Wavelets in Subspaces

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This work is on the connection between wavelet theory and operator theory. One can view this as a sequel to [4]. We parameterize the set of all multiresolution analyses by a set of unitary operators that satisfy certain local commutation relations (Theorem 3.5). We characterize the reducing subspaces of dilation and translation operators (Proposition 4.3). We prove that such a subspace always has an orthogonal wavelet (Theorem 4.4). Finally, we give examples of subspaces that are *not* reducing subspaces having orthogonal wavelets with regularity properties. Some other connections are provided, including parameterizing wavelets in subspaces.

1. Preliminaries

We use $\mathcal{H}$ for $L^2(\mathbb{R})$ ($=L^2(\mathbb{R}, m)$, where m is the Lebesgue measure). Let X be a nonzero closed subspace of $L^2(\mathbb{R})$. An *orthogonal wavelet* for X is a unit vector $\psi(t)$ in X such that $\{2^{n/2}\psi(2^n t-l): n, l\in\mathbb{Z}\}$ constitutes an orthonormal basis for X.

Let T and D be the translation and dilation (unitary) operators on $L^2(\mathbb{R})$ defined by

$$(Tf)(t)=f(t-1) \quad \text{and} \quad (Df)=\sqrt{2}f(2t).$$

We have $DT^2=TD$. A function ψ is an orthogonal wavelet for X if $\{D^nT^l\psi: n, l\in\mathbb{Z}\}$ is an orthonormal basis for X. Let $\ell^2(\mathbb{Z})$ be the Hilbert space with orthonormal basis $\{e_n: n\in\mathbb{Z}\}$. Let $\mathcal{Z}$ be the unitary operator on $\ell^2(\mathbb{Z})$ defined by $\mathcal{Z}e_n=e_{n+1}$, $n\in\mathbb{Z}$.

DEFINITION 1.1. Let X be a closed subspace of $L^2(\mathbb{R})$. A multiresolution analysis (*MRA*) in X is a set $\{\mathbf{V}_n: n\in\mathbb{Z}\}$ of closed subspaces in X that satisfies the following properties:

(i) $\mathbf{V}_n\subset\mathbf{V}_{n+1}$, for every integer n;
(ii) $\bigvee_{n\in\mathbb{Z}}\mathbf{V}_n=X$;
(iii) $\bigcap_{n\in\mathbb{Z}}\mathbf{V}_n=\{0\}$;

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(iv) $T\mathbf{V}_0 = \mathbf{V}_0$;
(v) $D^n \mathbf{V}_0 = \mathbf{V}_n$;
(vi) There exists an isomorphism Ψ from $\mathbf{V}_0$ onto $\ell^2(\mathbb{Z})$ such that $\mathcal{Z}\Psi = \Psi T|_{\mathbf{V}_0}$.

A *scaling function* related to the above MRA in X is a function $\phi \in X$ such that $\{T^n\phi : n \in \mathbb{Z}\}$ is an orthonormal basis for $\mathbf{V}_0$.

Let $\mathfrak{M} = \{\mathbf{V}_n : n \in \mathbb{Z}\}$ be an MRA. Let ϕ be a scaling function related to $\mathfrak{M}$ and let ψ be a wavelet from this MRA. Then $\mathbf{V}_0 = \overline{\text{span}}\{T^l\phi : l \in \mathbb{Z}\}$. Let $\mathbf{W}_n := \overline{\text{span}}\{D^n T^l \psi : l \in \mathbb{Z}\}$. Then (cf. [5; 2]) we have

$$\mathbf{W}_n \oplus \mathbf{V}_n = \mathbf{V}_{n+1}, \quad n \in \mathbb{Z},$$

and

$$\mathbf{V}_0 = \bigoplus_{-\infty}^{-1} \mathbf{W}_n = \left(\bigoplus_{n=0}^{\infty} \mathbf{W}_n \right)^{\perp}.$$

The wavelet ψ is in the *translation space* $\mathbf{W}_0$. Let P_ϕ be the orthogonal projection onto the space $\mathbf{V}_0$. Then $P_\phi \psi = 0$ and $P_\phi^{\perp} \psi = \psi$.

Let $\mathcal{S}$ be a set of operators in $\mathcal{B}(\mathcal{H})$ and let $x \in \mathcal{H}$. We define (see [4])

$$\mathcal{C}_x(\mathcal{S}) := \{A \in \mathcal{B}(\mathcal{H}) : (AS - SA)x = 0,\ S \in \mathcal{S}\}.$$

We call this the *local commutant* of $\mathcal{S}$ at x. Let ϕ be a scaling function for some MRA and let ψ be an orthogonal wavelet. We will use the following notation:

$$\mathcal{C}_\psi(D,T) := \mathcal{C}_\psi(\{D^n T^l : n, l \in \mathbb{Z}\});$$
$$\mathcal{C}_\phi(T) := \mathcal{C}_\phi(\{T^l : l \in \mathbb{Z}\}).$$

For a set $\mathcal{E}$ of operators we use $\mathcal{U}(\mathcal{E})$ to denote the subset of all unitary operators in $\mathcal{E}$. For disjoint sets E and F we will use "$E \,\dot{\cup}\, F$" for the union of E and F. We will use the similar notation "$\dot{\bigcup}_{j=1}^{\infty}$".

