Spectral Gaps and Rates to Equilibrium for Diffusions in Convex Domains

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1. Introduction

The purpose of this paper is to investigate how geometric quantities, such as inner radius and diameter, influence the gap between the first two eigenvalues for Schrödinger operators subject to Dirichlet and Neumann boundary conditions for domains in Euclidean space. The estimation of these gaps has important consequences for both physics and mathematics. For example, M. van den Berg [Be] uses gap results for the Dirichlet problem to give sufficient conditions for a free boson gas to macroscopically occupy the ground state alone under the thermodynamic limit. Mathematically, the problem of estimating the spectral gap has been of interest to analysts and geometers for several years [AB1; L; SWYY; YZ]. Probabilistically, the spectral gap determines the asymptotic exponential rate of convergence to equilibrium for the associated Markovian semigroup, and it is related to the log-Sobolev constant (see [D; Sa] for more on this). In the case of the Dirichlet gap, the relevant semigroup is the one for Brownian motion conditioned to remain forever in the domain. This interpretation was the original motivation for our study of the problem.

If we examine the eigenvalue problem for a Schrödinger operator with a smooth, nonnegative, convex potential V under Dirichlet boundary conditions in a smooth domain $\Omega \subset \mathbb{R}^n$,

$$\begin{aligned} -\Delta u + Vu &= \lambda u \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \end{aligned} \tag{1.1}$$

we see that the eigenvalues form a discrete set and can be arranged in nondecreasing order [D]:

$$0 < \lambda_1 < \lambda_2 \leq \cdots. \tag{1.2}$$

A conjecture which has been of interest for a number of years is that, with the above assumptions, for all convex domains of diameter d we have

$$\frac{3\pi^2}{d^2} < \lambda_2 - \lambda_1,$$

where the lower bound is approached for thin rectangles (see [Be; AB1]).

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The first substantial progress on this conjecture was made by Singer et al. [SWYY], who used a maximum principle technique as in [Sp] to show that

$$\frac{\pi^2}{4d^2} \le \lambda_2 - \lambda_1 \le \frac{n\pi^2}{r^2} + \frac{4(M-m)}{n} \tag{1.3}$$

for convex domains of diameter d and inner radius r (the radius of the largest inscribed ball), where $M = \sup_{\bar{\Omega}} V$ and $m = \inf_{\bar{\Omega}} V$. In particular, if $V = 0$ then (1.3) becomes

$$\frac{\pi^2}{4d^2} \le \lambda_2 - \lambda_1 \le \frac{n\pi^2}{r^2}. \tag{1.4}$$

Later, Yu and Zhong [YZ] refined the maximum principle method to improve the lower bound by a factor of four, and recently Ling [L], using the same approach, found the strict inequality

$$\frac{\pi^2}{d^2} < \lambda_2 - \lambda_1. \tag{1.5}$$

In Section 2 of this paper we obtain inequality (1.5) and improve it under certain assumptions of symmetry through a new approach, which we believe can be used to obtain a better lower bound even for the general case. Instead of appealing to the maximum principle, we use a variational method along the lines of what was done for the Neumann problem in a paper of Payne and Weinberger [PW]. Our proof was motivated by comparing the behavior of the diffusion conditioned to remain forever in the domain, where rate to equilibrium is measured by $\lambda_2 - \lambda_1$, with the behavior of the Brownian motion reflected at the boundary. This will be explained in more detail in Section 5. Next, in the case $V = 0$ and for $n = 2$, we use some recent results of Jerison [J] to improve the upper bound of [SWYY] to

$$\lambda_2 - \lambda_1 \le \frac{C}{d^{2/3} r^{4/3}} < \frac{C}{r^2}, \tag{1.6}$$

where C is a universal constant. We close Section 2 with sharper upper bounds for (1.3) by applying the proof of the Payne-Pólya-Weinberger conjecture found in [AB2].

In Section 3 we study the "gap" corresponding to Neumann boundary conditions. If $\Omega \subset \mathbb{R}^n$ is smooth and $\partial u/\partial n$ is the derivative of u in the outward normal direction, then the eigenvalues of the Neumann problem

$$\begin{aligned} -\Delta u &= \mu u \text{ in } \Omega, \\ \frac{\partial u}{\partial n} &= 0 \text{ on } \partial\Omega, \end{aligned} \tag{1.7}$$

can be written as

$$0 = \mu_1 < \mu_2 \le \mu_3 \le \cdots.$$

The gap $\mu_2 - \mu_1 = \mu_2$ is actually just the first nonzero eigenvalue, and as such has been studied much longer than the Dirichlet gap. In [W], Weinberger proved that among all domains of equal volume $|\Omega|$, the ball has the largest μ_2, which implies that

$$\mu_2 \leq \frac{c}{|\Omega|}, \tag{1.8}$$

where c is the second Neumann eigenvalue for the ball of volume 1. Subsequently, in collaboration with Payne [PW], it was proved that for all convex domains of diameter d,

$$\mu_2 \geq \frac{\pi^2}{d^2}, \tag{1.9}$$

with the lower bound being approached for long thin rectangles. Under the same hypotheses, we will show in Theorem 4 that

$$\mu_2 \leq \frac{c}{d^2}, \tag{1.10}$$

where c is a universal constant. This result allows us to conclude that the exponential decay in time to equilibrium for a reflected Brownian motion is proportional to $1/d^2$. (See Theorem 6 and its corollary.)

