On the Sum of Monotone Operators

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1. Introduction

Let X be a reflexive Banach space. Troyanski [28], using Asplund's averaging technique [1], has shown by means of a theorem of Lindenstrauss that there exists an equivalent norm on X that is everywhere Fréchet differentiable except at the origin and whose polar norm on its dual X^* is everywhere Fréchet differentiable except at the origin. For notational simplicity, we may assume throughout this paper that the given norm on X already has these special properties.

A set-valued operator $T: X \to X^*$ is a function sending each $x \in X$ to a (possibly empty) subset Tx of X^*. For a set-valued operator $T: X \to X^*$, the domain, the range, the graph, and the inverse of T are denoted, respectively, by

$$D(T) := \{x \in X; T(x) \neq \emptyset\},$$

$$R(T) := \bigcup\{x^* \in X^*; x^* \in T(x)\},$$

$$G(T) := \{(x, x^*) \in X \times X^*; x^* \in T(x)\},$$

and

$$T^{-1}(x^*) := \{x \in X; x^* \in T(x)\}.$$

Recall that a set-valued operator $T: X \to X^*$ is monotone, provided that

$$\langle x' - x, y' - y \rangle \geq 0 \quad \forall (x, y), (x', y') \in G(T).$$

T is maximal monotone provided T is monotone and there exists no other monotone set-valued operator whose graph properly contains the graph of T. Such operators have been studied extensively in both theory and applications; see, for example, the work by Brézis [3] and Phelps [19] and the references cited therein. It is known [20; 22; 23] that the subdifferential operator of a closed proper convex function is maximal monotone. The subdifferential operator of a convex function f on X is defined by

$$\partial f(x) := \{x^* \in X^*; f(z) - f(x) \geq \langle z - x, x^* \rangle \ \forall z \in X\}.$$

T is cyclically monotone, provided that

$$\langle x_1 - x_0, x_0^* \rangle + \langle x_2 - x_1, x_1^* \rangle + \cdots + \langle x_0 - x_m, x_m^* \rangle \leq 0$$

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for any set of pairs $(x_i, x_i^*) \in G(T)$, $i = 0, 1, 2, \ldots, m$. A maximal cyclically monotone operator is one whose graph is not properly contained in the graph of any other cyclically monotone operator. Indeed, Rockafellar [20, Cor. 1] proved that a set-valued operator T is maximal cyclically monotone if and only if $T = \partial f$ for some (closed) proper convex function f. We now introduce two new operators. T is called a (BH)-operator, provided that

$$\inf_{(x,x^*)\in G(T)} \langle x-\bar{x}, x^*-\bar{x}^*\rangle > -\infty \quad \forall \bar{x} \in D(T),\ \forall \bar{x}^* \in R(T).$$

For $Y^* \subset X^*$, T is called a Y^*-operator provided that, for all $\bar{y}^* \in Y^*$, there is some $\bar{x} \in X$ such that

$$\inf_{(x,x^*)\in G(T)} \langle x-\bar{x}, x^*-\bar{y}^*\rangle > -\infty.$$

We remark that the concepts of Y^*-operators and (BH)-operators were first introduced by Brézis [3] and Brézis and Haraux [4] in connection with monotone operators in Hilbert spaces, but they did not use that terminology. In addition, we denote the duality mapping by J, which is the Fréchet gradient of the function $j(x) := \frac{1}{2}\|x\|^2$. Thus, the mapping J assigns to each $x \in X$ the unique $J(x) \in X^*$ such that

$$\langle x, J(x)\rangle = \|x\|^2 = \|J(x)\|^2$$

(see [2; 3; 9; 12; 17; 18; 24; 28]). As is shown, J maps X one-to-one and onto X^* and is norm-to-norm continuous. Also, J is a strictly monotone operator, and for each $x \in X$ we have

$$\tfrac{1}{2}\|z\|^2 \geq \tfrac{1}{2}\|x\|^2 + \langle z-x, J(x)\rangle \quad \forall z \in X.$$

For simplicity, we shall denote by $\operatorname{co} C$, $\operatorname{cl} C$, and $\operatorname{int} C$ the convex hull, the closure, and the interior of a convex subset C of X, respectively. In addition, $\operatorname{ri} C$ denotes the relative interior of C, that is, the interior taken in the closed affine hull of C:

$$\operatorname{ri} C := \{x;\ \exists \epsilon > 0,\ \ni (x+B_\epsilon) \cap \operatorname{cl}\operatorname{aff} C \subset C\},$$

where $B_\epsilon := \{x;\ \|x\| < \epsilon\}$. It is easily seen that $\operatorname{cl}\operatorname{aff} C = x_0 + \operatorname{cl}(\operatorname{span}(C-C))$ for all $x_0 \in C$. Thus, we have

$$\operatorname{ri} C = \{x;\ \exists \epsilon > 0,\ \ni x + (B_\epsilon \cap \operatorname{cl}(\operatorname{span}(C-C)) \subset C\}.$$

The normality operator to a convex set C will be defined by

$$N_C(x) := \begin{cases} \{x^* \in X^*; \langle y-x, x^*\rangle \leq 0\ \forall y \in C\} & \text{if } x \in C, \\ \emptyset & \text{if } x \notin C. \end{cases}$$

Indeed, the normality operator to a closed convex set C is exactly the subdifferential of the indicator function δ_C, where

$$\delta_C(x) := \begin{cases} 0 & \forall x \in C, \\ +\infty & \forall x \notin C. \end{cases}$$

For a set-valued operator $T: X \to X^*$ and a nonempty closed convex subset C of X, the usual variational inequality is given as follows.

PROBLEM VI(T, C). Find $x \in C$ and $x^* \in T(x)$ such that

$$\langle y-x, x^* \rangle \geq 0 \quad \forall y \in C.$$

For the existence of the problem VI(T, C), see [6; 7; 10; 11; 12; 13; 14; 21; 27]. Under this notation, it is known that any solution to the variational inequality VI$(\partial f, C)$ is an optimal solution of the convex programming:

$$\min\{f(x); x \in C\}. \tag{P}$$

To motivate the notion, we remark that x is an optimal solution to (P) if and only if $0 \in \partial(f+\delta_C)(x)$, and in general we have

$$\partial f(x) + N_C(x) = \partial f(x) + \partial \delta_C(x) \subset \partial(f+\delta_C)(x). \tag{1}$$