We will use $\mathcal{F}$ for the Fourier–Plancherel transform on $L^2(\mathbb{R})$ (cf. [8, Vol. 1, Chap. 3]); this is a unitary operator. If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ then

$$(\mathcal{F}f)(s) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ist} f(t)\, dt := \hat{f}(s).$$

For an operator $S \in \mathcal{B}(\mathcal{H})$ we write $\hat{S} = \mathcal{F}S\mathcal{F}^{-1}$. $\hat{S}$ is called the *Fourier transform* of S. It is easy to verify that $\hat{T} = M_{e^{-is}}$, the multiplication operator by e^{-is}, and that $\hat{D} = D^{-1}$. For a set $B \subseteq L^2(\mathbb{R})$ we will write $\hat{B}$ for the set $\{\hat{f} : f \in B\}$. For a set $\mathcal{S} \in \mathcal{B}(\mathcal{H})$ we will use $\mathcal{S}'$ to denote the commutant of $\mathcal{S}$, the set of all operators in $\mathcal{B}(\mathcal{H})$ that commute with all elements in $\mathcal{S}$. For a set $E \subset \mathbb{R}$, χ_E will be the characteristic function of E.

2. Basic Facts

In this section, we will use operator theory to describe some basic known results in the theory of orthogonal wavelets. The following lemma is based on a known result in operator theory.

LEMMA 2.1. *Let $\mathbf{V}_0$ be a closed subspace of $L^2(\mathbb{R})$ with the property that $T\mathbf{V}_0 = \mathbf{V}_0$. Assume that there exists an isomorphism Ψ from $\mathbf{V}_0$ onto $\ell^2(\mathbb{Z})$ such that $\mathcal{Z}\Psi = \Psi T|_{\mathbf{V}_0}$, where $\mathcal{Z}$ is a bilateral shift of multiplicity 1 in $\ell^2(\mathbb{Z})$. Then there is a function ϕ in $\mathbf{V}_0$ such that $\{T^n\phi : n \in \mathbb{Z}\}$ is an orthonormal basis for $\mathbf{V}_0$.*

Proof. Operators $T|_{\mathbf{V}_0}$ and $\mathcal{Z}$ are unitary operators on $\mathbf{V}_0$ and $\ell^2(\mathbb{Z})$, respectively. By assumption they are similar. By Putnam's theorem (cf. [3, Cor. 6.11]), the above operators are unitarily equivalent. Hence there is a unitary operator U from $\mathbf{V}_0$ onto $\ell^2(\mathbb{Z})$ such that $\mathcal{Z}U = UT|_{\mathbf{V}_0}$. Let e_0 be an element in $\ell^2(\mathbb{Z})$ such that $\{\mathcal{Z}^n e_0 : n \in \mathbb{Z}\}$ is an orthonormal basis for $\ell^2(\mathbb{Z})$. Let $\phi := U^* e_0$. Then $\{T^n\phi : n \in \mathbb{Z}\}$ is an orthonormal basis for $\mathbf{V}_0$. □

REMARKS. Lemma 2.1 proves a known result [10] that an MRA yields a scaling function. This was also observed in [6]. Lemma 2.1 also works for an MRA in a subspace. Thus item (vi) in Definition 1.1 can be replaced by:

(vi′) there exists a scaling function $\phi \in \mathbf{V}_0$.

If ϕ_0 is a function in $\mathbf{V}_0$ such that $\{T^n\phi_0 : n \in \mathbb{Z}\}$ is a Schaulder basis (not necessarily a Riesz basis or, equivalently, an unconditional basis) for $\mathbf{V}_0$, by Lemma 2.1 there is a $\phi \in \mathbf{V}_0$ such that ϕ is a scaling function.

Let $\mathfrak{M}_0$ be an MRA for $L^2(\mathbb{R})$. Let $\mathcal{Z}$ be as in Lemma 2.1 and let $\mathcal{U}(\{\mathcal{Z}\}')$ be the set of unitary operators in $\mathcal{B}(\ell^2(\mathbb{Z}))$ that commute with $\mathcal{Z}$.

LEMMA 2.2. *Let $\mathfrak{M}_0$ be a given MRA in X, and let $\mathcal{S}$ be the set of all scaling functions corresponding to $\mathfrak{M}_0$ (in $\mathbf{V}_0$). Let U be as in the proof of Lemma 2.1. Then*

$$\mathcal{S} = U^*\mathcal{U}(\{\mathcal{Z}\}')e_0.$$

Proof. Let V be a unitary operator in $\{\mathcal{Z}\}'$ and let U be as in the proof of Lemma 2.1. Then $V\mathcal{Z}V^* = \mathcal{Z}$, so we have

$$T = U^*\mathcal{Z}U = U^*(V\mathcal{Z}V^*)U = (V^*U)^*\mathcal{Z}(V^*U).$$

By the proof of Lemma 2.1, $U^*Ve_0 = (V^*U)^*e_0$ is a scaling function.

