Local Homology Properties of Boundaries of Groups

MLADEN BESTVINA

0. Introduction

In this paper we formalize the concept of the boundary of a group (Definition 1.1). Even though a group might have different boundaries, certain global and local homological invariants of the boundary are determined by the cohomological invariants of the group. In particular, we show (Theorem 2.8) that the boundary of a Poincaré duality group is a homology manifold.

1. Boundaries of Groups

1.1. $\mathcal{Z}$-Structures on Groups

Recall that a compact metrizable space is a *Euclidean retract* (or ER) if it can be embedded in some Euclidean space as its retract, or (equivalently) if it is finite-dimensional, contractible, and locally contractible. A closed subset Z of a Euclidean retract $\tilde{X}$ is said to be a *Z-set* if any of the following equivalent conditions hold.

(a) There is a deformation $h_t: \tilde{X} \to \tilde{X}$ with $h_0 = \mathrm{id}$ and $h_t(\tilde{X}) \cap Z = \emptyset$ for $t > 0$.
(b) For every $\epsilon > 0$ there is a map $f: \tilde{X} \to \tilde{X}$ that is ϵ-close to the identity and whose image misses Z.
(c) For every open set $U \subset \tilde{X}$, the inclusion $U \setminus Z \hookrightarrow U$ is a homotopy equivalence.

DEFINITION 1.1. Let G be a group. A *$\mathcal{Z}$-structure* on G is a pair $(\tilde{X}, Z)$ of spaces satisfying the following four axioms.

(1) $\tilde{X}$ is an ER.
(2) Z is a Z-set in $\tilde{X}$.
(3) $X = \tilde{X} \setminus Z$ admits a covering space action of G with compact quotient.

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(4) The collection of translates of a compact set in X forms a null sequence in $\tilde{X}$; that is, for every open cover $\mathcal{U}$ of $\tilde{X}$, all but finitely many translates are $\mathcal{U}$-small.

A space Z is a *boundary* of G if there is a $\mathcal{Z}$-structure $(\tilde{X}, Z)$ on G.

It is an immediate consequence of the axioms that the orbit of each point in X accumulates on all of Z, and that for every $z \in Z$, every neighborhood $\tilde{U}$ of z in $\tilde{X}$, and every compact set $K \subset X$ there is $g \in G$ such that $g(K) \subset \tilde{U}$.

Similar axioms have been considered in the past, notably in attempts to establish the Novikov conjecture [CP; FH; FW].

In many examples the G-action on X extends to an action on $\tilde{X}$. It would seem natural to add this requirement to our definition (as in the articles just quoted), but we have found no use for it in the present work.

EXAMPLES 1.2. (i) Every torsion-free word-hyperbolic group G admits a $\mathcal{Z}$-structure in which X is the Rips complex of G and Z is the Gromov boundary ∂G [Gr]. The verification of the axioms in this case may be found in [BeMe].

(ii) If G admits a covering space action by isometries with compact quotient on a CAT(0)-space X, then G admits a $\mathcal{Z}$-structure via compactification by equivalence classes of geodesic rays. In particular, fundamental groups of nonpositively curved closed Riemannian manifolds admit a $\mathcal{Z}$-structure.

(iii) In many cases (braid groups, S-arithmetic groups, Baumslag–Solitar groups, ...) it is possible to give ad hoc constructions of structures satisfying axioms (1)–(3) (but not always (4)). We are not aware of any group G that admits a finite $K(G, 1)$ and does not have such a weak $\mathcal{Z}$-structure.

(iv) More concretely, the Baumslag–Solitar group

$$BS(1,2) = \langle x, t \mid t^{-1}xt = x^2 \rangle$$

has a $\mathcal{Z}$-structure with Z homeomorphic to the reduced suspension of the Cantor set (= "Cantor–Hawaiian earring"). The universal cover of the presentation 2-complex is contractible and can be viewed as the union of "sheets" homeomorphic to the plane. Each sheet can be compactified by adding a circle, and so that any two sheets share exactly one point at infinity. The compactification of sheets is *not* the usual compactification that results from viewing each sheet as a copy of the hyperbolic plane, but can be constructed from it by blowing up the special point on the circle (the parabolic fixed point) to an arc and collapsing the complementary arc to a point. Another way to construct such a compactification is to identify each sheet with the Euclidean plane so that the usual (CAT(0)) compactification satisfies axiom (4). A variation of this construction is described in Example 3.1.

(v) If G is a torsion-free, geometrically finite Kleinian group in $\mathrm{SO}(n, 1)$ with no parabolics, then $(\mathbb{H}^n \cup S^{n-1}, \Lambda)$ is a $\mathcal{Z}$-structure on G, where Λ denotes the limit set of G. Torsion-free uniform lattices in Lie groups admit a $\mathcal{Z}$-structure by (ii). Torsion-free nonuniform lattices in rank-1 Lie groups admit $\mathcal{Z}$-structure in which Z is a Sierpinski space. One blows up each parabolic fixed point on the sphere at infinity to the $(n-1)$-ball of directions leaving the point.

(vi) Benakli [Ben] has constructed groups whose boundaries are the Sierpinski curve and the Menger curve respectively. In [Dra] Dranishnikov constructs groups (finite-index subgroups of Coxeter groups) with boundary n-dimensional Menger compactum, or (alternatively) a space whose cohomological dimension depends on the ground field! See also [BeMe] and [Bes].

In practice, the most difficult axioms to verify are (1) and (2). The following [BeMe, Prop. 2.1] is useful.

LEMMA 1.3. *Let $(\tilde{X}, Z)$ be a pair of finite-dimensional compact metrizable spaces with Z nowhere dense in $\tilde{X}$, and such that $X = \tilde{X}\setminus Z$ is contractible and locally contractible and the following condition holds:*

(*) *For every $z \in Z$ and every neighborhood $\tilde{U}$ of z in $\tilde{X}$, there is a neighborhood $\tilde{V}$ of z contained in $\tilde{U}$ such that*

$$\tilde{V}\setminus Z \hookrightarrow \tilde{U}\setminus Z$$

is null-homotopic.

Then $\tilde{X}$ is an ER and Z is a Z-set in $\tilde{X}$.

