem, the leaves of the strong unstable foliation associated to the strong unstable plane $(x_1,\ldots,x_i)=0$ are the equivalence classes of the relation $p\sim q$ if and only if $\|f^{-n}(p)-f^{-n}(q)\|/s_i^n\to 0$ as $n\to\infty$. Since s_i increases with i, the leaves of each successive foliation are contained in the leaves of the previous (next higher-dimensional) one. Consequently, by first choosing a foliation chart for the highest-dimensional leaves and working toward the lowest, we can construct a common foliation chart.

In a common foliation chart, the family of planes $(x_1,\ldots,x_i)=\text{constant}$ are invariant for each i. This implies $f_1,\ldots,f_i$ are functions of $(x_1,\ldots,x_i)$ only. Since this holds for every i, $f(x_1,\ \ldots,\ x_m)=[f_1(x_1), f_2(x_1,x_2),\ \ldots,\ f_m(x_1,\ldots,x_m)]$. The function f is C^{r+1}, and if we group the variables as $x=(x_1,\ldots,x_{m-1})$, $y=x_m$, we see that we have precisely the set-up for Proposition 4. Therefore, there exists a C^r conjugacy from f to $[f_1(x_1),\ \ldots,\ f_{m-1}(x_1,\ldots,x_{m-1}), B_m x_m]$. Although the conjugacy is only C^r, $f_1(x_1),\ldots,f_{m-1}(x_1,\ldots,x_{m-1})$ are left completely unchanged so these functions are still C^{r+1}, and we can repeat the process to linearize f_{m-1}. Continuing in this manner, we linearize all of the components. (For f_1 we need to observe that the proof of Proposition 4 does not require that l be nonzero; that is, the splitting may be trivial.) Our desired conjugacy is then the composition of the individual ones.

We are left with showing uniqueness. Suppose G and G' are two conjugacies satisfying the conditions in Theorem 1. We want to show $H:=G'\circ G^{-1}=\text{id}$. We know that $(DH)_0=I$, and that H conjugates $(Df)_0$ to itself. Since the strong unstable foliations are unique and topological invariants, H preserves the strong unstable foliations of $(Df)_0$; that is, the families of planes $(x_1,\ldots,x_i)=\text{constant}$. However, this implies H is triangular; $H(x_1,\ldots,x_m)=[H_1(x_1), H_2(x_1,x_2),\ldots,H_m(x_1,\ldots,x_m)]$. Consequently, H_1 is a $C^{1+\theta}$ conjugacy of B_1x_1 to itself with derivative equal to the identity at zero. However, we already have the identity satisfying these properties, and by Proposition 4, this is unique. Therefore, $H_1=\text{id}$.

Continuing, we have $[H_1(x_1), H_2(x_1,x_2)]=[x_1, H_2(x_1,x_2)]$ is a $C^{1+\theta}$ conjugacy of $B_1\oplus B_2$ to itself with derivative equal to the identity at zero. Moreover, G and G' map the same weak unstable manifold to the weak unstable plane, so H preserves the weak unstable plane of $B_1\oplus B_2$. But again we know that the identity is the only conjugacy to do this. Continuing in this fashion, we conclude $H_i = \mathrm{id}$ for each i, and so the conjugacy is unique. □

Proof of Theorem 2. By Theorem 1 there exists a conjugacy G from φ_1 to e^B that takes the given weak unstable manifolds to the corresponding weak unstable planes, and whose derivative at the origin is the identity. We observe that, for any t, $e^{Bt}\circ G\circ\varphi_{-t}$ is also a conjugacy from φ_1 to e^B, and has the same properties. If $r>1$, then G is unique. Therefore, $e^{Bt}\circ G\circ\varphi_{-t}=G$, and G is a conjugacy for the flow.

The case $r=1$ is not as direct. However, in the proof of Proposition 4 (on which Theorem 1 is based), the uniqueness will be derived from a contraction mapping. Therefore, there is still uniqueness for $r=1$ relative to the appropriate function space. Since φ_t is at least C^2 and has the same strong unstable manifolds as φ_1, one can show that $e^{Bt}\circ G\circ\varphi_{-t}$ is again in this same function space, and so equals G. Hence, G is a conjugacy for the flow. We leave the details of this case to the reader. □

We are left with proving Proposition 4. For this we will need the following.

LEMMA 5 [HP] (fiber contraction theorem). *Let X be a topological space and Y a complete metric space. Let $\Psi: X\times Y\to X\times Y$ be a continuous function of the form $\Psi(x,y)=(\psi(x),\Psi_x(y))$. Suppose that ψ has a globally attracting fixed point $p\in X$ and, for each $x\in X$, that $\Psi_x: Y\to Y$ is Lipschitz with* $\mathrm{Lip}(\Psi_x)\le k$ *for some $k<1$ independent of x. Then Ψ has a globally attracting fixed point $(p,P)\in X\times Y$.*

Proof of Proposition 4. Let $\mathbb{E}^w := \mathbb{R}^l\times\{0\}$ denote the weak unstable subspace of $(Df)_0$, and let $\mathbb{E}^{uu} := \{0\}\times\mathbb{R}^{n-l}$ be the strong unstable subspace. Since the weak unstable manifold is tangent to $\mathbb{E}^w$, locally it is the graph of a C^r function $h:\mathbb{E}^w(\delta)\to\mathbb{E}^{uu}$. Make an initial change of variable by $H(x,y)=(x,y+h(x))$. $H^{-1}\circ f\circ H$ leaves $\mathbb{E}^w$ invariant and has the form $[f_1(x), f_2(x,y+h(x))-h\circ f_1(x)]$. Although $f\in C^{r+1}$, h is only C^r. Therefore, $H^{-1}\circ f\circ H$ is only C^r. On the other hand, the first component, f_1, is clearly still C^{r+1}, and the second component is C^{r+1} with respect to y in a very strong sense.