Conversely, let ϕ_1 be another scaling function. Then $\{T^n\phi_1 : n \in \mathbb{Z}\}$ is an orthonormal basis for $\mathbf{V}_0$. Let $R: T^n\phi \to T^n\phi_1$. For $n = 1$, we have $R\phi = \phi_1$. The map R extends to a unitary operator from $\mathbf{V}_0$ onto $\mathbf{V}_0$. By definition of R we have $RT|_{\mathbf{V}_0} = T|_{\mathbf{V}_0}R$. Letting $V = URU^*$, we have

$$V\mathcal{Z} = URU^*UTU^* = URTU^* = UTRU^* = UTU^*URU^* = \mathcal{Z}V.$$

Hence $V \in \{\mathcal{Z}\}'$ and $U^*Ve_0 = U^*URU^*e_0 = R\phi = \phi_1$. □

COROLLARY 2.3. *For a given MRA $\mathfrak{M}$, the set of corresponding scaling functions is norm path connected.*

Proof. The mapping $V \to (VU)^*e_0$ is one-to-one and obviously continuous. The set $\{\mathcal{Z}\}'$, the set of Laurent operators, is a von Neumann algebra. Recall that the unitary group of a von Neumann algebra is norm path connected, and the conclusion follows. □

It is known (cf. [10; 11]) that an MRA yields an orthogonal wavelet. In Proposition 2.4, we will provide an operator-theoretic construction for this.

Let $\ell^2(\mathbb{Z})$, $\{e_n : n \in \mathbb{Z}\}$, and $\mathcal{Z}$ be as defined in Section 1. Let

$$x = \sum_{n\in\mathbb{Z}} \lambda_n e_n \in \ell^2(\mathbb{Z}).$$

Define a mapping C on $\ell^2(\mathbb{Z})$ by

$$Cx := \sum_{n\in\mathbb{Z}} (-1)^n \bar{\lambda}_n e_{-n-1} = \sum_{n\in\mathbb{Z}} (-1)^{n+1} \bar{\lambda}_{-n-1} e_n.$$

It is easy to verify that:

(i) C is a *conjugate* linear isometry on $\ell^2(\mathbb{Z})$ and $C^2 = -I$;
(ii) $\langle x, Cx\rangle = \sum_{n\in\mathbb{Z}} \lambda_n \cdot (-1)^{n+1} \lambda_{-n-1} = 0$; and
(iii) $(\mathcal{Z}C)^2 = I$.

PROPOSITION 2.4. *Let $\mathfrak{M}$ be an MRA in X and let U be a unitary operator from $\mathbf{V}_0$ onto $\ell^2(\mathbb{Z})$ such that $\mathcal{Z}U = UT|_{\mathbf{V}_0}$. Let $\phi = U^*e_0$ and let $\psi = DU^*CUD^{-1}\phi$. Then ψ is an orthogonal wavelet for X.*

Proof. By (i)–(iii) and (v) in the definition of an MRA, X is a direct sum of subspaces $\{\mathbf{V}_{n+1} \ominus \mathbf{V}_n : n \in \mathbb{Z}\} = \{D^n \mathbf{W}_0 : n \in \mathbb{Z}\}$. It therefore suffices to prove that $\{T^l\psi : l \in \mathbb{Z}\}$ is an orthonormal basis for $\mathbf{W}_0$.

Let $f := UD^{-1}\phi$. We will show that $\{\mathcal{Z}^{2l}Cf : l \in \mathbb{Z}\}$ is an orthonormal basis of $U\mathbf{V}_0 \ominus U\mathbf{V}_{-1}$. If this is true, then $\{U^*\mathcal{Z}^{2l}Cf : l \in \mathbb{Z}\}$ is an orthonormal basis for $\mathbf{V}_0 \ominus \mathbf{V}_{-1}$ and therefore $\{DU^*\mathcal{Z}^{2l}Cf : l \in \mathbb{Z}\}$ is an orthonormal basis of $\mathbf{W}_0 = \mathbf{V}_1 \ominus \mathbf{V}_0$. We have

$$\begin{aligned}\{DU^*\mathcal{Z}^{2l}Cf : l \in \mathbb{Z}\} &= \{DT^{2l}U^*CUD^{-1}\phi : l \in \mathbb{Z}\}\\ &= \{T^l DU^*CUD^{-1}\phi : l \in \mathbb{Z}\}\\ &= \{T^l\psi : l \in \mathbb{Z}\}.\end{aligned}$$

Hence ψ will be an orthogonal wavelet for X.

We have $\mathcal{Z}^{2l}f = \mathcal{Z}^{2l}UD^{-1}\phi = UT^{2l}D^{-1}\phi = UD^{-1}T^l\phi$. Therefore, $\{\mathcal{Z}^{2l}f : l \in \mathbb{Z}\}$ is an orthonormal basis of $U\mathbf{V}_{-1}$. We need to prove that the set

$$\{\mathcal{Z}^{2l}f : l \in \mathbb{Z}\} \cup \{\mathcal{Z}^{2l}Cf : l \in \mathbb{Z}\}$$

is an orthonormal basis of $\ell^2(\mathbb{Z}) = U\mathbf{V}_0$. We use $\mathcal{B}$ to denote this union.