It is therefore clear that every solution of VI$(\partial f, C)$ is always an optimal solution of (P). On the other hand, once we have shown that

$$\partial f(x) + N_C(x) = \partial f(x) + \partial \delta_C(x) \supset \partial(f+\delta_C)(x), \tag{2}$$

every optimal solution of (P) is also a solution of VI(T, C), with $T = \partial f$. It is known that in a finite-dimensional space, [22, Thm. 23.8] implies the equalities (1) and (2) under the condition

$$\operatorname{ri}(\operatorname{dom} f) \cap \operatorname{ri} C \neq \emptyset,$$

where the term $\operatorname{dom} f$ is the effective domain of f, defined by

$$\operatorname{dom} f := \{x \in X; f(x) < +\infty\}.$$

Indeed, if one can show that $\partial f + N_C$ is a maximal monotone operator then equalities (1) and (2) hold. It is natural now to turn to the sum of maximal monotone operators. In [24, Thm. 2], Rockafellar showed that if T_1, T_2: $X \to X^*$ are maximal monotone set-valued operators, with $\dim X < \infty$, such that $\operatorname{ri} D(T_1) \cap \operatorname{ri} D(T_2) \neq \emptyset$, then $T_1 + T_2$ is maximal monotone. One of the main motivations behind this theorem is that such results make it possible, as Browder has remarked [5, p. 92], to derive theorems about variational inequalities from fundamental theorems about the ranges and effective domains of maximal monotone operators. For details, see for example [4; 6; 8; 10; 11; 12; 13; 20; 21; 23; 24; 25; 27]. In this paper we consider the milder constraint qualification

$$0 \in \operatorname{ri}(\operatorname{co} D(T_1) - \operatorname{co} D(T_2)),$$

under which we will show that $T_1 + T_2$ is maximal monotone, even if X is an infinite-dimensional reflexive Banach space. In general, for $n \geq 2$ we show that $T_1 + T_2 + \cdots + T_n$ is maximal monotone under the constraint qualification

$$(0, 0, \ldots, 0) \in \operatorname{ri}(\operatorname{co} C_2 \times \operatorname{co} C_3 \times \cdots \times \operatorname{co} C_n),$$

where $C_k := D(T_1) \cap D(T_2) \cap \cdots \cap D(T_{k-1}) - D(T_k)$ for all $k = 2, 3, \ldots, n$. Indeed, our main theorem (Corollary 3.5) relaxes and unifies the hypotheses

(i) X is finite-dimensional and $\operatorname{ri} D(T_1) \cap \operatorname{ri} D(T_2) \neq \emptyset$ and
(ii) $D(T_1) \cap \operatorname{int} D(T_2) \neq \emptyset$

appearing in [24]. However, Rockafellar's sum theorem (see Proposition 2.4 below) plays a crucial role in proving our main result. In a finite-dimensional space X, it is clear (by the simple fact that $\operatorname{ri}(A-B) = \operatorname{ri} A - \operatorname{ri} B$ for any convex subsets A and B of X [22, Cor. 6.6.2]) that the condition (i) is equivalent to our constraint qualification. Also, it should be noted that our constraint qualification is definitely weaker than that of Rockafellar. A simple example in a normed space X would be the following: Let A be a closed hyperplane in X, and let B be a 1-dimensional subspace such that $X = A + B$ and $A \cap B = \{0\}$. Let T_1, T_2 be the normality operators for the closed convex sets A, B, respectively. Then $0 \in \operatorname{int}(A-B) = \operatorname{ri}(\operatorname{co} D(T_1) - \operatorname{co} D(T_2))$, but $D(T_1) \cap \operatorname{int} D(T_2) = A \cap \operatorname{int} B = \emptyset$.

2. Preliminary Results

We begin with some general identities and inclusions.

PROPOSITION 2.1. *For any nonempty subsets A, B, C, D of X, one has:*

(a) $\operatorname{co} A \times \operatorname{co} B = \operatorname{co}(A \times B)$;
(b) $\operatorname{co} A + \operatorname{co} B = \operatorname{co}(A+B)$; *and*
(c) $A \times B - C \times D = (A-C) \times (B-D)$.

Moreover, if A and B are convex, then:

(d) $\operatorname{ri} A \cap \operatorname{ri} B \subset \operatorname{ri}(A \cap B)$;
(e) $\operatorname{ri}(A \times B) \subset \operatorname{ri} A \times \operatorname{ri} B$;
(f) *whenever* $\operatorname{ri} A \neq \emptyset$, *one has* $z \in \operatorname{ri} A$ *if and only if for* $x \in A$ *there is some* $\lambda > 1$ *such that* $(1-\lambda)x + \lambda z \in A$; *and*
(g) *whenever* $\operatorname{ri}(A-B) \neq \emptyset$, *one has* $\operatorname{ri} A - \operatorname{ri} B \subset \operatorname{ri}(A-B)$.

Proof. The proofs of the first four assertions are left to the reader. To prove (e), we note that for $(x,y) \in \operatorname{ri}(A \times B)$ there is some $\epsilon > 0$ such that

$$((x,y) + B_\epsilon) \cap \operatorname{cl} \operatorname{aff}(A \times B) \subset A \times B.$$

Since

$$\begin{aligned}((x+B_{\epsilon/2}) \cap \operatorname{cl} \operatorname{aff} A) \times ((y+B_{\epsilon/2}) \cap \operatorname{cl} \operatorname{aff} B) &\subset ((x,y)+B_\epsilon) \cap \operatorname{cl} \operatorname{aff}(A \times B) \\ &\subset A \times B,\end{aligned}$$

we obtain

$$(x+B_{\epsilon/2}) \cap \operatorname{cl} \operatorname{aff} A \subset A \quad \text{and} \quad (y+B_{\epsilon/2}) \cap \operatorname{cl} \operatorname{aff} B \subset B.$$

Here, we use the norm $\|(\cdot,\cdot)\|$ on $X \times X$, defined by

$$\|(x,y)\| := \sqrt{\|x\|^2 + \|y\|^2}.$$

It follows that $x \in \operatorname{ri} A$ and $y \in \operatorname{ri} B$. Thus, we establish (e).

Next, we prove (f). If $z \in \operatorname{ri} A$ and $x \in A$ with $x \neq z$, then there is some $\epsilon > 0$ such that

$$z + (B_\epsilon \cap \operatorname{cl}(\operatorname{span}(A-A))) \subset A.$$

It is clear that $(\lambda-1)(z-x) \in \operatorname{cl}(\operatorname{span}(A-A))$. Also, for $1 < \lambda < 1+\epsilon/\|z-x\|$ we have

$$\|(\lambda-1)(z-x)\| = (\lambda-1)\|z-x\| < \epsilon.$$

It follows that

$$(\lambda-1)(z-x) \in B_\epsilon \cap \operatorname{cl}(\operatorname{span}(A-A)).$$

This implies

$$(1-\lambda)x + \lambda z = z + (\lambda-1)(z-x) \in z + (B_\epsilon \cap \operatorname{cl}(\operatorname{span}(A-A))) \subset A.$$

Conversely, since $\operatorname{ri} A \neq \emptyset$, we have some $x_0 \in \operatorname{ri} A$. For $z \neq x_0$, there is some $\lambda > 1$ such that

$$x_1 := (1-\lambda)x_0 + \lambda z \in A.$$

It follows that

$$z = \frac{1}{\lambda}x_1 + \frac{\lambda-1}{\lambda}x_0.$$

On the other hand, since $x_0 \in \operatorname{ri} A$, we have some $\epsilon > 0$ such that

$$x_0 + (B_\epsilon \cap \operatorname{cl}(\operatorname{span}(A-A))) \subset A.$$