LEMMA 6. *For $k\le r$, $D_2D^k(H^{-1}\circ f\circ H)$ exists and is C^{r-k} in both variables.*

Proof. $(H^{-1}\circ f\circ H)(x,y)=(f\circ H)(x,y)-(0,h\circ f_1(x))$. This is C^r and so, by the chain rule,

$$D^k(H^{-1}\circ f\circ H)_{(x,y)}=(Df)_{H(x,y)}(D^kH)_{(x,y)}-D^k(0,h\circ f_1)_x+R_{(x,y)},$$

where R is a polynomial in the derivatives of f up to order k and the derivatives of H up to order only $k-1$. Therefore, R is C^{r-k+1} in x and y. (D^kH) and $D^k(0,h\circ f_1)$ depend only on x, so

$$D_2D^k(H^{-1}\circ f\circ H)=(D^2f)_{H(x,y)}[(D_2H)_{(x,y)},(D^kH)_{(x,y)}]+(D_2R)_{(x,y)}.$$

The least smooth terms here are (D^kH) and (D_2R), which are C^{r-k}. □

Consequently, without loss of generality we can assume that near 0 our original diffeomorphism has the form:

$$f(x,y) = (Ax+\rho_1(x), By+\rho_2(x,y)),$$

where $\rho_1 \in C^{r+1}$; ρ_2 is C^r, and C^{r+1} in y (in the above sense); $\rho_2(x,0) \equiv 0$ (f leaves $\mathbb{E}^w$ invariant); and if $\rho := (\rho_1, \rho_2)$, then $\rho(0) = 0$ and $(D\rho)_0 = 0$ $[(Df)_0 = A \oplus B]$. Clearly conjugating f by H also conjugates f^{-1} by H, and in the new coordinates f^{-1} leaves $\mathbb{E}^w$ invariant. It follows that f^{-1} can be written similarly:

$$f^{-1}(x,y) = (A^{-1}x+\sigma_1(x), B^{-1}y+\sigma_2(x,y)),$$

where $\sigma := (\sigma_1, \sigma_2)$ satisfies precisely the same properties as ρ. For notational convenience, write $\varphi := f^{-1} = A^{-1} \oplus B^{-1} + \sigma$.

Since $|\mathrm{sp}(B)| = \beta$, for any $\epsilon > 0$ we can choose an equivalent, adapted norm on $\mathbb{E}^{uu}$ such that $\|B\| < \beta+\epsilon$ and $\|B^{-1}\| < \beta^{-1}+\epsilon$. Likewise, $|\mathrm{sp}(A^{-1})| <$ 1, and so there is an equivalent norm on $\mathbb{E}^w$ such that $\|A^{-1}\| < 1$. Hence, given any $0 < \theta \leq 1$ and ϵ sufficiently small, we can choose these such that $\|B\|(\|B^{-1}\|+\epsilon)(\|A^{-1}\|+\epsilon)^\theta \leq \kappa < 1$. Fix this ϵ (and these new norms). Take the norm on $\mathbb{E}^w \times \mathbb{E}^{uu}$ to be the box norm, $\|(x,y)\| := \max\{\|x\|, \|y\|\}$.

$(D\rho)_p$ and $(D\sigma)_p$ are continuous, and both are zero at the origin. Therefore, $\|(D\rho)_p\|, \|(D\sigma)_p\| \leq \epsilon$ for all p in a sufficiently small neighborhood of the origin. By rescaling, we can assume for convenience that this neighborhood contains the unit ball $\mathbb{E}(1) = \{(x,y) : \|(x,y)\| \leq 1\} = \mathbb{E}^w(1) \times \mathbb{E}^{uu}(1)$. Since 0 is a repelling fixed point for f, a conjugacy will be uniquely determined by its values on any neighborhood of the origin. Therefore, it will suffice to find one on the unit ball.

We look for a conjugacy of the form $G(x,y) = (x, y+g(x,y))$ which leaves $\mathbb{E}^w$ invariant and has derivative I at the origin. That is, $g(x,0) \equiv 0$ and $(Dg)_0 = 0$. Let $F(x,y) := (f_1(x), By)$. If G conjugates f to F, then $G = F \circ G \circ f^{-1}$. Expanding, we have $g = Bg \circ f^{-1} + B\sigma_2 = Bg \circ \varphi + B\sigma_2$. For continuous $g : \mathbb{E}^w(1) \times \mathbb{E}^{uu}(1) \to \mathbb{E}^{uu}$, define $\Gamma(g) = Bg \circ \varphi + B\sigma_2$. A fixed point of Γ then corresponds to a semiconjugacy $G = \mathrm{id} + (0,g)$ from f to F. If g is C^1 and $(Dg)_0 = 0$, then G is a diffeomorphism on a neighborhood of zero. However, as already observed, this then determines G, and so G is a global diffeomorphism, and in particular a conjugacy.

Consequently, we look for a fixed point of Γ. Let

$$\mathcal{G}_0 := \{g \in C^0(\mathbb{E}(1), \mathbb{E}^{uu}) : g(x,0) \equiv 0\}.$$

For $g \in \mathcal{G}_0$, define $|||g||| := \sup\{\|g(x,y)\| / (\|y\| \cdot \|(x,y)\|^\theta) : y \neq 0\}$, where $|||g|||$ is possibly infinite. Set $\mathcal{G} := \{g \in \mathcal{G}_0 : |||g||| < \infty\}$. $|||\cdot|||$ is clearly a norm on $\mathcal{G}$, and it dominates the usual sup norm, $\|g\|_{\sup} \leq |||g|||$. From this it follows that $(\mathcal{G}, |||\cdot|||)$ is a Banach space.