First we prove that the set $\mathcal{B}$ is an orthonormal set. Let $k \neq l$ be arbitrary integers. Since $(\mathcal{Z}C)^2 = I$ and $C^2 = -I$, we have $\mathcal{Z}C = -C\mathcal{Z}^{-1}$ and so $\mathcal{Z}^nC = (-1)^n C\mathcal{Z}^{-n}$. We have

$$\mathcal{Z}^{2k}Cf = \mathcal{Z}^{k+l}\mathcal{Z}^{k-l}Cf = (-1)^{l-k}\mathcal{Z}^{k+l}C\mathcal{Z}^{l-k}f.$$

By property (ii) of the operator C we have $\mathcal{Z}^{l-k}f \perp C\mathcal{Z}^{l-k}f = -\mathcal{Z}^{k-l}Cf$, so $\mathcal{Z}^{l-k}f \perp \mathcal{Z}^{k-l}Cf$. Since $\mathcal{Z}^{k+l}$ is a unitary operator we have

$$\mathcal{Z}^{2l}f = \mathcal{Z}^{k+l}\mathcal{Z}^{l-k}f \perp \mathcal{Z}^{k+l}\mathcal{Z}^{k-l}Cf = \mathcal{Z}^{2k}Cf;$$

$$\mathcal{Z}^{2l}f \perp \mathcal{Z}^{2k}Cf.$$

Hence $\mathcal{B}$ is an orthonormal set.

Finally, we show that the span of $\mathcal{B}$ is $\ell^2(\mathbb{Z})$. Let P be the projection onto the span of $\mathcal{B}$. An element x is in the span if and only if $\|Px\| = \|x\|$. Therefore, the set $\mathcal{B}$ is a basis if and only if $\|Pe_n\| = 1$ for all $n \in \mathbb{Z}$. Let $f = \sum_{n\in\mathbb{Z}} \lambda_n e_n$. Then we have

$$\mathcal{Z}^{2k} f = \sum_{n\in\mathbb{Z}} \lambda_n e_{n+2k} = \sum_{n\in\mathbb{Z}} \lambda_{n-2k} e_n$$

$$\langle e_s, \mathcal{Z}^{2k} f\rangle = \bar{\lambda}_{s-2k};$$

$$\mathcal{Z}^{2k} Cf = \sum_{n\in\mathbb{Z}} (-1)^{n+1} \bar{\lambda}_{-n-1} e_{n+2k} = \sum_{n\in\mathbb{Z}} (-1)^{n-2k+1} \bar{\lambda}_{2k-n-1} e_n,$$

$$\langle e_s, \mathcal{Z}^{2k} Cf\rangle = (-1)^{s-2k+1} \lambda_{2k-1-s}.$$

As a result,

$$\begin{aligned}\|Pe_s\|^2 &= \sum_{k\in\mathbb{Z}} |\lambda_{s-2k}|^2 + \sum_{k\in\mathbb{Z}} |\lambda_{2k-1-s}|^2 \\ &= \sum_{k\in\mathbb{Z}} |\lambda_k|^2 = \|f\|^2 = \|\phi\|^2 = 1.\end{aligned}$$

This proves that $\mathcal{B}$ is an orthonormal basis of $\ell^2(\mathbb{Z})$. Proposition 2.4 is proven. □

REMARK. We have the following diagram:

$$\ell^2(\mathbb{Z}) \xrightarrow{U^*} \mathbf{V}_0 \xrightarrow{UD^{-1}} U\mathbf{V}_{-1} \xrightarrow{C} \ell^2(\mathbb{Z}) \ominus U\mathbf{V}_{-1} \xrightarrow{DU^*} \mathbf{V}_1 \ominus \mathbf{V}_0.$$

EXAMPLE 2.5. Let $\phi_n = \chi_{[n,n+1)}$ $(= T^n \phi_0)$ and let $\mathbf{V}_0 = [\phi_n : n \in \mathbb{Z}]$. Then $\{\mathbf{V}_n : n \in \mathbb{Z}\} = \{D^n \mathbf{V}_0 : n \in \mathbb{Z}\}$ is an MRA in $X = L^2(\mathbb{R})$ [5]. Define $U: \mathbf{V}_0 \to \ell^2(\mathbb{Z})$ by $U\phi_n = e_n$ and extend linearly. Then U is unitary and $UTU^* = \mathcal{Z}$, where $\mathcal{Z}$ is the bilateral shift on $\ell^2(\mathbb{Z})$ mapping e_n into e_{n+1} for each $n \in \mathbb{Z}$. The function $\phi_0 = U^* e_0$ is a scaling function. We have

$$UD^{-1}U^* e_0 = UD^{-1}\phi_0 = \frac{1}{\sqrt{2}} U\chi_{[0,2)} = \frac{1}{\sqrt{2}} U(\phi_0 + \phi_1) = \frac{1}{\sqrt{2}}(e_0 + e_1),$$

$$CUD^{-1}U^* e_0 = C\left(\frac{1}{\sqrt{2}}(e_0 + e_1)\right) = \frac{1}{\sqrt{2}}(e_{-1} - e_{-2}),$$

$$\psi = DU^*CUD^{-1}U^* e_0 = DU^*\left[\frac{1}{\sqrt{2}}(e_{-1} - e_{-2})\right] = \chi_{[-1/2,0)} - \chi_{[-1,-1/2)}.$$

Since ψ is a wavelet, $\psi_H = -T\psi$ is also a wavelet. This ψ_H is the Haar wavelet.