Letting $\epsilon_1 := ((\lambda-1)/\lambda)\epsilon$ and taking any $u \in B_{\epsilon_1} \cap \operatorname{cl}(\operatorname{span}(A-A))$, we then have

$$\frac{\lambda u}{\lambda-1} \in B_\epsilon \cap \operatorname{cl}(\operatorname{span}(A-A)$$

and

$$x_0 + \frac{\lambda}{\lambda-1}u \in A.$$

Since $u \in B_{\epsilon_1} \cap \operatorname{cl}(\operatorname{span}(A-A))$ and, by the convexity of A,

$$z + u = \frac{1}{\lambda}x_1 + \frac{\lambda-1}{\lambda}\left(x_0 + \frac{\lambda}{\lambda-1}u\right) \in A,$$

we conclude that $z + (B_{\epsilon_1} \cap \operatorname{cl}(\operatorname{span}(A-A))) \subset A$ and hence $z \in \operatorname{ri} A$.

Finally, we establish (g). Suppose that $x := x_1 - x_2$ where $x_1 \in \operatorname{ri} A$ and $x_2 \in \operatorname{ri} B$, and that $y := y_1 - y_2 \in A - B$ where $y_1 \in A$ and $y_2 \in B$. Then by (f) there is some $\lambda_1 > 1$ such that

$$(1-\lambda_1)y_1 + \lambda_1 x_1 \in A,$$

as well as some $\lambda_2 > 2$ such that

$$(1-\lambda_2)y_2 + \lambda_2 x_2 \in B.$$

By the convexity of A and B, we then have

$$(1-\alpha)y_1 + \alpha x_1 \in A \;\; \forall 1 < \alpha \leq \lambda_1 \quad \text{and} \quad (1-\beta)y_2 + \beta x_2 \in B \;\; \forall 1 < \beta \leq \lambda_2.$$

Let $\lambda := \min\{\lambda_1, \lambda_2\} > 1$. Then

$$(1-\lambda)y+\lambda x = (1-\lambda)y_1+\lambda x_1-((1-\lambda)y_2+\lambda x_2) \in A-B.$$

Again, (f) implies that $x \in \operatorname{ri}(A-B)$. Thus, the assertion (g) follows. □

We now state some well-known results, which we shall use in proving our main results.

PROPOSITION 2.2 ([24]). *If $T: X \to X^*$ is a monotone operator, then T is maximal monotone if and only if $R(T+J) = X^*$.*

The following proposition, essentially due to Browder [5], is a generalization of the fundamental Hilbert space theorem of Minty [15].

PROPOSITION 2.3. *If $T: X \to X^*$ is a maximal monotone operator and $\lambda > 0$, then $R(T+\lambda J) = X^*$ and $(T+\lambda J)^{-1}$ is a single-valued maximal monotone operator from X^* to X that is demicontinuous.*

The following proposition is called Rockafellar's sum theorem [24, Thms. 1(a) & 2].

PROPOSITION 2.4. *If $T_1, T_2: X \to X^*$ are maximal monotone operators such that either*

(i) *X is finite-dimensional and $\operatorname{ri} D(T_1) \cap \operatorname{ri} D(T_2) \neq \emptyset$, or else*
(ii) *$D(T_1) \cap \operatorname{int} D(T_2) \neq \emptyset$,*

then T_1+T_2 is a maximal monotone operator.

Next, we will show a fundamental property. From this, we can conclude that the sets $\operatorname{ri} \operatorname{dom} f$, $\operatorname{co} D(\partial f)$, and $\operatorname{dom} f$ have the same closed affine hull.

PROPOSITION 2.5. *Suppose that Y^* is a convex subset of X^* and*

$$\emptyset \neq \operatorname{ri} Y^* \subset S \subset \operatorname{cl} Y^* \subset X^*.$$

Then the sets S and Y^* have the same closed affine hull.

Proof. We first note that there is no loss in generality in assuming that the origin is in $\operatorname{ri} Y^*$. Let $A(S)$ and $V(S)$ denote (respectively) the closed affine hull of S and the closed subspace of X^* generated by S. Then it is easy to check that $A(S) = V(S)$. Similarly, $A(Y^*) = V(Y^*)$. Thus, in order to show that $A(S) = A(Y^*)$, we only need to show that $V(S) = V(Y^*)$. It therefore suffices to show that $V(\operatorname{cl} Y^*) \subset V(\operatorname{ri} Y^*)$. If not, by a simple application of the Hahn–Banach (or separation) theorem, there exists an element, say x, of X that vanishes on $V(\operatorname{ri} Y^*)$ but not on $V(\operatorname{cl} Y^*)$. For any $u \in \operatorname{cl} Y^*$ there exists a net (u_δ) in Y^* that converges to u. Since $0 \in \operatorname{ri} Y^*$, for $\lambda \in (0,1]$ we have (see [22, Thm. 6.1])

$$u_\delta(\lambda) := (1-\lambda)u_\delta = (1-\lambda)u_\delta + \lambda, \quad 0 \in \operatorname{ri} Y^*,$$

and hence

$$u_\delta = \lim_{\lambda \downarrow 0} u_\delta(\lambda) \in V(\operatorname{ri} Y^*).$$

It follows that

$$u = \lim u_\delta \in \operatorname{cl} V(\operatorname{ri} Y^*) = V(\operatorname{ri} Y^*).$$

Thus,

$$\langle x, u \rangle = 0 \quad \forall u \in \operatorname{cl} Y^*.$$

Therefore x vanishes on $V(\operatorname{cl} Y^*)$, which leads to a contradiction. Thus, we can conclude that $\operatorname{ri} Y^*$ and $\operatorname{cl} Y^*$ have the same closed affine hull, and the assertion follows. □

Using the preceding propositions, we now establish a technical result, which is the tool to prove our main theorems.