Combining (1.5), (1.9), and (1.10) yields

$$\frac{\pi^2}{d^2} \leq \mu_2 \leq \frac{c}{d^2} \leq C(\lambda_2 - \lambda_1). \tag{1.11}$$

This leads us to investigate an upper bound for $\lambda_2 - \lambda_1$ in terms of c/d^2 when the potential $V = 0$. We end Section 3 by constructing a sequence of domains $\{\Omega_j\}$ in $\mathbb{R}^2$ with diameter $d = 1$ such that $\lambda_2 - \lambda_1 \to \infty$ as $j \to \infty$, showing that no such upper bound is possible.

In Section 4 we consider the case of the gap for Robin boundary conditions:

$$\begin{gathered} -\Delta u = \lambda u \text{ in } \Omega, \\ \frac{\partial u}{\partial n} + \alpha u = 0 \text{ on } \partial\Omega, \quad 0 < \alpha < \infty \\ 0 < \lambda_1(\alpha) < \lambda_2(\alpha) \leq \cdots. \end{gathered} \tag{1.12}$$

These boundary conditions lie between the Neumann ($\alpha \to 0$) and Dirichlet ($\alpha \to \infty$) cases, so it is hoped that a study of them will shed further light on the extreme situations. (See [Sp] for basic references on this problem.)

Finally, in Section 5 we explicitly relate the calculations from the previous sections to the study of the time to equilibrium for the normalized Dirichlet heat kernel. It is well known that the rate to equilibrium is exponential and depends upon the magnitude of $\lambda_2 - \lambda_1$, provided the semigroup is intrinsically ultracontractive [D]. In Theorem 6 we provide a detailed proof of this dependence.

A word about the notation used throughout the paper: We use C or c to represent a universal constant that may change between lines of inequalities. Unless otherwise stated, in the rest of the paper $\Omega \subset \mathbb{R}^n$ will be a smooth, bounded convex domain and $V \in L^\infty(\Omega)$ a smooth, nonnegative, convex potential.

2. The Dirichlet Gap

The main result in this section is the following theorem.

THEOREM 1. *For λ_2, λ_1 defined as in* (1.1)–(1.2), *we have* $\lambda_2 - \lambda_1 > \pi^2/d^2$.

Proof. From Kirsch and Simon [KS], if $V \in L^\infty(\Omega)$ is smooth then

$$\lambda_2 - \lambda_1 = \inf \frac{\int_\Omega |\nabla f|^2 \varphi_1^2 \, dz}{\int_\Omega f^2 \varphi_1^2 \, dz},$$

where φ_1 is the lowest eigenfunction and the infimum is taken over all $C^\infty(\Omega)$ functions satisfying $\int_\Omega f\varphi_1^2 = 0$. In fact, this result applies for a much larger set of potentials known as the *Kato class*. Under our assumptions the infimum is achieved for $f = \varphi_2/\varphi_1$, where φ_2 is the second eigenfunction. This function f is smooth up to the boundary by (A) in [SWYY, Sec. 6].

Let $u \in C^\infty(\Omega)$ be such that both u and $u\varphi_1$ have bounded derivatives in Ω and

$$\int_\Omega u\varphi_1^2 \, dz = 0. \tag{2.1}$$

We then claim

$$\int_\Omega |\nabla u|^2 \varphi_1^2 \, dz > \frac{\pi^2}{d^2} \int_\Omega |u|^2 \varphi_1^2 \, dz. \tag{2.2}$$

Taking $u = f = \varphi_2/\varphi_1$ immediately proves the theorem, so we content ourselves with demonstrating (2.2). We do this in two dimensions only, but comment later on how the n-dimensional case follows similarly.

For the proof of (2.2) we need the following 1-dimensional lemma. This lemma is essentially contained in [PW]; we include its proof for completeness and because it shows the strictness of the inequality in Theorem 1.

LEMMA 1. *Let $p^*(y)$ be a nonnegative log-concave function on an interval $0 \le y \le L$. Then, for any piecewise C^1 function $\tilde{u}$,*

$$\int_0^L p^*(y)[\tilde{u}'(y)]^2 \, dy \ge \frac{\pi^2}{L^2} \left\{ \int_0^L p^*(y)\tilde{u}^2(y) \, dy - \frac{\left(\int_0^L p^*(y)\tilde{u}(y)\, dy\right)^2}{\int_0^L p^*(y)\, dy} \right\}. \tag{2.3}$$

Proof. By letting $u(x) = \tilde{u}(y) - \int_0^L p^*(y)\tilde{u}(y)\, dy / \int_0^L p^*(y)\, dy$, it is enough to show (2.3) for $p^*(y)$ nonnegative log-concave on $0 \le y \le L$ and $u(y)$ piecewise C^1 satisfying

$$\int_0^L p^*(y) u(y) \, dy = 0. \tag{2.4}$$

This reduces (2.3) to

$$\int_0^L p^*(y)[u'(y)]^2 \, dy \ge \frac{\pi^2}{L^2} \int_0^L p^*(y)[u(y)]^2 \, dy, \tag{2.5}$$

where we can assume that $p^* > 0$ on $[0, L]$ and is twice differentiable.

The minimizer v of

$$\frac{\int_0^L p^*(y)[u'(y)]^2\,dy}{\int_0^L p^*(y)[u(y)]^2\,dy} \tag{2.6}$$

subject to (2.4) satisfies the Sturm–Liouville problem

$$\begin{gathered}[p^*v']' + \lambda p^* v = 0,\\ v'(0) = v'(L) = 0,\end{gathered} \tag{2.7}$$

with λ the minimum in (2.6); see [CH].