3. Multiresolution Analysis

The main purpose of this section is to parameterize all MRAs in $L^2(\mathbb{R})$. Let $\mathcal{M}$ be an MRA in $L^2(\mathbb{R})$ and let ϕ and ψ be the related scaling function and orthogonal wavelet, respectively. We call $\langle \mathcal{M}, \phi, \psi\rangle$ an *MSW-triple* (MRA-scaling-wavelet triple) in $L^2(\mathbb{R})$. Let $\mathcal{W}(D,T)$ be the set of all orthogonal wavelets in $L^2(\mathbb{R})$.

LEMMA 3.1 [4, Lemma 3.1(i)]. *Let ψ_0 be an element in $\mathcal{W}(D,T)$. Then*

$$\mathcal{W}(D,T) = \mathcal{U}(\mathcal{C}_{\psi_0}(D,T))\psi_0.$$

The mapping $\theta: \mathcal{U}(\mathcal{C}_{\psi_0}(D,T)) \to \mathcal{W}(D,T)$ given by $\theta(U) = U\psi_0$ is one-to-one and onto.

For wavelets in subspace X we have the following lemma.

LEMMA 3.2. *Let ψ_0 be an orthogonal wavelet for $\mathcal{H} = L^2(\mathbb{R})$. Let X be a closed subspace of $\mathcal{H}$ and let P_X be the projection from $\mathcal{H}$ onto X. Assume that X has an orthogonal wavelet ψ. Then there is a unique isometry V in $\mathcal{C}_{\psi_0}(D,T)$ such that $VV^* = P_X$ and $V\psi_0 = \psi$. Every orthogonal wavelet (in a subspace) can be obtained in this way.*

Proof. Let ψ be a wavelet in X. Then $\{D^nT^l\psi: n,l \in \mathbb{Z}\}$ is an orthonormal basis for X. Define a mapping $V: D^nT^l\psi_0 \to D^nT^l\psi$. The map V extends to an isometry V from $\mathcal{H}$ onto X that maps ψ_0 into ψ. We therefore have $VD^nT^l\psi_0 = D^nT^l\psi = D^nT^lV\psi_0$, so $V \in \mathcal{C}_{\psi_0}(D,T)$.

Assume that

$$V \in \mathcal{C}_{\psi_0}(D,T)$$

is an isometry with final space X. Then $V\{D^nT^l\psi_0: n,l \in \mathbb{Z}\}$ is an orthonormal basis for X. Let $\psi = V\psi_0$. Then we have

$$\begin{aligned} V\{D^nT^l\psi_0: n,l \in \mathbb{Z}\} &= \{D^nT^lV\psi_0: n,l \in \mathbb{Z}\} \\ &= \{D^nT^l\psi: n,l \in \mathbb{Z}\}. \end{aligned}$$

Hence ψ is an orthogonal wavelet for X. □

COROLLARY 3.3. *Let ψ_0 be an orthogonal wavelet for $L^2(\mathbb{R})$. Let $\mathcal{U}_*$ be the set of all unitary operators or isometries in $\mathcal{C}_{\psi_0}(D,T)$. Let $\mathcal{W}_*$ be the set of all orthogonal wavelets (for $L^2(\mathbb{R})$ or for some subspaces). Then $\mathcal{W}_* = \mathcal{U}_*\psi_0$.*

The following lemma and remarks show that, by the same method of Lemma 3.1, one can obtain only a proper subset of all scaling functions.

LEMMA 3.4. *Let ϕ be a scaling function for an MRA in $L^2(\mathbb{R})$. Let $\mathcal{C}_\phi(D,T) = \mathcal{C}_\phi(\{D^nT^l: n,l \in \mathbb{Z}\})$. Then*

$$\mathcal{C}_\phi(D,T) = \{D,T\}'.$$

Proof. Let $S \in \mathcal{C}_\phi(D,T)$ and let $n \in \mathbb{N}$. By definition of $\mathcal{C}_\phi(D,T)$, we have

$$\begin{aligned} SDD^nT^l\phi &= SD^{n+1}T^l\phi = D^{n+1}T^lS\phi \\ &= DSD^nT^l\phi; \\ STD^nT^l\phi &= SD^nT^{2^n+l}\phi = D^nT^{2^n+l}S\phi \\ &= TD^nT^lS\phi = TSD^nT^l\phi. \end{aligned}$$

Since $\{D^nT^l\phi: l\in\mathbb{Z}\}$ is an orthonormal basis for $\mathbf{V}_n$, $STx = TSx$ and $SDx = DSx$ for $x\in\mathbf{V}_n$. Since $\bigcup_{n\in\mathbb{N}}\mathbf{V}_n$ is dense in $L^2(\mathbb{R})$ we have $S\in\{D,T\}'$, and $\mathcal{C}_\phi(D,T)\subseteq\{D,T\}'$. The "$\supseteq$" part is trivial. □

REMARKS. Let V be a unitary operator in $\{D,T\}'$. It is clear that $V\phi$ is a scaling function. By Theorem 3.5 in [4] (see Lemma 4.2 below), the Fourier transform of V, the operator $\hat{V}=\mathfrak{F}V\mathfrak{F}^{-1}$, is a multiplication operator M_g by a function g with $|g(t)|\equiv 1$. Hence $|\hat{V}\hat{\phi}(t)| = |g(t)\hat{\phi}(t)| = |\hat{\phi}(t)|$; in particular, $\hat{V}\hat{\phi}$ and $\hat{\phi}$ have the same support. Since there are scaling functions with different support sets [5], the set $\mathcal{U}(\{D,T\}')\phi$ is *not* the set of all scaling functions.