PROPOSITION 2.6. *If $T: X \to X^*$ is a maximal monotone Y^*-operator and $R(T) \subset \operatorname{cl}(\operatorname{co} Y^*)$, then $\operatorname{ri}(\operatorname{co} Y^*) \subset R(T)$. Moreover, if $\operatorname{ri}(\operatorname{co} Y^*) \neq \emptyset$ then $\operatorname{ri}(\operatorname{co} Y^*) = \operatorname{ri} R(T)$.*

Proof. We first show that T is a $\operatorname{co} Y^*$-operator. Let $\bar{y}^* = \sum_i \lambda_i \bar{y}_i^*$, where $\bar{y}_i^* \in Y^*$, $\lambda_i \geq 0$, and $\sum_i \lambda_i = 1$ for all $i = 1, 2, \ldots$. Since T is a Y^*-operator, for each i there are $\bar{x}_i \in X$ and $\mu_i > -\infty$ such that

$$\mu_i \leq \langle x - \bar{x}_i, x^* - \bar{y}_i^* \rangle \quad \forall (x, x^*) \in G(T).$$

Equivalently,

$$\mu_i - \langle \bar{x}_i, \bar{y}_i^* \rangle \leq \langle x, x^* \rangle - \langle x, \bar{y}_i^* \rangle - \langle \bar{x}_i, x^* \rangle \quad \forall (x, x^*) \in G(T).$$

Let $\bar{x} := \sum_i \lambda_i \bar{x}_i$. Then, for $(x, x^*) \in G(T)$, we have

$$\sum_i \lambda_i \mu_i - \sum_i \lambda_i \langle \bar{x}_i, \bar{y}_i^* \rangle \leq \langle x, x^* \rangle - \left\langle x, \sum_i \lambda_i \bar{y}_i^* \right\rangle - \left\langle \sum_i \lambda_i \bar{x}_i, x^* \right\rangle.$$

Define

$$\mu := \sum_i \lambda_i \mu_i - \sum_i \lambda_i \langle \bar{x}_i, \bar{y}_i^* \rangle.$$

It follows that

$$-\infty < \mu + \langle \bar{x}, \bar{y}^* \rangle \leq \langle x, x^* \rangle - \langle x, \bar{y}^* \rangle - \langle \bar{x}, x^* \rangle + \langle \bar{x}, \bar{y}^* \rangle = \langle x - \bar{x}, x^* - \bar{y}^* \rangle.$$

We can therefore conclude that T is a $\operatorname{co} Y^*$-operator, so now we may suppose without loss of generality that Y^* is convex. Note that for any $y^* \in \operatorname{ri} Y^*$ there is some $\alpha > 0$, so that whenever $z^* \in V := \operatorname{cl} \operatorname{span}(Y^* - Y^*)$ with $\|z^*\| \leq \alpha$ we have $y^* + z^* \in Y^*$. Since T is a Y^*-operator, there exist some $\bar{x}(z^*) \in X$ and $\mu(z^*) > -\infty$ such that

$$\mu(z^*) \leq \langle x - \bar{x}(z^*), x^* - y^* - z^* \rangle \quad \forall (x, x^*) \in G(T). \tag{3}$$

By Proposition 2.3, for $\epsilon > 0$ there is some $u_\epsilon \in X$ such that

$$y^* \in (T + \epsilon J)(u_\epsilon),$$

which implies

$$(u_\epsilon, y^* - \epsilon J u_\epsilon) \in G(T). \tag{4}$$

Combining (3) with (4), we then have

$$\mu(z^*) \le \langle u_\epsilon - \bar{x}(z^*), -\epsilon J u_\epsilon - z^* \rangle. \tag{5}$$

Since $J(\cdot) = \partial(\frac{1}{2}\|\cdot\|^2)$,

$$\langle \bar{x}(z^*) - u_\epsilon, \epsilon J u_\epsilon \rangle \le \frac{\epsilon}{2}\|\bar{x}(z^*)\|^2 - \frac{\epsilon}{2}\|u_\epsilon\|^2.$$

It follows that if $z^* \in V$ with $\|z^*\| \le \alpha$ and if $\epsilon > 0$,

$$\langle u_\epsilon, z^* \rangle \le \langle u_\epsilon, z^* \rangle + \frac{\epsilon}{2}\|u_\epsilon\|^2 \le \frac{\epsilon}{2}\|\bar{x}(z^*)\|^2 + \langle \bar{x}(z^*), z^* \rangle - \mu(z^*). \tag{6}$$

Using (6), we are now ready to prove that for all $z^* \in V$,

$$\sup_{0<\epsilon\le 1} |\langle u_\epsilon, z^* \rangle| < +\infty. \tag{7}$$

Define

$$\beta(z^*) := \tfrac{1}{2}\|\bar{x}(z^*)\|^2 + \langle \bar{x}(z^*), z^* \rangle - \mu(z^*)$$

and

$$\gamma(z^*) := \max\{|\beta(z^*)|, |\beta(-z^*)|\}.$$

By (6), for $z^* \in V$ with $\|z^*\| \le \alpha$ and for $0 < \epsilon \le 1$, we have

$$|\langle u_\epsilon, z^* \rangle| \le \gamma(z^*). \tag{8}$$

For $z^* \in V$ with $\|z^*\| > \alpha$, we let $\lambda := \|z^*\|/\alpha$ and $z_1^* := (1/\lambda)z^*$; then $z^* = \lambda z_1^*$ and $\|z_1^*\| = \alpha$. It follows from (8) that

$$|\langle u_\epsilon, z_1^* \rangle| \le \gamma(z_1^*) \quad \forall 0 < \epsilon \le 1.$$

Hence we have

$$|\langle u_\epsilon, z^* \rangle| = \lambda |\langle u_\epsilon, z_1^* \rangle| \le \lambda \gamma(z_1^*) = \frac{\|z^*\|}{\alpha}\gamma\left(\frac{\alpha}{\|z^*\|}z^*\right) \quad \forall 0 < \epsilon \le 1. \tag{9}$$

Combining (8) and (9) yields

$$\sup_{0<\epsilon\le 1} |\langle u_\epsilon, z^* \rangle| < +\infty \quad \forall z^* \in V.$$

Next, we define ${}^\perp V := \{x \in X;\ \langle x, y^* \rangle = 0\ \forall y^* \in V\}$ and let $U := X/{}^\perp V$. Since X is assumed to be reflexive, we may identify the dual space U^* (see [26, Thm. 4.9]) with $({}^\perp V)^\perp$, which is V. Thus, for $x \in X$, we may define $[x] := x + {}^\perp V \in U$ and define $\langle [u], \cdot \rangle\colon U^* \to R$ by $\langle [u], u^* \rangle := \langle u, u^* \rangle$. It is easy to check that the map is well-defined, since for all $y \in {}^\perp V$ and $u^* \in U^* = V$ we have

$$\langle [u], u^* \rangle = \langle u + y, u^* \rangle = \langle u, u^* \rangle.$$

By (7) we then have

$$\sup_{0<\epsilon\le 1} |\langle [u_\epsilon], u^* \rangle| < +\infty \quad \forall u^* \in U^*.$$

From the uniform boundedness principle we obtain

$$\sup_{0<\epsilon\le 1} \|[u_\epsilon]\| < +\infty. \tag{10}$$

Now we will show that for all $u \in X$ with $Ju \in V$,

$$\|u\| \le \|[u]\|. \tag{11}$$

For any $\delta > 0$, since

$$\|[u]\| := \inf\{\|u-y\|; y \in {}^{\perp}V\},$$

there exists some $y \in {}^{\perp}V$ such that

$$\|u-y\| < \delta + \|[u]\|.$$

It follows that for any $Ju \in V$ we have $\langle y, Ju\rangle = 0$, and therefore

$$\|u\|^2 = \langle u, Ju\rangle = \langle u-y, Ju\rangle \le \|u-y\|\|Ju\| \le (\delta + \|[u]\|)\|u\|.$$