LEMMA 7. *Let $g \in \mathcal{G}_0$. Suppose that $(D_2 g)$ exists and is θ-Hölder, and that $(D_2 g)_0 = 0$. Then $g \in \mathcal{G}$. In particular, if $g \in C^{1+\theta}$ and $(Dg)_0 = 0$, then $g \in \mathcal{G}$.*

Proof. If $g \in \mathcal{G}_0$, then $g(x,0) \equiv 0$. Therefore, $\|g(x,y)\| = \|g(x,y) - g(x,0)\|$, and by the mean value theorem, $\|g(x,y)\| \le (\sup_{0\le t\le 1}\|(D_2 g)_{(x,ty)}\|)\|y\|$. However, $(D_2 g)$ is θ-Hölder and $(D_2 g)_0 = 0$. Therefore, $\|(D_2 g)_{(x,ty)}\| \le K\|(x,ty)\|^\theta \le K\|(x,y)\|^\theta$ for some $K > 0$. Consequently, $|||g||| \le K$. □

LEMMA 8. *Γ is a contraction of $\mathcal{G}$ into itself.*

Proof. By our choices, $\|(D\varphi)_p\| \le \|A^{-1} \oplus B^{-1}\| + \|(D\sigma)_p\| \le \|A^{-1}\| + \epsilon < 1$ for all p in $\mathbb{E}^w(1) \times \mathbb{E}^{uu}(1)$. Consequently, $\varphi = f^{-1}$ maps $\mathbb{E}^w(1) \times \mathbb{E}^{uu}(1)$ inside itself, and $\Gamma(g)$ is well defined for any $g \in C^0(\mathbb{E}^w(1) \times \mathbb{E}^{uu}(1), \mathbb{E}^{uu})$. If $g \in \mathcal{G}_0$ then, since φ leaves $\mathbb{E}^w$ invariant, implying $\varphi(x,0) = (x',0)$, we have

$$\Gamma(g)(x,0) = B[g(x',0) + \sigma_2(x,0)] \equiv 0 \quad \text{and} \quad \Gamma(g) \in \mathcal{G}_0.$$

Now suppose $g \in \mathcal{G}$. $|||\Gamma(g)||| \le \|B\|\,|||g \circ \varphi||| + \|B\|\,|||\sigma_2|||$. By construction, $\sigma_2 \in \mathcal{G}_0$, $(D_2\sigma_2)_0 = 0$, and $(D_2\sigma_2)$ is C^r $(r \ge 1)$. Therefore, $\sigma_2 \in \mathcal{G}$ by Lemma 7, and we need only show $|||g \circ \varphi||| < \infty$ to conclude $\Gamma(g) \in \mathcal{G}$.

By definition, $\|(g\circ\varphi)(x,y)\| = \|g(\varphi(x,y))\| \le |||g|||\cdot\|\varphi_2(x,y)\|\cdot\|\varphi(x,y)\|^\theta$. Since $\sigma_2(x,0) = 0$, $\|\varphi_2(x,y)\| = \|B^{-1}y + \sigma_2(x,y) - \sigma_2(x,0)\| \le (\|B^{-1}\| + \epsilon)\|y\|$. Likewise, $\|\varphi(x,y)\| \le (\|A^{-1}\| + \epsilon)\|(x,y)\|$. Hence, $|||g \circ \varphi||| \le (\|B^{-1}\| + \epsilon) \times (\|A^{-1}\| + \epsilon)^\theta |||g|||$, and $\Gamma(g) \in \mathcal{G}$.

Finally, let $g_1, g_2 \in \mathcal{G}$. Then $\Gamma(g_2) - \Gamma(g_1) = B(g_1 - g_2)\circ\varphi$, and by the previous argument $|||\Gamma(g_2) - \Gamma(g_1)||| \le \|B\|(\|B^{-1}\| + \epsilon)(\|A^{-1}\| + \epsilon)^\theta |||g_1 - g_2||| \le \kappa |||g_1 - g_2|||$. Since $\kappa < 1$, Γ is a contraction. □

Since Γ is a contraction of the Banach space $\mathcal{G}$, by the contraction mapping theorem we can conclude that there exists a unique $\bar{g} \in \mathcal{G}$ such that $\Gamma(\bar{g}) = \bar{g}$. As observed, $G := \mathrm{id} + (0, \bar{g})$ is then a semiconjugacy from f to F. We need to show that $\bar{g}$ is C^r. However, first we show that under the assumption of smoothness, G is a conjugacy and is unique.

LEMMA 9. *If $G = \mathrm{id} + (0,g)$ is a C^1 semiconjugacy from f to F with $(Dg)_0 = 0$, then G is a conjugacy. If, in addition, $(D_2 g)$ is θ-Hölder and G leaves $\mathbb{E}^w$ invariant, then $g \in \mathcal{G}$ and is the unique fixed point of Γ.*

Proof. If $(Dg)_0 = 0$, then $(DG)_0 = I$ and so G is locally a diffeomorphism, which means that G is a conjugacy near zero. On the other hand, $G = F \circ G \circ f^{-1}$, and so $G = F^n \circ G \circ f^{-n}$ for all n. But f is an expansion, so for n sufficiently large, $f^{-n}[\mathbb{E}(1)]$ will be contained in the neighborhood of zero on which G is a diffeomorphism. Therefore, $F^n \circ G \circ f^{-n}$ is a diffeomorphism, and G is a conjugacy. Now, if G leaves $\mathbb{E}^w$ invariant then $g \in \mathcal{G}_0$ and, by Lemma 7, $g \in \mathcal{G}$. Therefore, g must be the fixed point of Γ. □

We are now prepared to prove that $\bar{g}$ is C^r. We begin with C^1, and because of the dissimilarity in the differentiability of f with respect to its variables, the two partial derivatives must be handled separately. If $\bar{g}$ is the fixed point of Γ and $\bar{g}$ is C^1, then we must have $(D\bar{g})_p = D(\Gamma\bar{g})_p = B(D\bar{g})_{\varphi(p)}(D\varphi)_p + B(D\sigma_2)_p$. For L in $C^0(\mathbb{E}(1), L(\mathbb{E}, \mathbb{E}^{uu}))$, define $\Gamma_1(L)$ by

$(\Gamma_1 L)_p := BL_{\varphi(p)}(D\varphi)_p + B(D\sigma_2)_p$. $\varphi = f^{-1}$ is C^r $(r \geq 1)$. Therefore, $\Gamma_1 L$ is continuous if L is. We want to show Γ_1 has a fixed point for some subset of $C^0(\mathbb{E}(1), L(\mathbb{E}, \mathbb{E}^{uu}))$ and that this is the derivative of $\bar{g}$.