Dividing (2.7) by p^*, differentiating with respect to y, and setting

$$w = v'(p^*)^{1/2} \tag{2.8}$$

shows that w is a solution of

$$\begin{gathered} w'' + \left[\frac{1}{2}\frac{(p^*)''}{p^*} - \frac{3}{4}\frac{[(p^*)']^2}{(p^*)^2}\right]w + \lambda w = 0,\\ w(0) = w(L) = 0.\end{gathered} \tag{2.9}$$

Since p^* is log-concave we have $\frac{1}{2}(\log p^*)'' \le 0$, and hence the second term in (2.9) is less than or equal to zero. Multiplying by w and integrating by parts, we obtain

$$\lambda \ge \frac{\int_0^L (w')^2\,dy}{\int_0^L w^2\,dy},$$

with $w(0) = w(L) = 0$. Therefore, a lower bound for λ is π^2/L^2, the first Dirichlet eigenvalue for the interval $[0, L]$, and the lemma is proved. We remark that if p^* is not constant in the lemma then the potential in (2.9) is strictly less than zero and there is strict inequality in (2.3) and (2.5). □

Having shown the 1-dimensional lemma, we now return to the proof of Theorem 1. Consider the set of lines that divide Ω into two pieces of equal area. Since Ω is connected, for any given $0 \le \theta < 180°$, at least one such line forms an angle θ with a ray in the direction of the positive x axis. No two of these lines can be parallel, for then they would divide Ω into three pieces, such that on each side of the parallel lines there is equal area, forcing the middle piece to have zero area. Hence, to every angle $0 \le \theta < 180°$ we can associate a unique line that divides Ω in two and makes an angle θ with the positive x axis. With u as in (2.1), it follows from the continuity of $u\varphi_1^2$ that there exists at least one angle θ for which the corresponding line divides Ω into two convex subdomains Ω_{11}, Ω_{12} of *equal area* with

$$\int_{\Omega_{1i}} u\varphi_1^2 = 0, \quad i = 1, 2.$$

Breaking each of the Ω_{1i} up similarly and so forth, we obtain for any $\nu = 1, 2, \ldots$ a collection $\{\Omega_{\nu k}\}$ of 2^ν convex sets, all of the same area and satisfying

$$\int_{\Omega_{\nu k}} u\varphi_1^2\,dz = 0, \quad k = 1, \ldots, 2^\nu.$$

Fixing ϵ, we take ν such that

$$A_\nu = |\Omega_{\nu k}| = \frac{1}{2^\nu}|\Omega| \le \pi\left(\frac{\epsilon}{4}\right)^2.$$

The inner radius of these sets is thus less than or equal to $\epsilon/4$, and because any convex domain may be inscribed in either a triangle or strip having the same inner radius, we can place each $\Omega_{\nu k}$ between parallel lines at a distance ϵ apart.

For each $\Omega_\nu \in \{\Omega_{\nu k}\}$ we introduce a coordinate system with the x axis tangent to Ω_ν and the positive y axis containing the diameter. If $p(s)$ is the length of the line segment (a_s, b_s), the intersection of the line $y = s$ with Ω_ν, then convexity of Ω_ν makes $p(s)$ a concave function on $[0, L_\nu]$ where L_ν is the length of the diameter of Ω_ν.

By the mean value theorem applied to $(\partial u/\partial y)^2\varphi_1^2$, we have $a_y < \xi_y < b_y$ such that

$$\begin{aligned}
&\left|\int_{\Omega_\nu}\left(\frac{\partial u(x,y)}{\partial y}\right)^2\varphi_1^2(x,y)\,dx\,dy - \int_0^{L_\nu} p(y)[u(0,y)']^2\varphi_1^2(0,y)\,dy\right| \\
&\quad = \left|\int_0^{L_\nu}\int_{a_y}^{b_y}\left[\left(\frac{\partial u(x,y)}{\partial y}\right)^2\varphi_1^2(x,y) - \left(\frac{\partial u(0,y)}{\partial y}\right)^2\varphi_1^2(0,y)\right]dx\,dy\right| \\
&\quad \le \int_0^{L_\nu}\int_{a_y}^{b_y}\left|(x-0)\left(\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\varphi_1\right)^2\right)(\xi_y, y)\right|dx\,dy. \qquad (2.10)
\end{aligned}$$

If we set

$$M_1 = \sup\left(\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\varphi_1\right)^2\right) \text{ in } \Omega,$$

the last quantity is bounded by

$$M_1\int_0^{L_\nu}\int_{a_y}^{b_y}|x|\,dx\,dy \le M_1 A_\nu \epsilon.$$

Similarly, there is an M_2 with

$$\left|\int_{\Omega_\nu} u^2\varphi_1^2\,dx\,dy - \int_0^{L_\nu} p(y)[u(0,y)]^2\varphi_1^2(0,y)\,dy\right| \le M_2 A_\nu \epsilon_\nu, \qquad (2.11)$$

and by assumption there is an M_3 such that

$$\begin{aligned}
&\left|\int_{\Omega_\nu} u\varphi_1^2\,dx\,dy - \int_0^{L_\nu} p(y)\varphi_1^2(0,y)u(0,y)\,dy\right| \\
&\quad = \left|\int_0^{L_\nu} p(y)\varphi_1^2(0,y)u(0,y)\,dy\right| \le M_3 A_\nu \epsilon. \qquad (2.12)
\end{aligned}$$