Equivalently, $\|u\| \le \delta + \|[u]\|$ for all $\delta > 0$. Thus, for all $u \in X$ with $Ju \in V$, we have $\|u\| \le \|[u]\|$. Note that by (4) we see

$$\begin{aligned} Ju_\epsilon \in \epsilon^{-1}(y^* - Tu_\epsilon) &\subset \epsilon^{-1}(Y^* - R(T)) \\ &\subset \epsilon^{-1}(Y^* - \operatorname{cl} Y^*) \subset \operatorname{cl}\operatorname{span}(Y^* - Y^*) = V. \end{aligned} \tag{12}$$

By (10), (11), and (12) we conclude that the set $\{u_\epsilon \mid 0 < \epsilon \le 1\}$ is bounded. Since

$$\|u_\epsilon\|^2 = \langle u_\epsilon, Ju_\epsilon\rangle = \|Ju_\epsilon\|^2,$$

the set $\{Ju_\epsilon \mid 0 < \epsilon \le 1\}$ is also bounded. So when ϵ converges to 0, we may assume (taking a subnet of (u_ϵ)) that $y^* - \epsilon Ju_\epsilon$ converges to y^* and that u_ϵ converges weakly to some $u \in X$. Since T is monotone, by (4) we have

$$0 \le \langle z - u_\epsilon, z^* - (y^* - \epsilon Ju_\epsilon)\rangle \quad \forall (z, z^*) \in G(T).$$

Letting $\epsilon \to 0$, we obtain

$$0 \le \langle z - u, z^* - y^*\rangle \quad \forall (z, z^*) \in G(T).$$

Since T is a maximal monotone operator, $(u, y^*) \in G(T)$; that is, $y^* \in R(T)$. Thus, $\operatorname{ri}(\operatorname{co} Y^*) \subset R(T)$.

Moreover, if $\emptyset \ne \operatorname{ri}(\operatorname{co} Y^*)$, then

$$\operatorname{ri}(\operatorname{co} Y^*) \subset R(T) \subset \operatorname{cl}(\operatorname{co} Y^*).$$

Applying Proposition 2.5 with $S := R(T)$, we obtain that the sets $R(T)$ and $\operatorname{co} Y^*$ have the same closed affine hull, say A. Working in A, we take the interior operation int_A and get

$$\operatorname{int}_A(\operatorname{co} Y^*) = \operatorname{int}_A(\operatorname{ri}(\operatorname{co} Y^*)) \subset \operatorname{int}_A R(T) \subset \operatorname{int}_A(\operatorname{cl}(\operatorname{co} Y^*)) = \operatorname{int}_A(\operatorname{co} Y^*).$$

It follows that

$$\operatorname{ri}(\operatorname{co} Y^*) \subset \operatorname{ri} R(T) \subset \operatorname{ri}(\operatorname{co} Y^*).$$

Thus, we conclude that $\operatorname{ri}(\operatorname{co} Y^*) = \operatorname{ri} R(T)$. □

We also need the following.

PROPOSITION 2.7. *If $T: X \to X^*$ is a monotone operator and $\lambda > 0$, then $T + \lambda J$ is a (BH)-operator.*

Proof. Let $\bar{x} \in D(T+\lambda J) = D(T)$ and $\widehat{x^*} \in R(T+\lambda J)$. First, we consider the case when $D(T)$ is bounded. Let $\mu > 0$, so that

$$D(T+\lambda J) = D(T) \subset \{x; \|x\| \le \mu\}.$$

Consider any $(x, x^*) \in G(T+\lambda J)$. Then, for any $\bar{x}^* \in (T+\lambda J)(\bar{x})$, the monotonicity of $T+\lambda J$ yields

$$\langle x-\bar{x}, x^*-\bar{x}^*\rangle \ge 0,$$

so that

$$\langle x-\bar{x}, x^*-\widehat{x^*}\rangle \ge \langle x-\bar{x}, \bar{x}^*-\widehat{x^*}\rangle \ge -\|x-\bar{x}\|\|\bar{x}^*-\widehat{x^*}\| \ge -2\mu\|\bar{x}^*-\widehat{x^*}\|.$$

Thus we have

$$\inf_{(x,x^*)\in G(T+\lambda J)} \langle x-\bar{x}, x^*-\widehat{x^*}\rangle \ge -2\mu\|\bar{x}^*-\widehat{x^*}\| \ge -\infty.$$

This implies that $T+\lambda J$ is a (BH)-operator.

Next, we consider the case when $D(T)$ is unbounded. Let $\bar{x}^* \in (T+\lambda J)(\bar{x})$, so that there exists some $\bar{y}^* \in T\bar{x}$ such that $\bar{x}^* = \bar{y}^* + \lambda J\bar{x}$. For any $(x, x^*) \in G(T+\lambda J)$ we can write $x^* = y^* + \lambda Jx$, where $y^* \in Tx$. Consequently,

$$\langle x-\bar{x}, x^*\rangle = \langle x-\bar{x}, y^*-\bar{y}^*\rangle + \lambda\langle x-\bar{x}, Jx\rangle + \langle x-\bar{x}, \bar{x}^*\rangle - \lambda\langle x-\bar{x}, J\bar{x}\rangle.$$

The first term on the right is nonnegative, so

$$\frac{\langle x-\bar{x}, x^*\rangle}{\|x\|} \ge \lambda\|x\| - \lambda\left\langle \bar{x}, \frac{Jx}{\|x\|}\right\rangle + \left\langle \frac{x-\bar{x}}{\|x\|}, \bar{x}^*\right\rangle - \lambda\left\langle \frac{x}{\|x\|}, J\bar{x}\right\rangle + \lambda\frac{\|\bar{x}\|^2}{\|x\|}.$$

Since each of the last four terms on the right is bounded, we can conclude that

$$\lim_{\substack{(x,x^*)\in G(T+\lambda J)\\ \|x\|\to+\infty}} \frac{\langle x-\bar{x}, x^*\rangle}{\|x\|} = +\infty.$$

Now, let $\alpha > 0$ be such that, for any $(x, x^*) \in G(T+\lambda J)$ with $\|x\| \ge \alpha$, we have

$$\frac{\langle x-\bar{x}, x^*\rangle}{\|x\|} \ge 1 + \|\widehat{x^*}\|.$$

If $\|x\| \ge \beta := \|\bar{x}\|\|\widehat{x^*}\|$, then

$$\frac{\langle x-\bar{x}, \widehat{x^*}\rangle}{\|x\|} \le \frac{\|x\|\|\widehat{x^*}\|}{\|x\|} + \frac{\|\bar{x}\|\|\widehat{x^*}\|}{\|x\|} \le \|\widehat{x^*}\| + 1.$$