LEMMA 1.4. *If $(\tilde{X}, Z)$ is a $\mathcal{Z}$-structure on G, and if K is a finite $K(G,1)$, then there is a natural $\mathcal{Z}$-structure $(\tilde{K}\cup Z, Z)$ on G, where $\tilde{K}$ denotes the universal cover of K.*

Proof. The topology on $\tilde{K}\cup Z$ comes from taking the closure of the diagonally embedded $\tilde{K}$ in $(\tilde{K}\cup\infty)\times\tilde{X}$ (the first factor is the one-point compactification of $\tilde{K}$, and the map to the second factor is any G-map $\tilde{K}\to X\hookrightarrow\tilde{X}$). That this is a $\mathcal{Z}$-structure on G follows from Lemma 1.3. □

Therefore, we may always assume without loss of generality that X is a cell complex, and that the G-action is cellular.

Similarly, one can show that if two groups G and G' with finite Eilenberg-MacLane spaces are quasi-isometric and if Z is a boundary of G, then Z is also a boundary of G'.

1.2. Global Invariants

Fix a PID $\mathbb{L}$. Throughout the paper, unless otherwise indicated, all homology and cohomology groups are taken with coefficients in $\mathbb{L}$. Also, Hom, $\otimes$, et cetera are over $\mathbb{L}$. We use the Steenrod homology theory (for a review see Section 1.3); the cohomology theory is Čech–Alexander–Spanier.

Fix a $\mathcal{Z}$-structure $(\tilde{X}, Z)$ on a group G. For the record, we state an immediate consequence of definitions, the long exact sequence of the pair $(\tilde{X}, Z)$, and the fact that $\tilde{X}$ is contractible.

PROPOSITION 1.5.

$$\tilde{H}^q(Z) \cong H_c^{q+1}(X) \cong H^{q+1}(G; \mathbb{L}G) \quad and \quad \tilde{H}_q(Z) \cong H_{q+1}^{\mathrm{LF}}(X).$$

The shape of Z is determined by G.

PROPOSITION 1.6. *If Z_1 and Z_2 are boundaries of G, then Z_1 and Z_2 have the same shape.*

Proof. The sequence of neighborhoods of Z_i in $\tilde{X}_i$ is pro-isomorphic to the sequence of complements of larger and larger compact sets in X. □

The following theorem computes the dimension of a boundary in terms of the cohomological dimension of the group. It was stated in [BeMe] in the case of hyperbolic groups. The proof is valid in the setting of $\mathcal{Z}$-structures. Recall that if the covering dimension $\dim Z$ of Z is finite, then $\dim Z = \dim_{\mathbb{Z}} Z$.

THEOREM 1.7. *Let Z be a boundary of G, and let $\mathbb{L}$ be a PID. Then*

$$dim_{\mathbb{L}} Z = \mathrm{cd}_{\mathbb{L}} G - 1.$$

In particular, $\dim Z = \mathrm{cd}\, G - 1$.

1.3. Review of the Steenrod Homology Theory

Let Z be a compact metrizable space. Embed Z as a Z-set in a compact absolute retract $\tilde{X}$ (e.g., if Z is finite-dimensional then we can embed it in the boundary of a ball; in general, we can embed it in a face of the Hilbert cube). Let $X := \tilde{X} \setminus Z$. The proper homotopy type of X depends only on Z. Following Steenrod [St], define the Steenrod reduced homology of Z by

$$\tilde{H}_i(Z) := \tilde{H}_{i+1}^{\mathrm{LF}}(X),$$

where the right-hand side stands for reduced singular homology based on locally finite chains. Steenrod did not shift dimensions. For more thorough expositions and generalizations of this concept, see [BoMo; Br; Mil; Fe]. When Z is a polyhedron (or, more generally, an ENR), we can take $\tilde{X}$ to be the cone on Z and easily see that $\tilde{H}_i(Z)$ agrees with the usual singular homology.

If Z' is another compact metric space embedded as a Z-set in an absolute retract $\tilde{X}'$, then any map $f: Z \to Z'$ can be extended to a map $F: \tilde{X} \to \tilde{X}'$ so that $F^{-1}(Z') = Z$. In particular, the restriction $X \to X' = \tilde{X}' \setminus Z'$ is proper and induces a homomorphism between homology groups based on locally finite chains. This homomorphism is declared to be $f_*: \tilde{H}_i(Z) \to \tilde{H}_i(Z')$. It depends only on f.

If A is a closed subset of Z, define

$$H_i(Z, A) := \tilde{H}_i(Z \cup cA).$$

It is a pleasant exercise to verify that this homology theory satisfies the usual axioms of Eilenberg and Steenrod.

If $z \in Z$, define local homology groups $H_i(Z, Z\backslash\{z\}) := \varinjlim_B H_i(Z, B)$ as B ranges over compact subsets of $Z\backslash\{z\}$. More generally, if $A \subseteq Z$ is a closed subspace, define $H_i(Z, Z\backslash A) := \varinjlim_B H_i(Z, B)$ as B ranges over compact subsets of $Z\backslash A$.

PROPOSITION 1.8. *Let Z be a Z-set in an AR $\tilde{X}$, and let $A \subseteq Z$ be a closed subspace. Choose a basis of neighborhoods $\{\tilde{U}_i\}$ of A in $\tilde{X}$ and let $U_i = X \cap \tilde{U}_i$. Then*

$$H_k(Z, Z\backslash A) = \varinjlim H_{k+1}^{\mathrm{LF}}(U_i).$$

Proof. Let $V_i := Z \cap \tilde{U}_i$. The pair $(\tilde{X}\backslash U_i, \tilde{X}\backslash\tilde{U}_i)$ is excision-equivalent to the pair $(Z, Z\backslash V_i)$ (via the pair $(\bar{V}_i, \partial V_i)$).