Define $\mathcal{E} \subset C^0(\mathbb{E}(1), L(\mathbb{E}^{uu}, \mathbb{E}^{uu}))$ and $\mathfrak{N} \subset C^0(\mathbb{E}(1), L(\mathbb{E}^{w}, \mathbb{E}^{uu}))$ by

$$E \in \mathcal{E} \Leftrightarrow |||E||| := \sup\{\|E_p\|/\|p\|^\theta : p \neq 0\} < \infty$$

and

$$N \in \mathfrak{N} \Leftrightarrow |||N||| := \sup\{\|N_{(x,y)}\|/\|y\| : y \neq 0\} < \infty.$$

This of course implies if $E \in \mathcal{E}$ then $E_0 = 0$, and if $N \in \mathfrak{N}$ then $N_{(x,0)} \equiv 0$. As with $\mathcal{G}$, these are norms which dominate the respective sup norms, and which make $\mathcal{E}$ and $\mathfrak{N}$ Banach spaces.

Set $\mathcal{L} = \mathcal{E} \times \mathfrak{N}$. $\mathcal{L}$ is naturally identified with a subset of $C^0(\mathbb{E}(1), L(\mathbb{E}, \mathbb{E}^{uu}))$, $(E, N) \leftrightarrow [N\ E]$; that is, for $(u, v) \in \mathbb{E}^w \times \mathbb{E}^{uu}$, $(E, N)(u, v) = Nu + Ev$.

LEMMA 10. *Γ_1 is a fiber contraction of $\mathcal{E} \times \mathfrak{N}$ covering a contraction on $\mathcal{E}$.*

Proof. $(D\varphi)$ is triangular, so $[N\ E](D\varphi) = [(N(D_1\varphi_1) + E(D_1\varphi_2))\ E(D_2\varphi_2)]$, or

$$\Gamma_1(E, N)_p = (BE_{\varphi(p)}(D_2\varphi_2)_p + B(D_2\sigma_2)_p,$$
$$BN_{\varphi(p)}(D_1\varphi_1)_p + BE_{\varphi(p)}(D_1\varphi_2)_p + B(D_1\sigma_2)_p).$$

Therefore, $\Gamma_1(E, N) =: \Psi(E, N) = (\psi(E), \Psi_E(N))$ has the form of a fiber map. For $\mathcal{E}$ we proceed as we did for $\mathcal{G}$. $(D_2\sigma_2)$ is C^r $(r \geq 1)$, and $(D_2\sigma_2)_0 = 0$. Therefore, $\|(D_2\sigma_2)_p\| \leq K\|p\|$ and $B(D_2\sigma_2) \in \mathcal{E}$. Moreover,

$$\|BE_{\varphi(p)}(D_2\varphi_2)_p\| \leq \|B\|(|||E||| \cdot \|\varphi(p)\|^\theta)(\|B^{-1}\| + \epsilon)$$
$$\leq \|B\|(\|B^{-1}\| + \epsilon)(\|A^{-1}\| + \epsilon)^\theta |||E||| \cdot \|p\|^\theta \leq \kappa |||E||| \cdot \|p\|^\theta.$$

Therefore, $|||BE_\varphi(D_2\varphi_2)||| \leq \kappa |||E|||$, and ψ maps $\mathcal{E}$ into itself. Likewise, $|||\psi(E_1) - \psi(E_2)||| \leq \kappa |||E_1 - E_2|||$ and $\kappa < 1$, so ψ is a contraction.

We need to show that Ψ_E maps $\mathfrak{N}$ into itself. Observe $(D_1\varphi_2)_p = (D_1\sigma_2)_p$. By assumption, $(D_2 D\sigma_2)$ exists and is continuous. In particular, $(D_2 D_1\sigma_2)$ exists and is continuous. Moreover, $\sigma_2(x, 0) \equiv 0$, so $(D_1\sigma_2)_{(x,0)} \equiv 0$. Therefore, by the mean value theorem,

$$\|(D_1\sigma_2)_{(x,y)}\| = \|(D_1\sigma_2)_{(x,y)} - (D_1\sigma_2)_{(x,0)}\|$$
$$\leq C\|y\| \quad \text{for } C = \sup_p \|(D_2 D_1\sigma_2)_p\|.$$

As a result, since $\|E_{\varphi(p)}\| \leq |||E||| \cdot \|\varphi(p)\|^\theta \leq |||E|||$ $(\|\varphi(p)\| \leq 1)$, both $B(D_1\sigma_2)$ and $BE_\varphi(D_1\varphi_2)$ are in $\mathfrak{N}$. We have

$$\|BN_{\varphi(p)}(D_1\varphi_1)_p\| \leq \|B\|(|||N||| \cdot \|\varphi_2(p)\|)(\|A^{-1}\| + \epsilon),$$

and again φ_2 is C^1 with $\varphi_2(x, 0) \equiv 0$. Therefore, $\|\varphi_2(x, y)\| \leq (\|B^{-1}\| + \epsilon)\|y\|$, and so $|||BN_\varphi(D_1\varphi_1)||| < \infty$. Hence, Ψ_E maps $\mathfrak{N}$ into itself. This also shows that $\Psi_2(E, N) = \Psi_E(N)$ is a bounded affine map in (E, N), and so $\Psi = \Gamma_1$

is continuous. Finally, Ψ_E contracts fibers by $\|B\|(\|B^{-1}\|+\epsilon)(\|A^{-1}\|+\epsilon) \leq \kappa < 1$. □