It follows that for $(x, x^*) \in G(T+\lambda J)$ with $\|x\| \ge \gamma := \max\{\alpha, \beta\}$, we have

$$\frac{\langle x-\bar{x}, x^*\rangle}{\|x\|} \ge 1 + \|\widehat{x^*}\| \ge \frac{\langle x-\bar{x}, \widehat{x^*}\rangle}{\|x\|},$$

which yields

$$\langle x-\bar{x}, x^*-\widehat{x^*}\rangle \ge 0. \tag{13}$$

On the other hand, by the monotonicity of $T+\lambda J$, for $(x, x^*) \in G(T+\lambda J)$ with $\|x\| \le \gamma$ and for $\bar{x}^* \in (T+\lambda J)(\bar{x})$, we have

$$\langle x-\bar{x}, x^*-\bar{x}^*\rangle \geq 0.$$

It follows that

$$\begin{aligned}\langle x-\bar{x}, x^*-\widehat{x^*}\rangle &\geq \langle x-\bar{x}, \bar{x}^*-\widehat{x^*}\rangle \geq -\|x-\bar{x}\|\|\bar{x}^*-\widehat{x^*}\| \\ &\geq -(\gamma+\|\bar{x}\|)\|\bar{x}^*-\widehat{x^*}\|. \end{aligned} \tag{14}$$

Combining (13) with (14), we obtain

$$\inf_{(x,x^*)\in G(T+\lambda J)} \langle x-\bar{x}, x^*-\widehat{x^*}\rangle \geq \min\{0, -(\gamma+\|\bar{x}\|)\|\bar{x}^*-\widehat{x^*}\|\} > -\infty.$$

This implies that $T+\lambda J$ is a (BH)-operator. □

3. Main Results for Two Maximal Monotone Operators

We begin with an extension of Brézis and Haraux [4] in a reflexive Banach space. Indeed, Brézis and Haraux [4, Thm. 3] show that if T_1 and T_2 are monotone (BH)-operators from a Hilbert space into itself such that T_1+T_2 is maximal monotone, then $R(T_1+T_2) \cong R(T_1)+R(T_2)$; that is,

$$\operatorname{cl} R(T_1+T_2) = \operatorname{cl}(R(T_1)+R(T_2))$$

and

$$\operatorname{int} R(T_1+T_2) = \operatorname{int}(R(T_1)+R(T_2)).$$

THEOREM 3.1. *If $T_1, T_2: X \to X^*$ are monotone (BH)-operators such that T_1+T_2 is maximal monotone, then*

$$\operatorname{ri}(\operatorname{co} R(T_1)+\operatorname{co} R(T_2)) \subset R(T_1+T_2) \subset R(T_1)+R(T_2).$$

Moreover, if $\operatorname{ri}(\operatorname{co} R(T_1)+\operatorname{co} R(T_2)) \neq \emptyset$, *then*

$$\operatorname{ri} R(T_1+T_2) = \operatorname{ri}(R(T_1)+R(T_2)) = \operatorname{ri}(\operatorname{co} R(T_1)+\operatorname{co} R(T_2)).$$

Proof. Let $T := T_1+T_2$ and $Y^* := R(T_1)+R(T_2)$. Then $R(T) = R(T_1+T_2) \subset R(T_1)+R(T_2) = Y^* \subset \operatorname{cl}(\operatorname{co} Y^*)$. For $\widehat{y^*} = \widehat{y_1^*}+\widehat{y_2^*}$, where $\widehat{y_i^*} \in R(T_i)$ for all $i = 1,2$, and for $\bar{x} \in D(T) = D(T_1) \cap D(T_2)$, since each T_i is a (BH)-operator we have

$$-\infty < \inf_{(x,y_i^*)\in G(T_i)} \langle x-\bar{x}, y_i^*-\widehat{y_i^*}\rangle =: \mu_i.$$

Thus, we have

$$\begin{aligned}-\infty < \mu_1+\mu_2 &\leq \inf_{(x,y_i^*)\in G(T_i)} \langle x-\bar{x}, (y_1^*+y_2^*)-(\widehat{y_1^*}+\widehat{y_2^*})\rangle \\ &= \inf_{(x,x^*)\in G(T)} \langle x-\bar{x}, x^*-\widehat{y^*}\rangle.\end{aligned}$$

It follows that T is a Y^*-operator. Notice that, by Proposition 2.1(b),

$$\begin{aligned}R(T_1+T_2) &\subset \operatorname{co} R(T_1)+\operatorname{co} R(T_2) = \operatorname{co}(R(T_1)+R(T_2)) \\ &\subset \operatorname{cl}(\operatorname{co}(R(T_1)+R(T_2))).\end{aligned}$$

Now, by Proposition 2.6 with $T = T_1+T_2$ and $Y^* = R(T_1)+R(T_2)$, we have

$$\begin{aligned}
&\operatorname{ri}(\operatorname{co} R(T_1)+\operatorname{co} R(T_2))\\
&\quad=\operatorname{ri}(\operatorname{co}(R(T_1)+R(T_2)))\\
&\quad=\operatorname{ri} R(T_1+T_2)\subset R(T_1+T_2)\subset R(T_1)+R(T_2)\subset \operatorname{co} R(T_1)+\operatorname{co} R(T_2)\\
&\quad=\operatorname{co}(R(T_1)+R(T_2))\subset \operatorname{cl}(\operatorname{co}(R(T_1)+R(T_2))).
\end{aligned}$$

Given the nonemptiness of $\operatorname{ri}(\operatorname{co} R(T_1)+\operatorname{co} R(T_2))$, we apply Proposition 2.5 and conclude that the above sets have the same closed affine hull. Taking the interiors with respect to this common closed affine hull yields

$$\operatorname{ri} R(T_1+T_2)=\operatorname{ri}(R(T_1)+R(T_2))=\operatorname{ri}(\operatorname{co} R(T_1)+\operatorname{co} R(T_2)). \qquad \square$$

The power of Theorem 3.1 is suggested by the following interesting result, which supplements certain results of Minty [15] and Rockafellar [25] concerning the convexity of $D(T)$ and $R(T)$.