The sequence $\{\tilde{H}_*(\tilde{X}\backslash\tilde{U}_i)\}$ is pro-trivial (i.e., sufficiently long compositions of bonding homomorphisms are trivial) since A is a Z-set in $\tilde{X}$. Hence, for every i, there is a $j > i$ such that $\tilde{X}\backslash\tilde{U}_i$ is null-homotopic in $\tilde{X}\backslash\tilde{U}_j$. It follows that the direct sequences

$$H_*(\tilde{X}\backslash U_i, \tilde{X}\backslash\tilde{U}_i) \cong H_*(Z, Z\backslash V_i)$$

and

$$\tilde{H}_*(\tilde{X}\backslash U_i) \cong H_{*+1}^{\mathrm{LF}}(U_i)$$

have isomorphic limits. □

REMARK 1.9. Let Y be a locally compact polyhedron. If $\mathbb{L}$ is a field, then $H_k^{\mathrm{LF}}(Y) \cong H_c^k(Y)^*$ (the right-hand side is the dual of compactly supported cohomology). More generally, if $\mathbb{L}$ is a PID, then the universal coefficient theorem provides a short exact sequence:

$$0 \to \mathrm{Ext}(H_c^{k+1}(Y); \mathbb{L}) \to H_k^{\mathrm{LF}}(Y) \to \mathrm{Hom}(H_c^k(Y); \mathbb{L}) \to 0.$$

1.4. Rigidity of Global Cycles in Boundaries

The following proposition imposes restrictions on the nature of the local topology of a boundary of a group.

PROPOSITION 1.10. *Suppose that $\mathbb{L}$ is a countable field, and that Z is a boundary of G.*

(1) *If $H^{q+1}(G; \mathbb{L}G)$ is finite-dimensional, then*

$$H^{q+1}(G; \mathbb{L}G)^* = H_{q+1}^{\mathrm{LF}}(X) = \tilde{H}_q(Z) \hookrightarrow H_q(Z, Z\backslash\{z\})$$

is injective for every $z \in Z$.

(2) *If there is a point $z \in Z$ such that the local homology module $H_q(Z, Z\backslash\{z\})$ is countable, then $H^{q+1}(G; \mathbb{L}G)$ is finite-dimensional.*

Proof. Let U_i be as in the proof of Proposition 1.8 with $A = \{z\}$.

(1) Observe first that, for every i, the inclusion-induced homomorphism

$$H_c^{q+1}(U_i) \to H_c^{q+1}(X)$$

is surjective. Indeed, since $\dim_{\mathbb{L}} H_c^{q+1}(X) < \infty$, there is a compact set $K \subset X$ such that each element of a basis of $H_c^{q+1}(X)$ has a representative cocycle with support in K. By axiom (4) of Definition 1.1, there is a group element $g \in G$ such that $g(K) \subset U_i$. Since g induces an isomorphism of $H_c^{q+1}(X)$, it follows that $g(K)$ contains the support of representative cocycles of a (possibly different) basis of $H_c^{q+1}(X)$, and the claim is established.

By taking duals, we see that $H_{q+1}^{\mathrm{LF}}(X) \to H_{q+1}^{\mathrm{LF}}(U_i)$ is injective for each i. Taking the limit as $i \to \infty$ implies (1).

(2) Consider the inverse sequence $\{H_c^{q+1}(U_i)\}_i$ of vector spaces. Then either (a) for every i there is a $j > i$ such that $\operatorname{Im}[H_c^{q+1}(U_j) \to H_c^{q+1}(U_i)]$ is finite-dimensional, or (b) the direct limit of the sequence $\{\operatorname{Hom}(H_c^{q+1}(U_i), \mathbb{L})\}_i$ is uncountable. This direct limit can be identified with $H_q(Z, Z\backslash\{z\})$ by Proposition 1.8 and Remark 1.9, and thus the former statement must hold. In particular, we can find i such that $s := \dim_{\mathbb{L}} \operatorname{Im}[H_c^{q+1}(U_i) \to H_c^{q+1}(X) = H^{q+1}(G; \mathbb{L}G)] < \infty$. If $\dim_{\mathbb{L}} H_c^{q+1}(X) > s$, then choose linearly independent classes $x_1, x_2, \dots, x_{s+1} \in H_c^{q+1}(X)$. There is a compact set $K \subset X$ such that each x_k has a representative cocycle with support in K. Let $g \in G$ be a group element such that $g(K) \subset U_i$. By the choice of s, the g-translates of the x_ks are linearly dependent in $H_c^{q+1}(X)$, contradicting the fact that g induces an isomorphism of $H_c^{q+1}(X)$. □

Thus, in the presence of finite dimensionality of global homology of Z, all global cycles are "hung up" on each point of Z; that is, the support of a nontrivial global class cannot be a proper subset of Z.

The following corollary was proved in [BeMe] for Gromov boundaries of hyperbolic groups. The proof there uses the dynamics of the action of the group on the boundary and does not apply in the nonhyperbolic setting.

COROLLARY 1.11. *Let $\mathbb{L}$ be a countable field. Suppose some $H^{q+1}(G; \mathbb{L}G)$ is nontrivial and finite-dimensional over $\mathbb{L}$. Then Z does not contain a cut point, that is, Z cannot be written nontrivially as the wedge of two compact spaces.*

Proof. Suppose that $Z = A \cup B$ and $A \cap B = \{z\}$, with A and B compact and not singletons. Then $\tilde{H}_q(Z) \cong \tilde{H}_q(A) \oplus \tilde{H}_q(B)$ and thus one of $\tilde{H}_q(A)$ or $\tilde{H}_q(B)$, say $\tilde{H}_q(A)$, is nontrivial. But then cycles in $\tilde{H}_q(A)$ are not hung up on points in $B\backslash\{z\}$. □

REMARK 1.12. In all examples known to the author, it is true that if

$$\dim_{\mathbb{L}} H_c^{q+1}(X) < \infty$$

then $\tilde{H}_q(Z) \to H_q(Z, Z\backslash\{z\})$ is an isomorphism for every $z \in Z$. In particular, the local homology sheaf $\mathcal{H}_q$ of Z is constant. (See [Br]; to an open set $V \subset Z$ the sheaf $\mathcal{H}_q$ assigns $H_q(Z, Z\backslash V)$, and the stalk over $z \in Z$ is $H_q(Z, Z\backslash\{z\})$). This is the case if the compactly supported cochain complex of X is regular (see Definition 2.5).

REMARK 1.13. There is an analog of Proposition 1.10 where one assumes only that $\mathbb{L}$ is a countable PID and that $H^k(G;\mathbb{L}G)$ is finitely generated as an $\mathbb{L}$-module.