Let C be as defined in Proposition 2.4. Let $\psi = DU^*CUD^{-1}\phi$. By Proposition 2.4, ψ is an orthogonal wavelet. Using property (i) of the operator C, we have $(DU^*CUD^{-1})^2 = -I$, or (equivalently) $\phi = -DU^*CUD^{-1}\psi$. Based on this and Lemma 3.1, one might expect to obtain a parameterization of the set of all scaling functions. However, there exists an orthogonal wavelet ψ that has no corresponding scaling function [10]. This can happen because the unitary operator U does not exist in this case.

Let $\langle\mathfrak{M}_0,\phi_0,\psi_0\rangle$ be an MSW-triple in $L^2(\mathbb{R})$. Let us define

$$\mathcal{C}(\mathfrak{M}_0,\phi_0,\psi_0) := \mathcal{C}_{\phi_0}(T)\cap\{\mathcal{B}(\mathcal{H})P_{\phi_0}+\mathcal{U}(\mathcal{C}_{\psi_0}(D,T))P_{\phi_0}^{\perp}\}$$

and

$$\mathcal{U}(\mathfrak{M}_0,\phi_0,\psi_0) := \mathcal{U}(\mathcal{C}(\mathfrak{M}_0,\phi_0,\psi_0)).$$

It is clear that the set of unitary operators in $\{D,T\}'$ is a subset of $\mathcal{U}(\mathfrak{M}_0, \phi_0, \psi_0)$. Let V be a unitary operator. We will write

$$\langle\mathfrak{M},\phi,\psi\rangle = V\langle\mathfrak{M}_0,\phi_0,\psi_0\rangle,$$

where

$$\phi := V\phi_0, \qquad \psi := V\psi_0,$$
$$\mathbf{V}_0' := V\mathbf{V}_0, \qquad \mathbf{V}_n' := D^n\mathbf{V}_0',\ n\in\mathbb{Z},$$
$$\mathfrak{M} := \{\mathbf{V}_n' : n\in\mathbb{Z}\}.$$

The new triple $\langle\mathfrak{M},\phi,\psi\rangle$ is not necessarily an MSW-triple.

THEOREM 3.5. *Let $\langle\mathfrak{M}_0,\phi_0,\psi_0\rangle$ be an MSW-triple in $L^2(\mathbb{R})$ and let $V\in\mathcal{U}(\mathfrak{M}_0,\phi_0,\psi_0)$. Then $\langle\mathfrak{M},\phi,\psi\rangle = V\langle\mathfrak{M}_0,\phi_0,\psi_0\rangle$ is also an MSW-triple in $L^2(\mathbb{R})$. If $\langle\mathfrak{M},\phi,\psi\rangle$ is an arbitrary MSW-triple in $L^2(\mathbb{R})$ then there is a unique unitary operator $V\in\mathcal{U}(\mathfrak{M}_0,\phi_0,\psi_0)$ such that*

$$\langle\mathfrak{M},\phi,\psi\rangle = V\langle\mathfrak{M}_0,\phi_0,\psi_0\rangle.$$

Proof. Let $\langle\mathfrak{M}_0,\phi_0,\psi_0\rangle$ be an MSW-triple and let $V\in\mathcal{U}(\mathfrak{M}_0,\phi_0,\psi_0)$. Then $V = WP_{\phi_0}+V'P_{\phi_0}^{\perp}$ for some unitary $V'\in\mathcal{C}_{\psi_0}(D,T)$ and some operator $W\in\mathcal{B}(\mathcal{H})$. Since $\psi_0\in\mathbf{V}_0^{\perp}$, we have $V\psi_0 = V'\psi_0$. Let $\psi = V\psi_0$. By Lemma 3.1, $\psi = V\psi_0$ is an orthogonal wavelet; the wavelet basis is $\{D^nT^l\psi: n,l\in\mathbb{Z}\}$. We write

$$\mathbf{V}_0' = \overline{\operatorname{span}}\{D^n T^l \psi : n < 0, l \in \mathbb{Z}\}.$$

We have $VD^nT^l\psi_0 = V'D^nT^l\psi_0 = D^nT^lV'\psi_0 = D^nT^lV\psi_0 = D^nT^l\psi$ for $n \geq 0$, so

$$\mathbf{V}_0'^{\perp} = V\mathbf{V}_0^{\perp}.$$

Since V is unitary we must have $\mathbf{V}_0' = V\mathbf{V}_0$. Since D is unitary we have

$$\begin{aligned}(D\mathbf{V}_0')^{\perp} &= D(\mathbf{V}_0'^{\perp}) \\ &= \overline{\operatorname{span}}\{D^nT^l\psi : n \geq 1, l \in \mathbb{Z}\} \\ &\subset \overline{\operatorname{span}}\{D^nT^l\psi : n \geq 0, l \in \mathbb{Z}\} \\ &= \mathbf{V}_0'^{\perp}.\end{aligned}$$

Thus $D\mathbf{V}_0' \supset \mathbf{V}_0'$. So, for arbitrary $n \in \mathbb{Z}$, we have $D^n\mathbf{V}_0' \subset D^{n+1}\mathbf{V}_0'$ or (equivalently) $\mathbf{V}_n' \subset \mathbf{V}_{n+1}'$.