COROLLARY 3.2. *If $T\colon X\to X^*$ is a maximal monotone operator with*

$$\operatorname{ri}(\operatorname{co} D(T))\neq\emptyset \quad \textit{and} \quad \operatorname{ri}(\operatorname{co} R(T))\neq\emptyset$$

then

$$\operatorname{ri}(\operatorname{co} D(T))=\operatorname{ri} D(T) \quad \textit{and} \quad \operatorname{ri}(\operatorname{co} R(T))=\operatorname{ri} R(T).$$

Proof. Define $S_1, S_2\colon X^*\to X$ by

$$S_1(x^*):=(T+\tfrac{1}{2}J)^{-1}(x^*)\ \ \forall x^*\in X^* \quad \text{and} \quad S_2(x^*):=0\ \ \forall x^*\in X^*.$$

Then S_1 and S_2 are maximal monotone and (by Proposition 2.3) $D(S_i)=X^*$ for all $i=1,2$. Thus,

$$D(S_1)\cap \operatorname{int} D(S_2)=X^*\neq\emptyset.$$

It follows from Proposition 2.4 that S_1+S_2 is maximal monotone. Also, Proposition 2.7 implies that S_1 and S_2 are (BH)-operators. Thus, Theorem 3.1 implies that

$$\operatorname{ri}(\operatorname{co} R(S_1)+\operatorname{co} R(S_2))\subset R(S_1+S_2).$$

Since $R(S_1)=D(T)$ and $R(S_2)=\{0\}$, we have

$$\operatorname{ri}(\operatorname{co} D(T))\subset R(S_1+S_2)=R(S_1)=D(T)\subset \operatorname{co} D(T).$$

By Proposition 2.5, $D(T)$ and $\operatorname{co} D(T)$ have the same closed affine hull. We then have

$$\operatorname{ri}(\operatorname{co} D(T))\subset \operatorname{ri} D(T)\subset \operatorname{ri}(\operatorname{co} D(T)).$$

Thus, we conclude that $\operatorname{ri}(\operatorname{co} D(T))=\operatorname{ri} D(T)$. On the other hand, T^{-1} is maximal monotone and $R(T)=D(T^{-1})$. Applying the above result to T^{-1}, we conclude that $\operatorname{ri}(\operatorname{co} R(T))=\operatorname{ri} R(T)$. $\square$

LEMMA 3.3. *If $T\colon X\to X^*$ is a maximal monotone operator and $A\colon X\to X$ is an invertible continuous linear operator, then A^*TA is maximal monotone.*

Proof. We first notice that A^*TA is monotone, since for all $(x,y),(\bar{x},\bar{y})\in G(A^*TA)$ there exist $x^*\in TA(x)$ and $y^*\in TA(\bar{x})$ such that $y=A^*x^*$ and $\bar{y}=A^*y^*$. Thus,

$$\langle x-\bar{x}, y-\bar{y}\rangle = \langle x-\bar{x}, A^*(x^*-y^*)\rangle = \langle Ax-A\bar{x}, x^*-y^*\rangle \geq 0.$$

The last inequality holds because T is monotone. Now, suppose that

$$\langle x-x_1, y-y_1\rangle \geq 0 \quad \forall (x,y) \in G(A^*TA).$$

To prove that A^*TA is maximal monotone, it suffices to show that $(x_1, y_1) \in G(A^*TA)$. Notice that

$$\begin{aligned} 0 &\leq \langle x-x_1, y-y_1\rangle \\ &= \langle Ax-Ax_1, (A^*)^{-1}y-(A^*)^{-1}y_1\rangle \quad \forall (x,y) \in G(A^*TA). \end{aligned} \tag{15}$$

Now, for $(\bar{x}, \bar{y}) \in G(T)$, there is some $\bar{x}_0$ such that $A\bar{x}_0 = \bar{x}$ and $\bar{y} \in TA\bar{x}_0$. Thus, $(\bar{x}_0, A^*\bar{y}) \in G(A^*TA)$. By (15) we have

$$\langle \bar{x}-Ax_1, \bar{y}-(A^*)^{-1}y_1\rangle \geq 0 \quad \forall (\bar{x}, \bar{y}) \in G(T).$$

Since T is maximal monotone, we conclude that $(Ax_1, (A^*)^{-1}y_1) \in G(T)$ and hence $(x_1, y_1) \in G(A^*TA)$. Thus, the proof is complete. □

We are now ready to establish the main theorem for the case of two maximal monotone operators.

THEOREM 3.4. *If $T_1, T_2: X \to X^*$ are maximal monotone operators and $A_1, A_2: X \to X$ are invertible continuous linear operators satisfying the constraint qualification*

$$0 \in \operatorname{ri}(\operatorname{co} A_1^{-1}D(T_1) - \operatorname{co} A_2^{-1}D(T_2)),$$

then the operator $A_1^*T_1A_1 + A_2^*T_2A_2$ is maximal monotone.

Proof. It is clear that $A_1^*T_1A_1 + A_2^*T_2A_2$ is monotone. By Proposition 2.2, it is sufficient to prove that

$$R(A_1^*T_1A_1 + A_2^*T_2A_2 + J) = X^*.$$

For any fixed $x_0^* \in X^*$, we define $S_1, S_2: X^* \to X$ by

$$S_1(x^*) := (A_1^*T_1A_1 + \tfrac{1}{2}J)^{-1}(x^*)$$

and

$$S_2(x^*) := -(A_2^*T_2A_2 + \tfrac{1}{2}J)^{-1}(x_0^* - x^*).$$

It is clear from Proposition 2.3 and Lemma 3.3 that S_1 and S_2 are maximal monotone single-valued operators. Thus,

$$D(S_1) \cap \operatorname{int} D(S_2) = X^* \neq \emptyset.$$

By Proposition 2.4, we conclude that $S_1 + S_2$ is also a maximal monotone operator. Since by Proposition 2.7 each S_i^{-1} is a (BH)-operator, each S_i is also a (BH)-operator. It follows from Theorem 3.1 that

$$\operatorname{ri}(\operatorname{co} R(S_1) + \operatorname{co} R(S_2)) \subset R(S_1 + S_2).$$

Since $R(S_1) = A_1^{-1}D(T_1)$ and $R(S_2) = -A_2^{-1}D(T_2)$, it follows that

$$0 \in \operatorname{ri}(\operatorname{co} A_1^{-1}D(T_1) - \operatorname{co} A_2^{-1}D(T_2)) = \operatorname{ri}(\operatorname{co} R(S_1) + \operatorname{co} R(S_2)) \subset R(S_1+S_2).$$

Thus, $0 \in R(S_1+S_2)$. Let $x^* \in X^*$ be such that $0 \in S_1(x^*)+S_2(x^*)$. It follows that there is some $y^* \in S_1(x^*)$ such that $-y^* \in S_2(x^*)$. We then have

$$\begin{aligned} x_0^* &= x^* + (x_0^* - x^*) \in (A_1^*T_1A_1 + \tfrac{1}{2}J)(y^*) + (A_2^*T_2A_2 + \tfrac{1}{2}J)(y^*) \\ &= (A_1^*T_1A_1 + A_2^*T_2A_2 + J)(y^*). \end{aligned}$$

This completes the proof. □

As a consequence, by taking each A_k to be the identity mapping, we obtain the following result, which generalizes [24, Thm. 2] to a reflexive Banach space.