(1) goes through with no changes. By construction, $H_c^{q+1}(U_i) \to H_c^{q+1}(X)$ is a *split* surjection. This implies that

$$\mathrm{Hom}(H_c^{q+1}(X), \mathbb{L}) \to \mathrm{Hom}(H_c^{q+1}(U_i), \mathbb{L})$$

and

$$\mathrm{Ext}(H_c^{q+1}(X), \mathbb{L}) \to \mathrm{Ext}(H_c^{q+1}(U_i), \mathbb{L})$$

are injective, and then Remark 1.9 and a diagram chase show that

$$H_{q+1}^{\mathrm{LF}}(X) \to H_{q+1}^{\mathrm{LF}}(U_i)$$

is also injective.

(2) is more technical. A theorem of Mitchell [Mit] states that if $\mathbb{L}$ is a countable PID and if $\{P_i\}_i$ is an inverse sequence of countable $\mathbb{L}$-modules, then the following two statements are equivalent:

(a) for every i there is a $j > i$ such that $\mathrm{Im}[P_j \to P_i]$ is finitely generated;
(b) $\lim \mathrm{Hom}(P_i, \mathbb{L})$ and $\lim \mathrm{Ext}(P_i, \mathbb{L})$ are countable.

Applying this to the sequence $\{H_c^{q+1}(U_i)\}_i$ we see that either (a) for every i there is a $j > i$ such that $\mathrm{Im}[H_c^{q+1}(U_j) \to H_c^{q+1}(U_i)]$ is finitely generated, or (b) one of $\lim \mathrm{Hom}(H_c^{q+1}(U_i), \mathbb{L})$ and $\lim \mathrm{Ext}(H_c^{q+1}(U_i), \mathbb{L})$ is uncountable. In the first case we conclude that every finitely generated submodule of $H^{q+1}(G;\mathbb{L}G)$ embeds into a fixed finitely generated module. For example, when $\mathbb{L} = \mathbb{Z}$, this implies that $H^{q+1}(G;\mathbb{Z}G)$ is the direct sum of a finite abelian group and a subgroup H of a finite-dimensional vector space over the rationals. (There are also restrictions on the kind of subgroups H allowed. There must be a finitely generated group of automorphisms of H such that every element of H can be transformed into a fixed lattice by one of the automorphisms in the group; e.g., if $\mathrm{rk}\, H = 1$ then $H \cong \mathbb{Z}[r_1, \ldots, r_m]$ for some $r_1, \ldots, r_m \in \mathbb{Q}$.) In the second case we conclude, by Remark 1.9, that either $H_q(Z, Z\backslash\{z\})$ or $H_{q-1}(Z, Z\backslash\{z\})$ is uncountable.

The dual statement to Proposition 1.10(1) about cohomology goes through for PIDs with no change.

PROPOSITION 1.14. *If $H^{q+1}(G;\mathbb{L}G)$ is finitely generated as an $\mathbb{L}$-module and $q > 0$, then each class in $H^q(Z)$ is carried by every nonempty open set U. That is,*

$$H^q(Z, Z\backslash U) \to H^q(Z)$$

is onto. In particular, if $H^{q+1}(G;\mathbb{L}G)$ and $H^{r+1}(G;\mathbb{L}G)$ are finitely generated, $q, r > 0$, and $u \in H^q(Z)$ and $v \in H^r(Z)$, then $u \cup v = 0$.

If *$H^{q+1}(G;\mathbb{L}G)$ is not finitely generated, then for every nonempty open set $U \subset Z$ the image of $H^q(Z, Z\backslash U) \to H^q(Z)$ contains an isomorphic copy of every finitely generated submodule of $H^{q+1}(G;\mathbb{L}G)$. In particular, if $\mathbb{L}$ is a field then this image is infinite-dimensional.*

Proof. Let $h_t\colon \tilde{X} \to \tilde{X}$ be a small deformation of the identity to a map that misses Z. Use a deck transformation $g \in G$ to translate cocycles representing a finite subset of $H_c^{q+1}(X)$ close to a point in U. The inverse image under h_t of the compact supports of these cocycles intersects Z in a subset of U and represents the carrier of the g-translates of the finite collection in $H_c^{q+1}(X) \cong \tilde{H}^q(Z)$. □

Proposition 1.14 immediately yields Corollary 1.11 in the more general situation when $\mathbb{L}$ is a PID.

EXAMPLE 1.15. The Cantor–Hawaiian earring (= the reduced suspension of the Cantor set) is a boundary of $BS(1,2)$. It is not locally connected, and has a cut point.

The amalgamated product of two surface groups over $\mathbb{Z}$ has a natural boundary (coming from the CAT(0) structure) made up of circles. In particular, there are separating arcs. Collapse one to obtain a boundary with a cut point. This construction is not equivariant.

COROLLARY 1.16. *For every nonempty open set $U \subset Z$,*

$$\dim_{\mathbb{L}}(U) = \dim_{\mathbb{L}}(Z) \quad (= \mathrm{cd}_{\mathbb{L}}\, G - 1).$$

Proof. If $n = \mathrm{cd}_{\mathbb{L}}(G)$ then $\tilde{H}^{n-1}(Z) = H^n(G; \mathbb{L}G) \neq 0$. Thus every U carries a cohomology class in $H^{n-1}(Z)$ and its dimension (over $\mathbb{L}$) cannot be smaller than $n-1$. □

1.5. Axiom H

Consider the following axiom. A sequence $U_1 \supset U_2 \supset \cdots$ of open sets in X is *basic* for $z \in Z$ if there is a sequence $\tilde{W}_1 \supset \tilde{W}_2 \supset \cdots$ of neighborhoods of $z \in \tilde{X}$ forming a basis at z so that sequences $\{\tilde{W}_i \cap X\}$ and $\{U_i\}$ are cofinal in each other.

AXIOM H. For every $z \in Z$ there is a basic sequence $\{U_i\}$ such that, for every $n > 1$ and every compact set $K \subset X$, there exists $g \in G$ such that

(a) $g(U_1 \cup K) \subset U_n$ and
(b) $g(U_n) \supset U_m$ for some $m > n$.

The letter H stands for "hyperbolic" (see Proposition 1.18). For example, Axiom H fails for the standard boundary of $\mathbb{Z}^n$ for $n > 1$.