By a result of Brascamp and Lieb [BL] we now have φ_1 log-concave, which implies the same for φ_1^2. Because our p is concave we have $p\varphi_1^2$ log-concave and, by setting $p^* = p\varphi_1^2$ in (2.3) and calling $M = \max(M_1, M_2, M_3)$, we deduce via (2.10), (2.11), and (2.12) that

$$
\begin{aligned}
&\int_{\Omega_\nu} |\nabla u|^2 \varphi_1^2\,dx\,dy \\
&\quad \ge \int_{\Omega_\nu} \left(\frac{\partial u}{\partial y}\right)^2 \varphi_1^2\,dx\,dy \\
&\quad \ge \int_0^{L_\nu} [u(0,y)']^2 p(y)\varphi_1^2(0,y)\,dy - \epsilon M A_\nu \\
&\quad \ge \frac{\pi^2}{L_\nu^2}\left\{\int_0^{L_\nu} [u(0,y)]^2 p(y)\varphi_1^2(0,y)\,dy - \frac{(\int_0^{L_\nu} p\varphi_1^2 u\,dy)^2}{\int_0^{L_\nu} p\varphi_1^2\,dy}\right\} - \epsilon M A_\nu \\
&\quad \ge \frac{\pi^2}{L_\nu^2}\left\{\int_{\Omega_\nu} u^2\varphi_1^2\,dx\,dy - \frac{(\int_0^{L_\nu} p\varphi_1^2 u\,dy)^2}{\int_0^{L_\nu} p\varphi_1^2\,dy}\right\} - \epsilon M A_\nu - \frac{\epsilon M A_\nu \pi^2}{L_\nu^2} \\
&\quad \ge \frac{\pi^2}{L_\nu^2}\left\{\int_{\Omega_\nu} u^2\varphi_1^2\,dx\,dy - \frac{M A_\nu \epsilon \int_0^{L_\nu} p\varphi_1^2 u\,dy}{\int_0^{L_\nu} p\varphi_1^2\,dy}\right\} - \epsilon M A_\nu - \frac{\epsilon M A_\nu \pi^2}{L_\nu^2} \\
&\quad \ge \frac{\pi^2}{L_\nu^2}\int_{\Omega_\nu} u^2\varphi_1^2\,dx\,dy - C(M)A_\nu\epsilon.
\end{aligned}
$$

The last inequality holds since for smooth u we can assume $u \le M$, so that

$$\left|\frac{\int_0^{L_\nu} p(y)\varphi_1^2(0,y)u(0,y)\,dy}{\int_0^{L_\nu} p(y)\varphi_1^2(0,y)\,dy}\right| \le M.$$

Summing over $\{\Omega_{\nu k}\}$ and recalling that $L_\nu \le d$ yields

$$\int_\Omega |\nabla u|^2\varphi_1^2\,dx\,dy \ge \frac{\pi^2}{d^2}\int_\Omega u^2\varphi_1^2\,dx\,dy - C(M)|\Omega|\epsilon.$$

Letting $\epsilon \to 0$ proves claim (2.2) with the exception of strict inequality, which may be derived by noticing that the $p^* = p\varphi_1^2$ used in our proof is not constant and using the remarks at the end of Lemma 1.

If instead of $\mathbb{R}^2$ we examine $\mathbb{R}^n$, the proof of the theorem can be modified as follows. We replace lines dividing Ω into equal areas with hyperplanes that break Ω into pieces of equal volume, dividing Ω into convex sets $\{\Omega_{\nu k}\}$ of equal volume, with each $\Omega_{\nu k}$ fitting between two hyperplanes (say, $x_1 = 0$ and $x_1 = \epsilon$) and tangent to the hyperplane $x_n = 0$. We then let $p(s)$ be the $(n-1)$-dimensional volume of $\Omega_{\nu k} \cap \{x_n = s\}$, and recall a well-known consequence of the Brunn-Minkowski inequalities [Sc], the so-called Busemann's theorem, which states that $p^{1/(n-1)}$ is concave. Consequently, p is log-concave on $0 < s < L_{\nu k}$ and therefore so is $p^* = p\varphi_1^2$. Theorem 1 is now complete. □

COROLLARY 1. *For a doubly symmetric domain in $\mathbb{R}^2$ with longer axis of symmetry of length b,*

$$\lambda_2 - \lambda_1 > \frac{\pi^2}{b^2}.$$

Proof. The symmetry assumptions on the domain are inherited by the first two eigenfunctions, allowing the partitioning in the proof of Theorem 1 to be done in such a way that the lines dividing the region are parallel to the longer axis. This means that each Ω_ν in the proof can be contained between parallel lines at a distance ϵ apart, and between the lines $y=0$ and $y=b$.

REMARK 1. Corollary 1 actually gives a better lower bound than (1.5) for domains such as a square. In this case we get

$$\lambda_2-\lambda_1 \geq \frac{\pi^2}{s^2} \quad \text{versus} \quad \lambda_2-\lambda_1 \geq \frac{\pi^2}{2s^2},$$

where s is the side length.

REMARK 2. If in Theorem 1 the convexity hypothesis is removed then we can find domains $D_r \subset B(0,1)$ with $\lambda_2^r-\lambda_1^r \to 0$. As an example, take a set of dumbbells connected by a thinner and thinner bar. In this case, the Brownian motion has a hard time getting through the passage and hence the time to equilibrium should be large. More generally, if a set of domains is tending to become disconnected then the gap will tend to zero, because (essentially) the second eigenfunction can be chosen approximately equal to the first on one part of the domain and a negative multiple of it on the other part of the disconnecting domain.