By the fiber contraction theorem, there exists a unique $L \in \mathcal{L}$ that is a globally attracting fixed point of Γ_1. Comparing to the usual sup norm on $\mathcal{L}$, we have $\|(E,N)\|_{\text{sup}} \leq \|E\|_{\text{sup}} + \|N\|_{\text{sup}} \leq |||E||| + |||N|||$. Consequently, L is globally attracting (in $\mathcal{L}$) relative to the sup norm. Now suppose $g \in \mathcal{G}$ is C^1 and $(Dg) \in \mathcal{L}$ (e.g., $g \equiv 0$ satisfies this). By construction, $\Gamma_1(Dg) = D(\Gamma g)$. Hence, the orbit of $(g,(Dg))$ in $\mathcal{G} \times \mathcal{L}$ under $(\Gamma \times \Gamma_1)$ is a sequence of 1-jets $(g_i,(Dg_i))$ that converge in $\mathcal{G} \times \mathcal{L}$ to $(\bar{g}, L)$. However, this implies uniform convergence in both factors, so $(g_i)_{i \geq 1}$ is converging in the C^1 topology. Consequently, $\bar{g}$ is C^1 and $(D\bar{g}) = L$.

For $1 < k \leq r$ we proceed by induction in much the same fashion. For each $i \leq k$, we choose $\mathcal{L}_i \subset C^0(\mathbb{E}(1), L^i_{\text{sym}}(\mathbb{E}, \mathbb{E}^{uu}))$ and a norm $|||\cdot|||$ on $\mathcal{L}_i$ which makes $\mathcal{L}_i$ a Banach space and which dominates the sup norm, $\|L^{(i)}\|_{\text{sup}} \leq |||L^{(i)}|||$ (so convergence in $\mathcal{L}_i$ implies uniform convergence). For each $i < k$, we assume a continuous function Γ_i mapping $\mathcal{L}_1 \times \cdots \times \mathcal{L}_i$ into itself such that Γ_i has a globally attracting fixed point, and if g is C^i and $j^i(g) \in \mathcal{G} \times \mathcal{L}_1 \times \cdots \times \mathcal{L}_i$ then $(\Gamma \times \Gamma_i)(j^i g) = j^i(\Gamma g)$, where $j^i(\cdot)$ denotes the ith jet of $(\cdot)$. The previous case showed these assumptions are satisfied for $i = 1$ if we set $|||(E,N)||| := |||E||| + |||N|||$.

Suppose we can extend this to $i = k$. Then, by the same argument as for the C^1 case, we will have that $\bar{g}$ is C^k. For if $g \in \mathcal{G}$ is C^k and $j^k(g) \in \mathcal{G} \times \mathcal{L}_1 \times \cdots \times \mathcal{L}_k$ (again $g \equiv 0$ will do), then the orbit of $j^k(g)$ under $\Gamma \times \Gamma_k$ will be a sequence of k-jets, $j^k(g_i)$. These converge uniformly to the unique fixed point of $\Gamma \times \Gamma_k$. Consequently, $(g_i)_{i \geq 1}$ is converging in the C^k topology, so $\bar{g}$ is C^k and the fixed point of $\Gamma \times \Gamma_k$ is $j^k(\bar{g})$.

For $i > 1$, define $\mathcal{L}_i$ as follows. Let $\pi_1: \mathbb{E}^w \times \mathbb{E}^{uu} \to \mathbb{E}^w \times \{0\}$ be the usual projection. Define $|||L^{(i)}||| := \max\{\|L^{(i)}\|_{\text{sup}}, \sup\{\|L^{(i)}_{(x,y)} \circ [\pi_1^i]\|/\|y\| : y \neq 0\}\}$; then $L^{(i)} \in \mathcal{L}_i$ if and only if $|||L^{(i)}||| < \infty$. Clearly $\|L^{(i)}\|_{\text{sup}} \leq |||L^{(i)}|||$, and if $L^{(i)} \in \mathcal{L}_i$ then $L^{(i)}_{(x,0)} \circ [\pi_1^i] \equiv 0$. One can again see that $|||\cdot|||$ is a norm, and $(\mathcal{L}_i, |||\cdot|||)$ is a Banach space.

Given Γ_i $(i < k)$ satisfying the above assumptions, construct Γ_k as follows. For $(L^{(1)}, \ldots, L^{(k)}) \in \mathcal{L}_1 \times \cdots \times \mathcal{L}_k$ and $p \in \mathbb{E}(1)$, let $\gamma_p: \mathbb{E} \to \mathbb{E}^{uu}$ be the polynomial $\gamma_p(q) := L^{(1)}_{\varphi(p)}(q - \varphi(p)) + \cdots + L^{(k)}_{\varphi(p)}(q - \varphi(p))^{(k)}$. Define $\Gamma_k(L^{(1)}, \ldots, L^{(k)})(p) := [\Gamma_{k-1}(L^{(1)}, \ldots, L^{(k-1)})(p), D^k(\Gamma\gamma_p)_p]$. If g is C^k and $(L^{(1)}, \ldots, L^{(k)}) = ((Dg), \ldots, (D^k g))$, then γ_p and g have the same derivatives at $\varphi(p)$ (the constant term does not matter). Hence, $D^k(\Gamma\gamma_p)_p = D^k(\Gamma g)_p$, and $\Gamma_k((Dg), \ldots, (D^k g)) = (D(\Gamma g), \ldots, D^k(\Gamma g))$ by the assumptions on Γ_{k-1}. Consequently, $(\Gamma \times \Gamma_k)(j^k g) = j^k(\Gamma g)$.