PROPOSITION 1.17. *Let Z be a boundary of G and assume Axiom H. If $\mathbb{L}$ is a countable field, $q \geq 0$, and $z \in Z$, then one of the following holds.*

(1) *The natural map $H_q(Z) \to H_q(Z, Z\backslash\{z\})$ is an isomorphism and the two vector spaces are finite-dimensional.*
(2) *$H_q(Z, Z\backslash\{z\})$ is uncountable.*

Proof. Let $\{U_i\}$ be a basic sequence at z as in Axiom H. Either there is an $n > 1$ such that $V = \mathrm{Im}[H_c^{q+1}(U_n) \to H_c^{q+1}(U_1)]$ has dimension $s < \infty$, or

$H_q(Z, Z\backslash\{z\}) = \lim H_c^{q+1}(U_i)$ is uncountable. We show that (1) is implied by the former possibility.

We may assume that n is chosen so that s is as small as possible. By the argument of Proposition 1.10, we conclude that $\dim H_c^{q+1}(X) \leq s$ (for otherwise $s+1$ linearly independent classes could be translated into U_n and would yield $s+1$ linearly independent classes in V) and that $H_c^{q+1}(U_1) \to H_c^{q+1}(X)$ maps V onto $H_c^{q+1}(X)$. We now claim that V maps isomorphically onto $H_c^{q+1}(X)$. Indeed, if not then there is a compact set $K \subset X$ such that the inclusion-induced homomorphism $V \to H_c^{q+1}(U_1 \cup \operatorname{int} K)$ has nontrivial kernel and image of dimension $< s$. By Axiom H, there is $g \in G$ and $m > n$ such that $U_m \subset g(U_n) \subset g(U_1 \cup \operatorname{int} K) \subset U_1$ and hence $H_c^{q+1}(U_m) \to H_c^{q+1}(U_1)$ has image of dimension $< s$, contradicting the choice of n.

If $i > n$ is given, by Axiom H there is a $j > i$ such that $H_c^{q+1}(U_j) \to H_c^{q+1}(U_i)$ has image of dimension $\leq s$ (since we can interpolate inclusion $g(U_n) \subset g(U_1)$ into $U_j \subset U_i$). But this image maps onto $H_c^{q+1}(X)$ as before. Thus the sequence $\{H_c^{q+1}(U_i)\}$ is pro-isomorphic to $H_c^{q+1}(X)$ and (1) follows. □

PROPOSITION 1.18. *If G is word-hyperbolic and Z the Gromov boundary, then Axiom H holds.*

Proof. To explain the idea, we first focus on the case when G is the fundamental group of a closed hyperbolic manifold M and Z is the sphere at infinity of the universal cover $\tilde{M}$. For a given $z \in Z$, choose a geodesic ray r in $\tilde{M}$ that converges to z. For each point x on the ray, consider the half-space U_x determined by the ray $[x, z)$, namely, the set of points y in $\tilde{M}$ such that $\angle yxz < \pi/2$. For a sequence of points x_i approaching z along r, these half-spaces $U_i := U_{x_i}$ form a basic sequence for z (this would not be true in Euclidean space). Since the unit tangent bundle T_1M of M is compact, we can choose such a sequence of points x_i so that the unit direction vectors at x_i pointing toward z are close when projected to T_1M. In particular, there is a covering transformation g_i that carries x_1 close to x_i and the unit direction vector at x_1 close to the one at x_i. If K and U_n are given, for all sufficiently large i we will have $g_i(U_1 \cup K) \subset U_n$ and $g_i(U_n)$ will contain some $U_{m(i)}$. Thus we have verified Axiom H in this case.

The general case follows similarly by the "deltafication" of the foregoing argument. Let G be a word-hyperbolic group and Γ its Cayley graph. Recall that by definition of word-hyperbolicity there exists δ such that for every geodesic triangle ABC in Γ the side AB is contained in the δ-neighborhood of the union $AC \cup BC$ of the other two sides. In particular, note the following consequence. If Q is a point on a geodesic segment PR and if $d(P, Q) > 2\delta$ and $d(Q, R) > 2\delta$, then for every $S \in \Gamma$ we have $d(Q, S) < \max\{d(P, S), d(R, S)\}$. (*Proof:* Apply the definition to the triangle PRS to conclude that Q is within δ of a point T in PS or RS, say PS. Since $d(P, Q) > 2\delta$, it follows by triangle inequality that $d(P, T) > \delta$ and hence $d(S, Q) \leq d(Q, T) + d(T, S) \leq \delta + d(T, S) < d(P, T) + d(T, S) = d(P, S)$.)

Now recall that the Rips complex $\tilde{M}$ is the simplicial complex with vertex set G, and that a collection of vertices spans a simplex if the pairwise distance between the vertices in the collection is $\leq d$, where d is a large integer. The Cayley graph Γ is naturally contained in $\tilde{M}$. A point z in the Gromov boundary is determined by a geodesic ray r, which may be taken to be contained in Γ. Instead of unit tangent vectors we will consider pairs of vertices (x, x') in $\tilde{M}$ at a fixed distance $L > 2\delta$ (measured in Γ) from each other. The half-space $U_{(x,x')}$ is defined to be the interior of the subcomplex of $\tilde{M}$ spanned by the vertices y in $\tilde{M}$ that are closer to x' than to x.

Now choose a sequence of vertices $x_1, x_1', x_2, x_2', x_3, x_3', \ldots$ along r occurring in that order so that $d(x_i, x_i') = L$ and $d(x_i', x_{i+1}) > 2\delta$. By passing to a subsequence we may also assume that there are group elements g_i with $g_i(x_1) = x_i$ and $g_i(y_1) = y_i$. Let $U_i = U_{(x_i, y_i)}$. By the preceding fact we have $U_i \supset U_{i+1}$. We now argue that the sequence $\{U_i\}$ is basic for z. This follows from the two following claims.

Claim 1: If y is a vertex that is closer to x_i than to x_i', then for all $n > i$ the distance between x_1 and any geodesic segment $[x_n, y]$ does not exceed $\delta + d(x_1, x_{i+1})$.