REMARK 3. The key to the lower bound is equation (2.11), which follows from the log-concavity of $p\varphi_1^2$ on line segments and a 1-dimensional comparison with the eigenvalues of the vibrating strings. We could thus improve our result if we could get better bounds for the first eigenvalue of the following Schrödinger problem:

$$\begin{gathered} w''+\left[\frac{1}{2}\frac{(p\varphi_1^2)''}{p\varphi_1^2}-\frac{3}{4}\frac{[(p\varphi_1^2)']^2}{[p\varphi_1^2]^2}\right]w+\lambda w=0, \\ w(0)=w(L_\nu)=0, \end{gathered} \tag{2.13}$$

where φ_1 is the first eigenfunction restricted to a line segment in Ω. What we know is that the potential in (2.13) is strictly less than 0, giving strict inequality both in Theorem 1 and in Remark 1. As a final remark, we emphasize having φ_1 log-concave is not enough to improve the bounds on the eigenvalue even if its boundary values are zero. To see this, simply take φ_1 to be trapezoidal in shape with long bases and a short height.

THEOREM 2. *If in* (1.1) *the potential V is* 0 *and $n=2$, we can obtain a better upper bound for the Dirichlet gap: $\lambda_2-\lambda_1 \leq C/d^{2/3}r^{4/3}$.*

Proof. Let η_1, η_2 be the first two eigenvalues for the Schrödinger operator $-d^2/dx^2+\pi^2/p(x)^2$ on $[0,d]$, where $p(x)$ is the cross-sectional length of Ω perpendicular to the diameter. Without loss of generality, we assume the domain to have inner radius 1. We can then apply Theorem A in [J], which states that

$$\eta_1 \leq \lambda_1 \quad \text{and} \quad \lambda_2 \leq \eta_2+\frac{c}{L^3},$$

where L is a natural length associated to Ω that satisfies $d^{1/3} \leq L \leq d$. Thus,

$$\lambda_2 - \lambda_1 \leq \eta_2 - \eta_1 + \frac{c}{L^3},$$

and by [J, Lemma 4.4], the right side can be majorized to obtain

$$\lambda_2 - \lambda_1 \leq \frac{c}{L^2} + \frac{c}{L^3} \leq \frac{C}{L^2} \leq \frac{C}{d^{2/3} r^{4/3}}. \qquad \square$$

Finally, if the potential $V \in L^\infty(\Omega)$ and only the inner radius is fixed, we have the following isoperimetric inequality.

THEOREM 3. *If $n = 2$ and $\lambda_1^{V,\Omega}, \lambda_2^{V,\Omega}$ are the first two Dirichlet eigenvalues for a domain Ω with inner radius r and potential $V \in L^\infty(\Omega)$, then* (1.3) *can be improved to*

$$\lambda_2^{V,\Omega} - \lambda_1^{V,\Omega} \leq \frac{8.899}{r^2} + 1.539(M-m).$$

Proof. The Payne–Pólya–Weinberger conjecture proved in [AB2] states that, for a nonnegative potential V^*,

$$\lambda_2^{V^*,\Omega} - \lambda_1^{V^*,\Omega} \leq 1.539 \lambda_1^{V^*,\Omega},$$

where equality occurs for a ball with potential $V = 0$. For given V, let $V^* = V - m \geq 0$. Then $\lambda_i^{V^*,\Omega} = \lambda_i^{V,\Omega} - m$ and the preceding result applied to V^* yields

$$\begin{aligned} \lambda_2^{V,\Omega} - \lambda_1^{V,\Omega} &= \lambda_2^{V^*,\Omega} - \lambda_1^{V^*,\Omega} \leq 1.539 \lambda_1^{V^*,\Omega} = 1.539(\lambda_1^{V,\Omega} - m) \\ &\leq 1.539(\lambda_1^{0,\Omega} + M - m) \leq 1.539(\lambda_1^{0,B_r} + M - m) \\ &= \lambda_2^{0,B_r} - \lambda_1^{0,B_r} + 1.539(M-m) \\ &\approx \frac{8.899}{r^2} + 1.539(M-m), \end{aligned}$$

with B_r the ball of radius r. The last inequality holds by domain monotonicity and the next-to-last by the variational formulation for the eigenvalues. $\square$

The same theorem continues to hold in higher dimension with 8.899 and 1.539 replaced by the appropriate constant of the gap for the n-dimensional ball of radius 1. If—in contrast to fixing the inner radius—we have only the area fixed, the gap need not be bounded even for convex sets, as we will soon see.

3. Neumann Conditions versus Dirichlet Conditions

In this section we discuss geometric results that hold for the Neumann gap μ_2 corresponding to (1.7). One of the early results in the area was the discovery by Payne and Weinberger [PW] of the following optimal inequality:

$$\mu_2 \geq \frac{\pi^2}{d^2}.$$

It is also true that there exists an upper bound in terms of the diameter. This result, which we prove below using elementary ideas, may also be shown by considering the convex domain under Neumann conditions as a submanifold of a manifold with positive Ricci curvature, and then applying a theorem of Cheng [C].

THEOREM 4. *For all convex domains in $\mathbb{R}^2$ of diameter d,*

$$\mu_2 \leq \frac{48}{d^2}.$$

Proof. We can take Ω in the (x, y) plane with the diameter on the x axis and

$$\int_\Omega x\,dy\,dx = 0.$$

The variational formulation of μ_2,

$$\mu_2 = \inf_{\int_\Omega u\,dz=0} \frac{\int_\Omega |\nabla u|^2\,dz}{\int_\Omega u^2\,dz}, \tag{3.1}$$

along with the choice $u(z) = u(x, y) = x$, allows us to reduce the proof to demonstrating

$$\frac{\int_\Omega dy\,dx}{\int_\Omega x^2\,dy\,dx} \leq \frac{c}{d^2}. \tag{3.2}$$