We need to show that Γ_k maps into $\mathcal{L}_1 \times \cdots \times \mathcal{L}_k$ and has an attracting fixed point. For the former we need $D^k(\Gamma\gamma_p)_p \in \mathcal{L}_k$ for any $(L^{(1)}, \ldots, L^{(k)}) \in \mathcal{L}_1 \times \cdots \times \mathcal{L}_k$. We have $D^k(\Gamma\gamma_p)_p = B(D^k(\gamma_p \circ \varphi)_p + (D^k \sigma_2)_p)$. By assumption, $(D_2 D^k \sigma_2)_p$ exists, and it and $(D^k \sigma_2)_p$ are continuous. Moreover,

$(D^k\sigma_2)_p \circ [\pi_1^k] = (D_1^k\sigma_2)_p$ and $\sigma_2(x, 0) \equiv 0$, so we have $(D_1^k\sigma_2)_{(x,0)} \equiv 0$. Therefore $\|(D^k\sigma_2)_{(x,y)} \circ [\pi_1^k]\| = \|(D_1^k\sigma_2)_{(x,y)}\| \le \|(D_2 D^k\sigma_2)\|_{\sup}\|y\|$ and $(D^k\sigma_2) \in \mathfrak{L}_k$. From the higher-order chain rule, $D^k(\gamma_p \circ \varphi)_p$ is the sum of terms of the form

$$(D^i\gamma_p)_{\varphi(p)}[(D^{j_1}\varphi)_p, \ldots, (D^{j_i}\varphi)_p] = L^{(i)}_{\varphi(p)}[(D^{j_1}\varphi)_p, \ldots, (D^{j_i}\varphi)_p],$$

where $1 \le i \le k$ and $j_1 + \cdots + j_i = k$. Therefore, it will be sufficient to show these are in $\mathfrak{L}_k$. These are clearly continuous functions of p. Moreover,

$$\|L^{(i)}_{\varphi(p)}[(D^{j_1}\varphi)_p, \ldots, (D^{j_i}\varphi)_p]\|_{\sup} \le [\|(D^{j_1}\varphi)\|_{\sup} \cdots \|(D^{j_i}\varphi)\|_{\sup}]\,|||L^{(i)}|||.$$

Consider $L^{(i)}_{\varphi(p)}[(D^{j_1}\varphi)_p, \ldots, (D^{j_i}\varphi)_p] \circ [\pi_1^k] = L^{(i)}_{\varphi(p)}[(D_1^{j_1}\varphi)_p, \ldots, (D_1^{j_i}\varphi)_p]$.

$$(D_1^j\varphi)_p = [(D_1^j\varphi_1)_p \;\; 0]^{\mathrm{T}} + [0 \;\; (D_1^j\varphi_2)_p]^{\mathrm{T}},$$

which for notational convenience we will write as $(D_1^j\varphi_1)_p + (D_1^j\varphi_2)_p$. Using the multilinearity of $L^{(i)}_{\varphi(p)}$, we have

$$\begin{aligned} &L^{(i)}_{\varphi(p)}[(D_1^{j_1}\varphi)_p, \ldots, (D_1^{j_i}\varphi)_p] \\ &\quad = L^{(i)}_{\varphi(p)}[(D_1^{j_1}\varphi_1)_p, \ldots, (D_1^{j_i}\varphi_1)_p] \\ &\qquad + \sum_{0 \le t \le i} L^{(i)}_{\varphi(p)}[\ldots, (D_1^{j_{t-1}}\varphi_1)_p, (D_1^{j_t}\varphi_2)_p, (D_1^{j_{t+1}}\varphi)_p, \ldots], \end{aligned}$$

$$\|(D_1^j\varphi_2)_{(x,y)}\| = \|(D_1^j\sigma_2)_{(x,y)}\| \le \|(D_2 D^j\sigma_2)\|_{\sup}\|y\|,$$

$$L^{(i)}_{\varphi(p)}[(D_1^{j_1}\varphi_1)_p, \ldots, (D_1^{j_i}\varphi_1)_p] = L^{(i)}_{\varphi(p)} \circ [\pi_1^i] \circ [(D_1^{j_1}\varphi)_p, \ldots, (D_1^{j_i}\varphi)_p],$$

and $\|L^{(i)}_{\varphi(x,y)} \circ [\pi_1^i]\| \le |||L^{(i)}||| \cdot \|\varphi_2(x,y)\|$. Since $\|\varphi_2(x,y)\| \le (\|B^{-1}\| + \epsilon)\|y\|$, every term has a factor of $\|y\|$. Consequently,

$$\|L^{(i)}_{\varphi(p)}[(D^{j_1}\varphi)_p, \ldots, (D^{j_i}\varphi)_p] \circ [\pi_1^k]\| \le (\text{constant})\,|||L^{(i)}||| \cdot \|y\|.$$

Since this holds for all the terms in $D^k(\gamma_p \circ \varphi)_p$, we have $D^k(\gamma_p \circ \varphi)_p \in \mathfrak{L}_k$ and $D^k(\Gamma\gamma_p)_p \in \mathfrak{L}_k$. The above computations also showed that $D^k(\Gamma\gamma_p)_p$ is a bounded affine map in the $L^{(i)}$, so Γ_k is continuous.

We are left showing that Γ_k has an attracting fixed point. Recall from the induction hypotheses that Γ_{k-1} has a globally attracting fixed point. To show that Γ_k has an attracting fixed point, we split the $\mathfrak{L}_k$ term into two pieces as we did for the $k = 1$ case. Since elements in $\mathfrak{L}_k$ are symmetric, there is considerable redundancy in their entries, and it is not necessary to observe all of the entries to know that a sequence in $\mathfrak{L}_k$ is convergent. In particular, for any $L^{(k)} \in \mathfrak{L}_k$, define subblocks $N := L^{(k)} \circ [\pi_1^k]$ and $E := L^{(k)} \circ [\pi_2, I, \ldots, I]$; in other words, for $v_i = w_i + u_i \in \mathbb{E}$ $(i = 1, \ldots, k)$, $w_i = \pi_1(v_i)$, and $u_i = \pi_2(v_i)$, let $N[v_1, \ldots, v_k] = L^{(k)}[w_1, \ldots, w_k]$ and $E[v_1, \ldots, v_k] = L^{(k)}[u_1, v_2, \ldots, v_k]$.