Indeed, let P and Q be points on $[x_n, y] \cup [y, x_1]$ at distance $\leq \delta$ from x_{i+1} and x_i, respectively. If one of P, Q belongs to $[x_n, y]$ we are done, so assume that they both belong to $[y, x_1]$. Since $d(x_i, x_{i+1}) > 4\delta$, it must be that P is between y and Q and $d(P, Q) > 2\delta$. Now we have $d(y, x_{i+1}) \leq d(y, P) + \delta = d(y, Q) - d(P, Q) + \delta \leq d(y, x_i) + \delta - d(P, Q) + \delta < d(y, x_i)$, contradicting the fact established before.

Claim 2: If y is a vertex that is closer to x_i' than to x_i, then for all $n > i$ the distance between x_1 and any geodesic segment $[x_n, y]$ is at least $d(x_1, x_i) - \delta$.

Indeed, let P be a vertex of $[x_n, y]$ closest to x_1. Let R and S be points in $[x_n, P] \cup [P, x_1]$ at distance $\leq \delta$ from x_i and x_{i+1}, respectively. If one of R, S belongs to $[P, x_1]$ we are done, so assume that they belong to $[x_n, P]$. Since $d(x_i, x_{i+1}) > 4\delta$, the triangle inequality shows that R is between P and S and $d(P, S) > 2\delta$. We now have $d(y, x_i) \leq \delta + d(R, y) = \delta + d(S, y) - d(S, R) \leq \delta + d(x_{i+1}, y) + \delta - d(S, R) < d(x_{i+1}, y)$, a contradiction.

Because $g_i(U_1) = U_i$, Axiom H follows. □

EXAMPLE 1.19. A "dendrite" made up of a null-sequence of 2-spheres with a dense collection of cut points is not an Axiom H boundary, for it has points where local first homology is trivial, and other points (the cut points) where it is nonzero and countable.

2. The Boundary of a Poincaré Duality Group

DEFINITION 2.1. A compact metrizable space Z of finite covering dimension is said to be a *homology n-manifold* over $\mathbb{L}$ if $H_*(Z, Z\setminus\{z\}) \cong H_*(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\})$ for all $z \in Z$. Further, Z is a *homology n-sphere* if in addition $\tilde{H}_*(Z) \cong \tilde{H}_*(S^n)$.

DEFINITION 2.2. A group G is a *Poincaré duality group of dimension n* over $\mathbb{L}$ (or a PD^n-group) if it acts freely, properly discontinuously, and cocompactly on a contractible cell complex Y with $H_c^*(Y) \cong H_c^*(\mathbb{R}^n)$. The induced action of G on $H_c^*(Y)$, that is, the homomorphism ρ from G into the group of units of $\mathbb{L}$, is the *orientation character.*

EXAMPLES 2.3. Every PD^n-group over $\mathbb{Z}$ is a PD^n-group over any field.

There exist groups that are PD^n over some fields but not over others. One way to construct such groups is to start with a closed PL-manifold M^{n-1} that is a homology sphere over some fields but not over others (e.g. a lens space), and then hyperbolize [CD; DJ; Gr] the suspension of M.

2.1. Regular Chain Complexes

Fix a group G with a $\mathcal{Z}$-structure $(\tilde{X}, Z)$, and let $X = \tilde{X} \backslash Z$.

DEFINITION 2.4. Let

$$\underline{C} = (0 \to C_l \to C_{l-1} \to \cdots \to C_{k+1} \to C_k \to 0)$$

be a chain complex of finitely generated free $\mathbb{L}G$-modules. The subscript denotes dimension. For every C_i, choose an $\mathbb{L}G$-isomorphism ϕ_i between C_i and the free abelian group on points in finitely many distinct orbits in X (the latter is an $\mathbb{L}G$-module via the natural action of G). The *support* of $c \in C_i$, denoted Supp(c), is the finite subset of X consisting of those points in the selected orbits that have nonzero coefficient in the expression of $\phi_i(c)$ in the basis.

Note that if Supp$'$ is the notion of support with respect to different choices of orbits and ϕ_i, then there is a constant $M > 0$ such that, for any c, any point in Supp(c) is connected to a point in Supp$'$(c) by a chain of fewer than M fundamental domains, and vice versa.

DEFINITION 2.5. We say that $\underline{C}$ is *regular* if, for every $z \in Z$ and every open set $\tilde{U} \subset \tilde{X}$ containing z, there exists a smaller open set $\tilde{V} \subset \tilde{X}$ containing z such that, for all i, if $c \in \mathrm{Im}(\partial\colon C_{i+1} \to C_i)$ and Supp(c) is contained in $\tilde{V}$ then there exists a $d \in C_{i+1}$ such that $c = \partial d$ and Supp(d) is contained in $\tilde{U}$.

In light of axiom (4) of Definition 1.1, the notion of regularity does not depend on the choice of ϕ_is.

Topological interpretation: Let Y be a CW complex such that G acts on Y freely, cellularly, and cocompactly, and let Y_0 be an equivariant subcomplex of Y. We can compactify Y as in Lemma 1.4 by Z. To say that the relative cellular chain complex of (Y, Y_0) is *regular* means that if a relative cycle near a boundary point z bounds in all of Y, then it bounds near z. In general, of course, there will be cycles near z that do not bound at all.

EXAMPLES 2.6. (i) Suppose X is a cell complex and G acts on X cellularly. Then the cellular chain complex $(C_i(X))$ is regular. The reason is that $c \in C_i$

from the definition can be identified with a (reduced) i-cycle near a boundary point $z \in Z$. Since $\tilde{X}$ is locally contractible at z, we can cone off c, staying close to z. Further, since Z is a Z-set, we may assume that the cone misses Z and thus can be interpreted as a chain in C_{i+1} with boundary c.

(ii) If $\underline{C}$ and $\underline{D}$ are homotopy equivalent (and both are free and finitely generated), and one is regular, then so is the other.

PROPOSITION 2.7. *Let $(\tilde{X}, Z)$ be a $\mathcal{Z}$-structure on an orientable Poincaré duality group G of dimension n (over $\mathbb{L}$). Also assume that $X = \tilde{X} \setminus Z$ is a cell complex and that G acts cellularly. Then the cellular cochain complex*

$$0 \to C_c^0(X) \to C_c^1(X) \to \cdots \to C_c^m(X) \to 0$$

is regular (*where* $m = \dim X$).