Let f, g be concave and convex functions (respectively) such that

$$\Omega = \{(x, y) : a < x < b,\ g(x) < y < f(x)\},$$

and set $h(x) = f(x) - g(x)$. We know that $h \geq 0$ and concave, so (3.2) will be proven if we can show

$$\int_a^b h(x)\,dx \leq \frac{c}{(b-a)^2} \int_a^b x^2 h(x)\,dx. \tag{3.3}$$

By considering $y^* = \sup_{a \leq x \leq b} h(x)$ and $x^* = h^{-1}(y^*)$, we define the piecewise linear interpolation

$$T(x) = \begin{cases} \dfrac{y^*}{x^* - a}(x - a), & a < x \leq x^*, \\[2ex] \dfrac{y^*}{b - x^*}(b - x), & x^* \leq x < b. \end{cases}$$

The left-hand side of (3.3) is majorized by $(b-a)y^*$, while by concavity the right-hand side is minorized by

$$\frac{c}{(b-a)^2} \int_a^b x^2 T(x)\,dx.$$

Hence (3.3) can be further reduced to showing

$$(b-a)y^* \le \frac{c}{(b-a)^2}\left[\int_a^{x^*} x^2 \frac{y^*}{(x^*-a)}(x-a)\,dx + \int_{x^*}^b x^2 \frac{y^*}{b-x^*}(b-x)\,dx\right], \tag{3.4}$$

or equivalently

$$(b-a)^3 \le \frac{c}{12}[b^3+b^2x^*+b(x^*)^2-a(x^*)^2-a^2x^*-a^3]. \tag{3.5}$$

Dividing (3.5) by $b-a$ yields

$$(b-a)^2 \le \frac{c}{12}[b^2+ab+a^2+(b+a)x^*+(x^*)^2],$$

and choosing $c=48$ we have

$$b^2-2ab+a^2 \le 4b^2+4ab+4a^2+4(a+b)x^*+4(x^*)^2.$$

The previous inequality is true because

$$0 \le 2(a+b)^2+(a+b+2x^*)^2,$$

so

$$\mu_2 \le \frac{48}{d^2}.$$

Notice that, if we fix the inner radius, we can apply the ideas of Theorem 3 to get the isoperimetric inequality

$$\mu_2^{\Omega} \le \mu_2^{\Omega^*} \le \frac{c}{r^2},$$

where $c=\mu_2(B_1)\approx 3.39$ is the eigenvalue for the ball of radius 1, and Ω^* is the ball of same volume as Ω.

We now give a brief table of $\mu_2 d^2$ for various convex domains, showing how the gap increases as a diamond shape region is approached.

Domain	Value
thin rectangle	$\mu_2 d^2 \sim \pi^2$
circle	$\mu_2 d^2 \sim 13.56$
thin sector	$\mu_2 d^2 \sim 14.682$
square	$\mu_2 d^2 = 2\cdot\pi^2$

An immediate corollary to Theorem 4 is the existence of a constant c for which

$$\mu_2 \le c(\lambda_2-\lambda_1).$$

On the other hand, there is no reverse inequality, as (1.10) and the next theorem demonstrate.

THEOREM 5. *For the sectors $\Omega_j=\{(r,\theta): 0\le r<1,\ 0<\theta<\pi/j\}$, the Dirichlet eigenvalue gap is unbounded as $j\to\infty$.*

Proof. By separation of variables, we find that the Dirichlet eigenfunctions of Ω_j are $u_{jm,k}(r,\theta)=\sin(jm\theta)J_{jm}(\sqrt{\lambda_{jm,k}}r)$, where $J_{jm}(r)$ is the Bessel function of order jm whose zeros are $\sqrt{\lambda_{jm,k}}$. The eigenvalues of Ω_j are thus $\lambda_{jm,k}$. In addition, for j large the first two eigenvalues have $m=1$, so λ_1 and λ_2 are the first and second zeroes of $J_j(\sqrt{\lambda})$.

By approximating the first two zeros of J_j (see [Ol]), we see that

$$\lambda_1 \sim j^2+3.712j^{4/3} \quad \text{and} \quad \lambda_2 \sim j^2+6.4892j^{4/3}.$$

From this we have

$$\lambda_2-\lambda_1 \sim 2.777j^{4/3} \to \infty.$$

With $|\Omega|$ and d the area and diameter of a domain Ω, Theorem 5 has the following consequence: There can not be a constant with

$$\lambda_2-\lambda_1 < \frac{c}{d^2} \quad \text{or even} \quad \lambda_2-\lambda_1 \le \frac{c}{|\Omega|}$$

for all convex domains Ω, since in our example $d=1$ and $|\Omega_j| \approx cj$.

4. The Robin Boundary Value Gap

If we fix $\alpha>0$ and consider the eigenvalue problem

$$\Delta u+\lambda u=0 \text{ in } \Omega, \tag{4.1}$$

$$\frac{\partial u}{\partial n}+\alpha u=0 \text{ on } \partial\Omega, \tag{4.2}$$

we obtain, as in the Dirichlet case, eigenvalues satisfying

$$0<\lambda_1(\alpha)<\lambda_2(\alpha)\le\lambda_3(\alpha)\le\cdots.$$

This third type of boundary condition, the *Robin boundary,* can be thought of as intermediary between that of the Neumann ($\alpha\to 0$) and that of the Dirichlet ($\alpha\to\infty$), and in terms of a physical interpretation it represents a boundary that is partially reflecting (insulated) and partially absorbing.