Let $\mathfrak{N} := \{N = L^{(k)} \circ [\pi_1^k] : L^{(k)} \in \mathfrak{L}_k\}$ and $\mathcal{E} := \{E = L^{(k)} \circ [\pi_2, I^{k-1}] : L^{(k)} \in \mathfrak{L}_k\}$. Give these the norms inherited from $\mathfrak{L}_k$, and $\mathcal{E} \times \mathfrak{N}$ the Box norm. Let $\Pi : L^{(k)} \mapsto (E, N)$ be the map from $\mathfrak{L}_k$ onto $\mathcal{E} \times \mathfrak{N}$. Since $E \circ [\pi_1^k] = L^{(k)} \circ [\pi_2 \circ \pi_1, \pi_1^{k-1}] \equiv 0$, $|||E||| = \|E\|_{\sup} \le |||L^{(k)}|||$. Likewise,

$$|||N||| = \sup\{\|N_{(x,y)}\| / \|y\| : y \ne 0\} \le |||L^{(k)}|||.$$

Therefore, Π is linear with $\|\Pi\| \le 1$. On the other hand, $L^{(k)}$ can be rewritten in terms of E and N. We have

$$L^{(k)}[v_1, \ldots, v_k] = L^{(k)}[w_1, \ldots, w_k] + \sum_i L^{(k)}[\ldots, w_{i-1}, u_i, v_{i+1}, \ldots],$$

and by symmetry

$$\begin{aligned} L^{(k)}[\ldots, w_{i-1}, u_i, v_{i+1}, \ldots] &= L^{(k)}[u_i, v_{i+1}, \ldots, v_k, w_1, \ldots, w_{i-1}] \\ &= E[v_i, \ldots, v_k, \pi_1(v_1), \ldots]. \end{aligned}$$

Therefore, if P_i denotes the cyclic permutation (to the left) by i places, then $L^{(k)} = N + \sum_{i=0,\ldots,k-1} E \circ P_i \circ [\pi_1^i, I^{k-i}]$, and so Π^{-1} exists and $\|\Pi^{-1}\| \le k+1$. Consequently, Π is an isomorphism between $\mathcal{L}_k$ and $\mathcal{E} \times \mathcal{N}$.

Via this isomorphism, Γ_k induces a map $\bar{\Gamma}_k$ from $\mathcal{L}_1 \times \cdots \times \mathcal{L}_{k-1} \times \mathcal{E} \times \mathcal{N}$ into itself. It will now suffice to show $\bar{\Gamma}_k$ has a globally attracting fixed point. This will be guaranteed by the following lemma together with the fiber contraction theorem.

LEMMA 11. *$\bar{\Gamma}_k$ is a fiber contraction on both*

$$(\times_{i \le k-1} \mathcal{L}_i) \times \mathcal{E} \quad \textit{and} \quad (\times_{i \le k-1} \mathcal{L}_i) \times \mathcal{E} \times \mathcal{N}.$$

Proof. Let $(L^{(1)}, \ldots, L^{(k)}) \in \mathcal{L}_1 \times \cdots \times \mathcal{L}_k$, and let $(E, N) = \Pi[L^{(k)}]$ as above. The $\mathcal{E}$-component of $\bar{\Gamma}_k[L^{(1)}, \ldots, L^{(k-1)}, E, N](p)$ is $D^k(\Gamma\gamma_p)_p \circ [\pi_2, I^{k-1}]$, and $D^k(\Gamma\gamma_p)_p = BL^{(k)}_{\varphi(p)}[(D\varphi)_p^k] +$ terms involving only $(L^{(1)}, \ldots, L^{(k-1)})$. Since $(D\varphi)_p$ is triangular,

$$(D\varphi)_p \circ \pi_2 = \pi_2 \circ (D_2\varphi_2)_p.$$

Hence, $D^k(\Gamma\gamma_p)_p \circ [\pi_2, I^{k-1}] = BE_{\varphi(p)}[(D_2\varphi_2)_p, (D\varphi)_p^{k-1}] +$ terms involving only $(L^{(1)}, \ldots, L^{(k-1)})$. Therefore, $\bar{\Gamma}_k$ is a fiber map on $(\times_{i \le k-1} \mathcal{L}_i) \times \mathcal{E}$. If E, E' are two elements in $\mathcal{E}$, then

$$\begin{aligned} &\|(\bar{\Gamma}_k[\ldots, L^{(k-1)}, E] - \bar{\Gamma}_k[\ldots, L^{(k-1)}, E'])(p)\| \\ &\quad = \|B(E_{\varphi(p)} - E'_{\varphi(p)}) \circ [(D_2\varphi_2)_p, (D\varphi)_p^{k-1}]\| \\ &\quad \le (\beta+\epsilon)(\beta^{-1}+\epsilon)(\|A^{-1}\|+\epsilon)^{k-1} \|E_{\varphi(p)} - E'_{\varphi(p)}\| \\ &\quad \le \kappa(\|A^{-1}\|+\epsilon)^{k-1-\theta} |||E-E'|||. \end{aligned}$$

Since $k \ge 2$, it follows that $\kappa(\|A^{-1}\|+\epsilon)^{k-1-\theta} < 1$ and $\bar{\Gamma}_k$ is a fiber contraction on $(\times_{i \le k-1} \mathcal{L}_i) \times \mathcal{E}$.