Proof. First assume that $m = n$. The two resolutions of the trivial $\mathbb{L}G$-module $\mathbb{L}$,

$$0 \to C_c^0(X) \to C_c^1(X) \to \cdots \to C_c^n(X) \to \mathbb{L} \to 0$$

and

$$0 \to C_n(X) \to C_{n-1}(X) \to \cdots \to C_0(X) \to \mathbb{L} \to 0,$$

are homotopy equivalent. Since the cellular chain complex is regular, so is the compactly supported cochain complex.

If $m > n$ then, using the standard cell-trading arguments, one trades away cells of dimension $> n$ and constructs a chain complex of finitely generated free $\mathbb{L}G$-modules,

$$0 \to C_c^0(X) \to C_c^1(X) \to \cdots \to C_c^{n-2}(X) \to \hat{C}_c^{n-1}(X) \to \hat{C}_c^n(X) \to 0,$$

homotopy equivalent to the compactly supported cochain complex (shifted $m-n$ places). This complex is a free resolution of $\mathbb{L}$ and thus is regular, since it is homotopy equivalent to the cellular chain complex of X. Thus the compactly supported cochain complex of X is also regular. □

2.2. *The Boundary of a PD^n-Group*

In the proof of Theorem 2.8 we must assume that the image of the orientation character is finite (i.e., that G is *virtually orientable*).

Questions: Is there a PD^n-group (say, over $\mathbb{Q}$) such that the image of the orientation character is infinite (or at least has more than two elements)? Is there an orientable $\mathbb{Q}$-homology manifold Z and a homeomorphism $Z \to Z$ that acts on the fundamental class by doubling it?

THEOREM 2.8. *Let G be a Poincaré duality group of dimension n over $\mathbb{L}$ such that the image of its orientation character is finite. If Z is a boundary of G, then Z is a homology n-manifold over $\mathbb{L}$ and moreover Z is a homology $(n-1)$-sphere over $\mathbb{L}$.*

Proof. Fix a $\mathcal{Z}$-structure $(\tilde{X}, Z)$ on G. By passing to a subgroup of finite index, we may assume that G is orientable (i.e., that the orientation character is trivial). First, $\tilde{H}_*(Z) \cong \tilde{H}^{\mathrm{LF}}_{*+1}(X)$ (by definition of the Steenrod homology and axioms (1) and (2) of Definition 1.1) and hence $\tilde{H}_*(Z) \cong \tilde{H}_*(S^{n-1})$ by the universal coefficient theorem.

Let $z \in Z$ be an arbitrary point, and choose a nested sequence of basic open sets $\tilde{U}_i$ at z in $\tilde{X}$. Let $U_i = X \cap \tilde{U}_i$. We claim that the inverse sequence $\{H^k_c(U_i)\}_i$ is pro-trivial when $k \neq n$ and that it is pro-isomorphic to $\mathbb{L}$ when $k = n$. The theorem then follows from Proposition 1.8 and Remark 1.9.

By axiom (4) of Definition 1.1, for every compactly supported cocycle c in X there are elements $g \in G$ that translate c arbitrarily close to $z \in Z$. By assumption, these translated cocycles represent the same class as c. Thus, to prove the claim, it suffices to show that whenever x is a compactly supported cocycle in X with support near z that cobounds a compactly supported cochain in X, then it cobounds a compactly supported cochain with support near z. That this is true is the content of Proposition 2.7. $\square$

REMARK 2.9. Since homology manifolds of dimension ≤ 2 are manifolds [Wi], Z is homeomorphic to S^{n-1} when $n \leq 3$. Thus the previous result generalizes a theorem in [BeMe]. When $n > 3$, Z may not be homeomorphic to S^{n-1} (or even locally simply connected). The examples stem from the work of Davis [Da; DJ].

3. Questions

3.1. Existence

There seems to be no systematic method of constructing boundaries of groups in general, so the following is probably hopeless.

Question: Does every group G with finite $K(G, 1)$ have a $\mathcal{Z}$-structure?

Chapman and Siebenmann [CS] have developed an obstruction theory for compactifying noncompact I^∞-manifolds to compact I^∞-manifolds. By a theorem of Edwards, a product of an ER and I^∞ is homeomorphic to I^∞, so the vanishing of Chapman–Siebenmann obstructions for $X \times I^\infty$ is a necessary condition for the existence of a $\mathcal{Z}$-structure. One way to state this condition is to say that X is proper simple homotopy equivalent to the mapping telescope of an inverse sequence of compact polyhedra.

3.2. Uniqueness

In general, a boundary Z is not uniquely determined by G. For example, take the standard $\mathcal{Z}$-structure for $\mathbb{Z}^n$ in which $Z = S^{n-1}$, and then quotient out a cell-like subset of Z. When $n > 3$ we can choose the subset so that the decomposition space is not even a manifold. Even more disturbing is the following example.

EXAMPLE 3.1. Let G have a $\mathcal{Z}$-structure $(\tilde{X}, Z)$ and fix a proper map h: $X \to [1, \infty)$. Compactify $X \times \mathbb{R}$ by the suspension $Z \times [-\infty, \infty]/\sim$ of Z so that a sequence (x_i, t_i) converges to $[z, \tau]$ if and only if $t_i/h(x_i) \to \tau$ and $(x_i \to z$ or $\tau = \pm\infty)$. If h is chosen so that the variation of h over translates of a compact set forms a null sequence, then this yields a $\mathcal{Z}$-structure on $G \times \mathbb{Z}$.

Now let $f: G \to G$ be an automorphism. Denote by X_f the universal cover of the mapping torus of the map $X/G \to X/G$ induced by f. There is a natural map $X_f \to \mathbb{R}$ whose point preimages are copies of X, and the covering translation corresponding to f descends to translation by 1. There is a proper homotopy equivalence $\hat{f}: X_f \to X \times \mathbb{R}$ which is given by f^k on the point preimage of k. Using $\hat{f}$, we can pull back the compactification of $X \times \mathbb{R}$. To obtain a $\mathcal{Z}$-structure on the semidirect product G_f of G and $\mathbb{Z}$ via f, we need to ensure that orbits of compact sets are null-sequences. This can be done by choosing h to have very small variation near infinity. For example, $h \sim$ $\log\log(\text{word length})$ will do.