Because

$$\lambda_2(\alpha)-\lambda_1(\alpha)\to\mu_2 \quad \text{as } \alpha\to 0$$

and

$$\lambda_2(\alpha)-\lambda(\alpha)\to\lambda_2-\lambda_1 \quad \text{as } \alpha\to\infty,$$

we expect the lower bounds, which hold under both Neumann and Dirichlet boundary conditions, to remain valid when Robin boundary conditions are considered. For example, in convex domains we should have $\lambda_2(\alpha)-\lambda_1(\alpha)\ge \pi^2/d^2$. One possible approach to this is to mimic our proof of Theorem 1. However, we do not know if the first Robin eigenfunction is log-concave, which was a key point in our proof.

On an interval, say $(-\pi/2,\pi/2)$ in $\mathbb{R}^1$, we know that $\lambda_1(\alpha)$ is the first root of $\tan(\pi/2)\sqrt{x}=\alpha/\sqrt{x}$ and $\lambda_2(\alpha)$ is the first root of

$$\tan\left(\frac{\pi}{2}\sqrt{x}-\frac{\pi}{2}\right)=\frac{\alpha}{\sqrt{x}}.$$

Using implicit differentiation, we obtain

$$\frac{d\lambda_i}{d\alpha}=\frac{4\lambda_i}{\pi(\lambda_i+\alpha^2)+2\alpha},$$

and $\lambda_2(\alpha)>\lambda_1(\alpha)$ implies $d\lambda_2/d\alpha-d\lambda_1/d\alpha>0$, so that $\lambda_2(\alpha)-\lambda_1(\alpha)$ is increasing. We actually have this increasingness for all gaps $\lambda_j(\alpha)-\lambda_k(\alpha)$, $k<j$. We have also verified that the gap $\lambda_2(\alpha)-\lambda_1(\alpha)$ is increasing for rectangles and circles, which leads us to the following question: Is $\lambda_2(\alpha)-\lambda_1(\alpha)$ increasing for all bounded, convex domains in $\mathbb{R}^n$? We do not know the answer to this question.

5. Rates to Equilibrium

In the following we show how estimates from Sections 2 and 3 may be used to derive bounds on the rate at which heat kernels in convex domains tend uniformly to a fixed function as time increases.

If a domain $\Omega\subset\mathbb{R}^n$ is minimally smooth (convex is more than enough) then the Dirichlet heat semigroup of $\Delta+V$ is intrinsic ultracontractive, which is equivalent to having its heat kernel $K(t,x,y)$ satisfy

$$0\le K(t,x,y)\le C_t\varphi_1(x)\varphi_1(y) \tag{5.1}$$

for all $(x,y,t)\in D\times D\times(0,\infty)$, where $\varphi_1(x)$ is the base eigenfunction with $\|\varphi_1\|_{L^2}=1$ and C_t depends only on t and the domain (for this and related topics, see [B1]).

If we set

$$\tilde{K}(t,x,y)=\frac{e^{\lambda_1 t}K(t,x,y)}{\varphi_1(x)\varphi_1(y)},$$

we obtain a new semigroup on $L^2(\varphi_1^2)$. This is the semigroup for Brownian motion conditioned to remain forever in D, the Doob h-process associated with φ_1. The conditioned Brownian motion tends to spend most of its time in the "center" of the domain since this is where the eigenfunction is largest (see [B2, Prop. 2] for a precise statement of this). For smooth convex domains it is also reasonable to expect that the reflected Brownian motion will tend to be in the "center" much of the time, particularly in the long run. This motivated our comparison of time-to-equilibrium results, and hence gaps results, for the two processes.

Following Saloff-Coste [Sa], for each $\epsilon>0$ we define the time to equilibrium for the Brownian motion conditioned to remain forever in Ω to be

$$T_\epsilon=\inf\{t>0\colon \sup_{x,y\in\Omega}|\tilde{K}(t,x,y)-1|\le\epsilon\}$$

and for the reflected Brownian motion to be

$$\tilde{T}_\epsilon=\inf\{t>0\colon \sup_{x,y\in\Omega}|P(t,x,y)-1/|\Omega||\le\epsilon\},$$

where $P(t,x,y)$ denotes the heat kernel for the Neumann problem. The next theorem relates the spectral gaps $\lambda_2-\lambda_1$ and μ_2 to T_ϵ and $\tilde{T}_\epsilon$.

THEOREM 6. *Given a convex domain $\Omega \subset \mathbb{R}^n$, there exist constants C_1 and C_2 such that for all $t \geq 1$,*

$$e^{-(\lambda_2-\lambda_1)t} \leq \sup_{x,y\in\Omega}|\tilde{K}(t,x,y)-1| \leq C_1 e^{-(\lambda_2-\lambda_1)t}$$

and

$$e^{-\mu_2 t} \leq \sup_{x,y\in\Omega}|P(t,x,y)-1/|\Omega|| \leq C_2 e^{-\mu_2 t}.$$

Proof. Since $e^{-\lambda_1 t}\varphi_1(x) = \int_\Omega K(t,x,z)\varphi_1(z)\,dz$, the semigroup property of $K(t,x,y)$ gives

$$\begin{aligned}
|\tilde{K}(t,x,y)-1| &= \left|\int_\Omega \left(\tilde{K}\left(\frac{t}{2},x,z\right)-1\right)\left(\tilde{K}\left(\frac{t}{2},y,z\right)-1\right)\varphi_1^2(z)\,dz\right| \\
&\leq \left\|\tilde{K}\left(\frac{t}{2},x,z\right)-1\right\|_{L^2(\varphi_1^2 dz)} \left\|\tilde{K}\left(\frac{t}{2},y,z\right)-1\right\|_{L^2(\varphi^2 dz)} \\
&= \sqrt{\tilde{K}\left(\frac{t}{2},x,x\right)-1}\,\sqrt{\tilde{K}\left(\frac{t}{2},y,y\right)-1}.
\end{aligned} \tag{5.2}$$