We are left with showing $\bar{\Gamma}_k$ contracts on $\mathcal{N}$-fibers. The $\mathcal{N}$-component of $\bar{\Gamma}_k$ is $D^k(\Gamma\gamma_p)_p \circ [\pi_1^k] = BL^{(k)}_{\varphi(p)}[(D\varphi)_p^k] \circ [\pi_1^k] +$ terms involving only $(L^{(1)}, \ldots, L^{(k-1)})$. Hence $BL^{(k)}_{\varphi(p)}[(D\varphi)_p^k] \circ [\pi_1^k] = BL^{(k)}_{\varphi(p)}[(D_1\varphi)_p^k] = BN_{\varphi(p)}[(D_1\varphi_1)_p^k] +$ terms involving E. Therefore,

$$\|(\bar{\Gamma}_k[\ldots, E, N] - \bar{\Gamma}_k[\ldots, E, N'])(p)\| \le (\beta+\epsilon)(\|A^{-1}\|+\epsilon)^k \|N_{\varphi(p)} - N'_{\varphi(p)}\|.$$

However,

$$\|N_{\varphi(x,y)} - N'_{\varphi(x,y)}\| \le |||N-N'||| \cdot \|\varphi_2(x,y)\| \le (\beta^{-1}+\epsilon)|||N-N'||| \cdot \|y\|.$$

Consequently,

$$\|\|\bar{\Gamma}_k[\ldots,E,N]-\bar{\Gamma}_k[\ldots,E,N']\|\| \leq (\beta+\epsilon)(\beta^{-1}+\epsilon)(\|A^{-1}\|+\epsilon)^k \|\|N-N'\|\| < \kappa\|\|N-N'\|\|,$$

and again this is a fiber contraction. ❑

We now observe that this completes the proof of Proposition 4. By induction we have shown that $\bar{g}$ is C^k for each k, $1 \leq k \leq r$, where $\bar{g} \in \mathcal{G}$ is the unique fixed point of Γ. Moreover, $G = \mathrm{id} + (0, \bar{g})$ is a semiconjugacy from f to F. Since $\bar{g}$ is at least C^1 and $\bar{g}(x, 0) \equiv 0$, then certainly $(D_1\bar{g})_0 = 0$. Since $\bar{g} \in \mathcal{G}$ and $\bar{g}(0) = 0$, we have $\|\bar{g}(0, y) - \bar{g}(0)\| \leq \|y\|^{1+\theta}$, and so $(D_2\bar{g})_0 = 0$. Therefore, $(D\bar{g})_0 = 0$ and, by Lemma 9, G is a conjugacy. Converting back to the original coordinates, $G \circ H^{-1}$ is then a local conjugacy between our original f and $(f_1(x), By)$ that takes the given weak unstable manifold to the weak unstable plane, $\mathbb{E}^w$. Both G and H are C^r, so $G \circ H^{-1}$ is C^r. Moreover, since $H^{-1}(x, y) = (x, y - h(x))$ and since the graph of h is the weak unstable manifold, implying $h(0) = 0$ and $(Dh)_0 = 0$, we have $D(G \circ H^{-1})_0 = I$. We can see directly that $G \circ H^{-1}$ has the general form $\mathrm{id} + (0, g)$. Therefore, we have exhibited the desired conjugacy.

Finally, suppose $G' = \mathrm{id} + (0, g')$ is a C^1 semiconjugacy with the same properties. Then $G' \circ H$ has the form $\mathrm{id} + (0, g)$, leaves $\mathbb{E}^w$ invariant, is C^1, and $D(G' \circ H)_0 = I$, which implies that $(Dg)_0 = 0$. If $(D_2 g')$ is θ-Hölder for some $\theta > 0$, then $(D_2 g)_p = (D_2 g')_{H(p)}$ is also θ-Hölder. In the above proof θ was any number with $1 \geq \theta > 0$. Therefore, for this specific θ, we have by Lemma 9 that $g = \bar{g} \in \mathcal{G}(\theta)$ is the unique fixed point of Γ on $\mathcal{G}(\theta)$. On the other hand, the sets $\mathcal{G}(\theta)$ are a nested family, so if $\bar{g} \in \mathcal{G}(\theta_1) \subset \mathcal{G}(\theta_2)$ is a fixed point of Γ on $\mathcal{G}(\theta_1)$, then it is also a fixed point of Γ on $\mathcal{G}(\theta_2)$. Hence $\bar{g}$ does not depend upon θ, and so $G' = G \circ H^{-1}$ and the conjugacy is unique.

References

[B1] G. R. Belitskii, *Functional equations and conjugacy of local diffeomorphisms of a finite smoothness class,* Functional Anal. Appl. 7 (1973), 268–277.

[B2] ———, *Equivalence and normal forms of germs of smooth mappings,* Russian Math. Surveys 11 (1978), 107–177.

[H] P. Hartman, *On local homeomorphisms of Euclidean space,* Bol. Soc. Mat. Mexicana (2) 5 (1960), 220–241.

[HP] M. Hirsch and C. Pugh, *Stable manifolds and hyperbolic sets,* Proc. Sympos. Pure. Math. 14, pp. 133–164, Amer. Math. Soc., Providence, RI, 1970.

[HPS] M. Hirsch, C. Pugh, and M. Shub, *Invariant manifolds,* Lecture Notes in Math. 583, Springer, New York, 1977.

[M] P. McSwiggen, *On the smoothness of strong unstable foliations,* preprint.

[P] H. Poincaré, *Oeuvres,* vol. 1, XCIX–CV.

[Se] G. Sell, *Smooth linearization near a fixed point,* Amer. J. Math. 107 (1985), 1035–1091.

[S1] S. Sternberg, *Local contractions and a theorem of Poincaré,* Amer. J. Math. 79 (1957), 809–824.

[S2] ———, *On the structure of local homeomorphisms of Euclidean n-space,* Amer. J. Math. 80 (1958), 623–631.

[St] D. Stowe, *Linearization in two dimensions,* J. Differential Equations 63 (1986), 183–226.

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