COROLLARY 3.5. *If $T_1, T_2: X \to X^*$ are maximal monotone operators satisfying the constraint qualification*

$$0 \in \operatorname{ri}(\operatorname{co} D(T_1) - \operatorname{co} D(T_2)),$$

then the operator T_1+T_2 is maximal monotone.

As previously remarked, together with the well-known result [23, Thm. A] that every subdifferential operator of a closed convex function on a Banach space is maximal monotone, we establish the following motivating theorem.

THEOREM 3.6. *If $f: X \to R$ is a closed proper convex function on X and if C is a nonempty closed convex subset of X such that*

$$0 \in \operatorname{ri}(\operatorname{dom} f - C),$$

then x is a solution of VI$(\partial f, C)$ *if and only if x is an optimal solution of* (P).

4. Main Results for n Maximal Monotone Operators

In this section we will extend the main theorem to the case of n maximal monotone operators.

THEOREM 4.1. *If $T_1, T_2, \ldots, T_n: X \to X^*$ are maximal monotone operators and $A_1, A_2, \ldots, A_n: X \to X$ are invertible continuous linear operators satisfying the constraint qualification*

$$(0, 0, \ldots, 0) \in \operatorname{ri}(\operatorname{co} C_2 \times \operatorname{co} C_3 \times \cdots \times \operatorname{co} C_n),$$

where

$$C_k := A_1^{-1}D(T_1) \cap A_2^{-1}D(T_2) \cap \cdots \cap A_{k-1}^{-1}D(T_{k-1}) - A_k^{-1}D(T_k) \qquad \forall k = 2, 3, \ldots, n,$$

then the operator $A_1^*T_1A_1 + A_2^*T_2A_2 + \cdots + A_n^*T_nA_n$ is maximal monotone.

Proof. Observe that $0 \in \operatorname{ri}(\operatorname{co} C_k)$ for all $k = 2, 3, \ldots, n$, since

$$\begin{aligned} (0, 0, \ldots, 0) &\in \operatorname{ri}(\operatorname{co} C_2 \times \operatorname{co} C_3 \times \cdots \times \operatorname{co} C_n) \\ &\subset \operatorname{ri}(\operatorname{co} C_2) \times \operatorname{ri}(\operatorname{co} C_3) \times \cdots \times \operatorname{ri}(\operatorname{co} C_n). \end{aligned}$$

We now prove the assertion by induction. In the case $n=2$, this is exactly the conclusion of Theorem 3.4, since

$$\begin{aligned} 0 \in \operatorname{ri}(\operatorname{co} C_2) &= \operatorname{ri}(\operatorname{co}(A_1^{-1}D(T_1) - A_2^{-1}D(T_2))) \\ &= \operatorname{ri}(\operatorname{co} A_1^{-1}D(T_1) - \operatorname{co} A_2^{-1}D(T_2)). \end{aligned}$$

Suppose that the assertion holds for $n=k$ and that we have a maximal monotone operator $T := A_1^* T_1 A_1 + A_2^* T_2 A_2 + \cdots + A_k^* T_k A_k$. Now, let $n = k+1$. Notice that

$$\begin{aligned} 0 &\in \operatorname{ri}(\operatorname{co} C_{k+1}) \\ &= \operatorname{ri}(\operatorname{co}(A_1^{-1}D(T_1) \cap A_2^{-1}D(T_2) \cap \cdots \cap A_k^{-1}D(T_k) - A_{k+1}^{-1}D(T_{k+1}))) \\ &= \operatorname{ri}(\operatorname{co}(A_1^{-1}D(T_1) \cap A_2^{-1}D(T_2) \cap \cdots \cap A_k^{-1}D(T_k)) - \operatorname{co} A_{k+1}^{-1}D(T_{k+1})) \\ &= \operatorname{ri}(\operatorname{co} D(T) - \operatorname{co} A_{k+1}^{-1}D(T_{k+1})). \end{aligned}$$

Again, Theorem 3.4 implies that $T + A_{k+1}^* T_{k+1} A_{k+1} = I^* T I + A_{k+1}^* T_{k+1} A_{k+1}$ is maximal monotone; that is, the operator

$$A_1^* T_1 A_1 + A_2^* T_2 A_2 + \cdots + A_{k+1}^* T_{k+1} A_{k+1}$$

is maximal monotone, where I denotes the identity mapping on X. This completes the proof. □

As a consequence, by taking each A_k to be the identity mapping on X, we obtain the following corollary.

COROLLARY 4.2. *If $T_1, T_2, \ldots, T_n : X \to X^*$ are maximal monotone operators satisfying the constraint qualification*

$$(0, 0, \ldots, 0) \in \operatorname{ri}(\operatorname{co} C_2 \times \operatorname{co} C_3 \times \cdots \times \operatorname{co} C_n),$$

where

$$C_k := D(T_1) \cap D(T_2) \cap \cdots \cap D(T_{k-1}) - D(T_k) \quad \forall k = 2, 3, \ldots, n,$$

then the operator $T_1 + T_2 + \cdots + T_n$ is maximal monotone.

From the general inclusion $\operatorname{ri} \operatorname{dom} f \subset D(\partial f) \subset \operatorname{dom} f$, for any closed proper convex function f on X we have the following.

COROLLARY 4.3. *If $f_1, f_2, \ldots, f_n : X \to R$ are closed proper convex functions satisfying the constraint qualification*

$$0 \in \operatorname{ri}(\operatorname{dom} f_1 \cap \operatorname{dom} f_2 \cap \cdots \cap \operatorname{dom} f_{k-1} - \operatorname{dom} f_k) \quad \forall k = 2, 3, \ldots, n,$$

then the operator $\partial f_1 + \partial f_2 + \cdots + \partial f_n$ is maximal monotone.

Finally, we give an extensive form concerning maximal cyclical monotonicity as follows.

THEOREM 4.4. *If $T_1, T_2, \ldots, T_n : X \to X^*$ are maximal cyclically monotone operators satisfying the constraint qualification*

$$(0,0,\ldots,0)\in \operatorname{ri}(\operatorname{co} C_2\times \operatorname{co} C_3\times\cdots\times \operatorname{co} C_n),$$

where

$$C_k := D(T_1)\cap D(T_2)\cap\cdots\cap D(T_{k-1})-D(T_k)\quad \forall k=2,3,\ldots,n,$$

then the operator $T_1+T_2+\cdots+T_n$ *is maximal cyclically monotone.*

Proof. Since each T_i is a maximal cyclically monotone operator, by [23, Cor. 1] there is some closed proper convex function f_i on X such that $T_i=\partial f_i$ for $i=1,2,\ldots,n$. Since by [23, Thm. A] each subdifferential operator of a closed proper convex function is maximal monotone, we have

$$T_1+T_2+\cdots+T_n=\partial f_1+\partial f_2+\cdots+\partial f_n=\partial(f_1+f_2+\cdots+f_n).$$

The last equality follows from Corollaries 4.2 and 4.3. Thus, the assertion now follows from [23, Cor. 1]. □

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