Let f be a function in $\bigcap_{n\in\mathbb{Z}} D^n\mathbf{V}_0'$. Then $f \perp (D^n\mathbf{V}_0')^{\perp}$ for each $n \in \mathbb{Z}$. Because $(D^n\mathbf{V}_0')^{\perp} = \overline{\operatorname{span}}\{D^mT^l\psi : m \geq n \text{ and } l \in \mathbb{Z}\}$, we have $f \perp D^nT^l\psi$, $n, l \in \mathbb{Z}$. Since $\{D^nT^l\psi : n, l \in \mathbb{Z}\}$ is an orthonormal basis, we have $f = 0$. This proves that

$$\bigcap_{n\in\mathbb{Z}} D^n\mathbf{V}_0' = \{0\}.$$

We must show that $\{T^l\phi : l \in \mathbb{Z}\}$ is an orthonormal basis for $\mathbf{V}_0'$. Since $\{T^l\phi, D^nT^l\psi : n \geq 0, l \in \mathbb{Z}\}$ and $\{D^nT^l\psi : n, l \in \mathbb{Z}\}$ are two orthonormal bases for $\mathcal{H} = L^2(\mathbb{R})$, we have $\overline{\operatorname{span}}\{T^l\phi : l \in \mathbb{Z}\} = \overline{\operatorname{span}}\{D^nT^l\psi : n < 0, l \in \mathbb{Z}\} = \mathbf{V}_0'$. Therefore, ϕ is a scaling function and so $\langle \mathfrak{M}, \phi, \psi \rangle$ is an MSW-triple in $L^2(\mathbb{R})$.

Let $\langle \mathfrak{M}, \phi, \psi \rangle$ be an arbitrary given MSW-triple in $L^2(\mathbb{R})$. Define a mapping V from the set $\{T^l\phi_0, D^nT^l\psi_0 : n \geq 0, l \in \mathbb{Z}\}$ onto the set $\{T^l\phi, D^nT^l\psi : n \geq 0, l \in \mathbb{Z}\}$ by

$$VT^l\phi_0 = T^l\phi, \quad l \in \mathbb{Z}$$

and

$$VD^nT^l\psi_0 = D^nT^l\psi, \quad n \geq 0, l \in \mathbb{Z}.$$

This V extends to a unitary operator, which is also denoted by V. It is clear that $V \in \mathcal{C}_{\phi_0}(T)$. Since ψ_0 and ψ are orthogonal wavelets, by Lemma 3.1 there is a unique operator $V' \in \mathcal{U}_{\psi_0}(D,T)$ such that $V'\psi_0 = \psi$ and $V'D^nT^l\psi_0 = D^nT^lV'\psi_0 = D^nT^l\psi$. For $n \geq 0$ and $l \in \mathbb{Z}$ we have

$$\begin{aligned}VD^nT^l\psi_0 &= D^nT^lV\psi_0 = D^nT^l\psi \\ &= D^nT^lV'\psi_0 = V'D^nT^l\psi_0.\end{aligned}$$

Hence V and V' "coincide" on $\mathbf{V}_0^{\perp}$. Therefore

$$VP_{\phi_0}^{\perp} = V'P_{\phi_0}^{\perp}$$

for $V \in \mathcal{B}(\mathcal{H})P_{\phi_0} + \mathcal{U}(\mathcal{C}_{\psi_0}(D,T))P_{\phi_0}^{\perp}$. Theorem 3.5 is proven. □

LEMMA 3.6. *Let $\langle \mathcal{M}_0, \phi_0, \psi_0 \rangle$ be an MSW-triple for $L^2(\mathbb{R})$. Then:*

(i) $\mathcal{C}(\mathcal{M}_0, \phi_0, \psi_0) = \{T\}' \cap \{\mathcal{B}(\mathcal{H})P_{\phi_0} + \mathcal{U}(\mathcal{C}_{\psi_0}(D,T))P_{\phi_0}^{\perp}\}$;

(ii) $\mathcal{C}(\mathcal{M}_0, \phi_0, \psi_0) \not\subset \{D\}'$; *and*

(iii) $\mathcal{C}(\mathcal{M}_0, \phi_0, \psi_0) \cap \mathcal{C}_{\psi_0}(D,T) \subseteq \{D,T\}'$.

Proof. (i) Since $\mathcal{C}_{\phi_0}(T) \supseteq \{T\}'$, the "$\supseteq$" part is obvious. For the reverse inclusion, it suffices to show that $\mathcal{C}(\mathcal{M}_0, \phi_0, \psi_0) \subseteq \{T\}'$. Let V be an operator in $\mathcal{C}(\mathcal{M}_0, \phi_0, \psi_0)$. We have

$$VT(T^l\phi_0) = VT^{n+1}\phi_0 = T^{n+1}V\phi_0 = TV(T^n\phi_0)$$

for $n \in \mathbb{Z}$. Let $n \geq 0$. Then

$$\begin{aligned} VTD^nT^l\psi_0 &= VD^nT^{2^n+l}\psi_0 = D^nT^{2^n+l}V\psi_0 \\ &= TD^nT^lV\psi_0 = TVD^nT^l\psi_0 \\ &= TVD^nT^l\psi_0. \end{aligned}$$

Hence V and T commute at the orthonormal basis $\{T^l\phi_0, D^nT^l\psi_0 : n \geq 0, l \in \mathbb{Z}\}$ and so $V \in \{T\}'$.