In particular, if G is a free group of finite rank and $f: G \to G$ an automorphism such that G_f is word-hyperbolic, then the latter group has two very different boundaries, namely the Gromov boundary (which looks like it ought to be the 1-dimensional Menger curve) and the boundary coming from the previous construction (homeomorphic to the suspension of the Cantor set).

Questions: If Z_1 and Z_2 are boundaries of G, then is there a compactum Z and cell-like maps $Z \to Z_1$ and $Z \to Z_2$? If so, can Z be chosen to be a boundary of G?

Other obstructions to uniqueness come from the work of Bowers and Ruane [BR]. Thus it seems reasonable to restrict our attention to boundaries satisfying Axiom H.

Questions: If Z_1 and Z_2 are Axiom H boundaries of G, then is there a compactum Z and cell-like maps $Z \to Z_1$ and $Z \to Z_2$? If so, can Z be chosen to be an Axiom H boundary of G?

Question: If Z is an Axiom H boundary of a word-hyperbolic group G, is Z (equivariantly) homeomorphic to the Gromov boundary of G?

3.3. Regularity

In general, regularity fails for arbitrary $\mathcal{Z}$-structures. Let $G = BS(1,2) \times \mathbb{Z}$. Then G has boundary equal to the suspension of the Cantor–Hawaiian earring. We may collapse the arc through the cut point, resulting in a cut point. The cut point has uncountable local H_1, while other points have trivial local H_1. Thus there are compactly supported 2-cocycles supported near the cut point that do not vanish near the cut point.

However, observe that, in any $\mathcal{Z}$-structure, any compactly supported 1-cocycle supported near $z \in Z$ that cobounds in X also cobounds near z. The same is true for compactly supported n-cocycles provided $n = \mathrm{cd}_{\mathbb{L}} G$ and $\dim_{\mathbb{L}} H^n(G; \mathbb{L}G) < \infty$.

Questions: Let G be a hyperbolic group and Z the Gromov boundary. Is every chain complex of finitely generated free $\mathbb{L}G$-modules regular? Is the compactly supported cochain complex regular?

3.4. Local Homology and Local Connectedness

As pointed out earlier, the following two questions have a negative answer in general. The first fails for $BS(1,2)\times\mathbb{Z}$, and the second for $BS(1,2)$. But the author does not know the answer under such additional hypotheses as:

(a) Z is the Gromov boundary of a hyperbolic group G; or
(b) Axiom H; or
(c) G acts on Z as a convergence group.

Question: If $H_c^{q+1}(X)$ is finitely generated over $\mathbb{L}$, is

$$H_q(Z)\to H_q(Z, Z\setminus\{z\})$$

an isomorphism for each $z\in Z$?

The preceding question is equivalent to the regularity question for the compactly supported cochain complex.

Question: If $H^k(G;\mathbb{L}G)=0$ for $k\leq m+1$, is Z locally homologically m-connected? That is, for every $z\in Z$ and every neighborhood A of z in Z, is there a smaller neighborhood B such that $\tilde{H}_i(B)\to\tilde{H}_i(A)$ is trivial for $i\leq m$?

4. Dictionary between Groups and Compacta

There are some striking analogies between group theory and the theory of finite-dimensional compact metrizable spaces.

EXAMPLE 4.1—Hopf theorem on number of ends versus local H_1.

A theorem of Hopf [Ho] states that if G is a finitely generated and infinite group, then $H^1(G;\mathbb{L}G)$ is either 0 (G has one end), $\mathbb{L}$ (G has two ends), or infinitely generated (G has infinitely many ends).

This can be compared to the elementary fact that if Z is compact and metrizable and $z\in Z$, then $H_0(Z, Z\setminus\{z\})$ is either $\mathbb{L}$ (if z is an isolated point), infinitely generated (if there is a sequence of distinct components of Z converging to z), or 0 (otherwise).

EXAMPLE 4.2—Farrell's theorem on $H^2(G;\mathbb{Z}G)$ versus Whyburn's theorem about cut points.

THEOREM [Wh]. *Let Z be a compact metrizable space. Then Z has at most countably many local cut points of order >2.*

THEOREM [Fl]. *Let G be a finitely presented group. Then $H^2(G;\mathbb{Z}G)$ is either 0, $\mathbb{Z}$, or infinitely generated.*

The boundary of a group G with $H^2(G; \mathbb{Z}G) = \mathbb{Z} \times \mathbb{Z}$ (say) would (in the presence of regularity) be a space with all points local cut points of order 3.

EXAMPLE 4.3—The Conner–Floyd characterization of homology manifolds versus Farrell's characterization of PD^n-groups.

THEOREM [CF; Br]. *If Z is a connected finite-dimensional compact metrizable space, if all local homology groups $H_*(Z, Z\backslash\{z\}; \mathbb{Z})$ are finitely generated, and if the local homology sheaves are (locally) constant, then Z is a homology manifold.*

THEOREM [F2]. *If $H^*(G; \mathbb{Z}G)$ is finitely generated as a $\mathbb{Z}$-module, then G is a Poincaré duality group over $\mathbb{Z}$.*

EXAMPLE 4.4—Farrell spectral sequence versus Bredon spectral sequence.

Assuming that the compactly supported cochain complex of X is regular, the local homology sheaf $\mathcal{H}_*$ is constant and equal to $H_*(Z)$ whenever the latter is finite-dimensional. The Bredon spectral sequence [Br]

$$E_2^{p,q} = H^p(Z; \mathcal{H}_{-q}) \Rightarrow H_{-p-q}(Z)$$

contains the finite-dimensional pieces of the limit (except in dimension 0) in the first column. Deleting those, we obtain a spectral sequence very much like the Farrell spectral sequence [F2],

$$E_2^{p,q} \cong H^p(G; \mathrm{Hom}(H^q(G; \mathbb{L}G), \mathbb{L}G))$$

and converging to $\mathbb{L}$ in dimension 0 and to 0 in other dimensions.

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Department of Mathematics
University of Utah
Salt Lake City, UT 84112

bestvina@math.utah.edu