Furthermore, the heat kernel expansion in terms of the orthogonal eigenfunctions $\varphi_n(x)$,

$$K(t,x,y) = \sum_{n=1}^{\infty} e^{-\lambda_n t}\varphi_n(x)\varphi_n(y), \tag{5.3}$$

implies

$$\begin{aligned}
|\tilde{K}(t,x,x)-1| &= \sum_{n=2}^{\infty} e^{-(\lambda_n-\lambda_1)t}\frac{\varphi_n^2(x)}{\varphi_1^2(x)} \\
&\leq \exp\left\{-\left(\frac{\lambda_2-\lambda_1}{2}\right)t\right\} \sum_{n=2}^{\infty} \exp\left\{-\frac{(\lambda_n-\lambda_1)t}{2}\right\}\frac{\varphi_n^2(x)}{\varphi_1^2(x)} \\
&\leq \exp\left\{-\frac{(\lambda_2-\lambda_1)t}{2}\right\}\left|\tilde{K}\left(\frac{t}{2},x,x\right)-1\right|
\end{aligned}$$

$$\Rightarrow |\tilde{K}(t,x,x)-1| \leq \exp\left\{-(\lambda_2-\lambda_1)\left(t-\frac{t}{2^j}\right)\right\}\left|\tilde{K}\left(\frac{t}{2^j},x,x\right)-1\right|. \tag{5.4}$$

Picking j an integer with $t/2^j \in [\frac{1}{2}, 1]$ (i.e. $t \geq 1$) and calling

$$c = \sup_{t\in[1/2,1],\,x\in\Omega} |\tilde{K}(t,x,x)-1| < \infty,$$

by (5.4) we have

$$|\tilde{K}(t,x,x)-1| \leq c\exp\{-(\lambda_2-\lambda_1)t\}\exp\left\{(\lambda_2-\lambda_1)\frac{t}{2^j}\right\}, \quad t \geq 1, \tag{5.5}$$

and, using (5.2) and (5.5) together with $t/2^j \leq 1$, we have

$$|\tilde{K}(t,x,y)-1| \le Ce^{-(\lambda_2-\lambda_1)t}, \quad t \ge 1.$$

On the other hand, by (5.3) we may write

$$\begin{aligned}
\frac{e^{\lambda_1 t-\lambda_2 t}\varphi_2(x)}{\varphi_1(x)} &= \frac{e^{\lambda_1 t}}{\varphi_1(x)}\int_\Omega K(t,x,y)\frac{\varphi_2(y)}{\varphi_1^2(y)}\varphi_1^2(y)\,dy \\
&= \int_\Omega \tilde{K}(t,x,y)\frac{\varphi_2(y)}{\varphi_1(y)}\varphi_1^2(y)\,dy - \int_\Omega \frac{\varphi_2(y)}{\varphi_1(y)}\varphi_1^2(y)\,dy \\
&\le \int_\Omega |\tilde{K}(t,x,y)-1|\left|\frac{\varphi_2(y)}{\varphi_1(y)}\right|\varphi_1^2(y)\,dy \\
&\le \sup_{y\in\Omega}\left|\frac{\varphi_2(y)}{\varphi_1(y)}\right| \sup_{x,y\in\Omega}|\tilde{K}(t,x,y)-1|.
\end{aligned}$$

Taking the supremum on the left-hand side yields

$$e^{\lambda_1 t-\lambda_2 t} = e^{-(\lambda_2-\lambda_1)t} \le \sup_{x,y}|\tilde{K}(t,x,y)-1|,$$

and we conclude that $\sup|\tilde{K}(t,x,y)-1| \sim e^{-(\lambda_2-\lambda_1)t}$, $t \ge 1$. □

The same proof would apply equally under Neumann conditions. Indeed, the key point in our proof is (5.1). Such an estimate holds for the Neumann heat kernel in our domains by [D, Thm. 2.4.4].

Theorem 6 and the results of Sections 2 and 3 allow us to estimate T_ϵ and $\tilde{T}_\epsilon$ by d^2 and r^2. For example, we have the following result.

COROLLARY. *Let Ω be a convex domain in $\mathbb{R}^2$ of diameter d and inner radius r, and assume that $V=0$. Then there exist C_Ω^1 and C_Ω^2 such that*

$$\frac{r^2}{8.899}\log\frac{1}{\epsilon} \le T_\epsilon \le C_\Omega^1 + \frac{d^2}{\pi^2}\log\frac{1}{\epsilon}$$

and

$$\frac{d^2}{48}\log\frac{1}{\epsilon} \le \tilde{T}_e \le C_\Omega^2 + \frac{d^2}{\pi^2}\log\frac{1}{\epsilon}.$$

From Theorem 6 we see that the conjectured lower bound on the Dirichlet gap is equivalent to the statement that among all convex domains of fixed diameter the rate to equilibrium is slowest for a thin rectangle, while Theorem 5 shows there are arbitrarily large rates to equilibrium even for fixed area. The situation is quite different in the Neumann case, where for fixed area uniformity of the heat kernel occurs with maximum rate for the ball. We also note that for domains of fixed inner radius 1, the fastest rate to equilibrium for either the Neumann or Dirichlet kernel is given by that of the unit ball.

Finally, the Robin problem does not seem to have been studed probabilistically. However, intuitively this should correspond to a diffusion which, upon hitting the boundary, escapes or reflects with a certain probability depending on the parameter α. On the domain Ω, the gap $\lambda_2(\alpha)-\lambda_1(\alpha)$ should then be a measure on the time to stability for this diffusion conditioned by the first strictly positive eigenfunction φ_1^α.

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