(ii) Assume that $\mathcal{C}(\mathcal{M}_0, \phi_0, \psi_0) \subseteq \{D\}'$. By the first part we have

$$\mathcal{C}(\mathcal{M}_0, \phi_0, \psi_0) \subseteq \{D,T\}'.$$

By the remark after Lemma 3.4 this is impossible; a contradiction.

(iii) This is a direct consequence of (i) and $\mathcal{C}_{\psi_0}(D,T) \subseteq \{D\}'$ (cf. [4, Lemma 3.1(iii)]). □

4. Reducing Subspaces of D and T

In this section we will describe the reducing subspaces of dilation and translation operators. We will show that in each nonzero reducing subspace, the wavelet set is nonempty.

Let $S \in \mathcal{B}(\mathcal{H})$ and let P be a (orthogonal) projection in $\mathcal{B}(\mathcal{H})$. We say that the subspace $X = P\mathcal{H}$ *reduces* S if X and $X^{\perp}$ are invariant under S. This occurs if and only if X is invariant under both S and S^*, if and only if $P \in \{S\}'$, the commutant of S. We say that a subspace X is a *reducing subspace* of $\{D,T\}$ if X reduces both operators D and T simultaneously.

LEMMA 4.1. *Let X be a closed subspace of $L^2(\mathbb{R})$ having a multiresolution analysis. Then X is a reducing subspace of $\{D,T\}$.*

Proof. By Lemma 2.1 and Proposition 2.4, X has an MSW-triple $\langle \mathcal{M}, \phi, \psi \rangle$. Hence $\mathcal{B}_1 := \{T^l\phi : l \in \mathbb{Z}\} \cup \{D^nT^l\psi : n \geq 0 \text{ and } l \in \mathbb{Z}\}$ is an orthonormal basis for X. We have $T\mathcal{B}_1 = \{T^{l+1}\phi : l \in \mathbb{Z}\} \cup \{D^nT^{2^n+l}\psi : n \geq 0 \text{ and } l \in \mathbb{Z}\} = \mathcal{B}_1$. Therefore, $TX = X$. Since T is unitary, $T^*X = X$. Thus X reduces T. Let $\mathcal{B}_2$ be the wavelet basis $\{D^nT^l\psi : n, l \in \mathbb{Z}\}$ for X. We have $D\mathcal{B}_2 = D\{D^nT^l\psi : n, l \in \mathbb{Z}\} = \{D^{n+1}T^l\psi : n, l \in \mathbb{Z}\} = \mathcal{B}_2$. By the same reasoning as for T, X reduces D. □

LEMMA 4.2 (cf. [4, Thm. 3.5]).

$$\mathfrak{F}\{D,T\}'\mathfrak{F}^{-1} = \{M_g\colon g \in L^\infty(\mathbb{R}) \text{ and } g(s) = g(2s) \text{ a.e.}\}.$$

Let X be a reducing subspace of $\{D,T\}$ and let P be the projection onto X. Then $\hat{P} = \mathfrak{F}P\mathfrak{F}^{-1}$ is a projection in $\mathfrak{F}\{D,T\}'\mathfrak{F}^{-1}$. Let $\hat{P} = M_g$. Then $\hat{P}$ is a projection if and only if $g^2 = g$ for some real-valued function g if and only if $g = \chi_\Omega$, where χ_Ω is a characteristic function of some measurable set Ω. By Lemma 4.2, g must satisfy the relation $g(t) = g(2t)$, so the set Ω must satisfy the relation $2\Omega = \Omega$. This proves the following result.

PROPOSITION 4.3. *A closed subspace X of $L^2(\mathbb{R})$ is a reducing subspace of $\{D,T\}$ if and only if there is a measurable set $\Omega \subseteq \mathbb{R}$ with $\Omega = 2\Omega$ such that*

$$\hat{X} = L^2(\mathbb{R})\cdot\chi_\Omega.$$

Next we will show that in each reducing subspace of $\{D,T\}$ the set of wavelets is nonempty.

Let E be a subset of $\mathbb{R}$. We write

$$E \,(\mathrm{mod}\, 2\pi) := \bigcup_{n\in\mathbb{Z}} \{E \cap [2n\pi, 2n\pi + 2\pi) - 2(n-1)\pi\}.$$

It is clear that $E\ (\mathrm{mod}\, 2\pi)$ is a subset of $[2\pi, 4\pi)$. A set E is said to be 2π-*congruent* to $[2\pi, 4\pi)$ if $E\ (\mathrm{mod}\, 2\pi) = [2\pi, 4\pi)$ *and* the sets $\{E_n\colon n \in \mathbb{Z}\}$ are disjoint, where $E_n = E \cap [2n\pi, 2n\pi + 2\pi) - 2(n-1)\pi$. It is easy to verify that E is 2π-congruent to $[2\pi, 4\pi)$ if and only if $E\ (\mathrm{mod}\, 2\pi) = [2\pi, 4\pi)$ and $m(E) = 2\pi$.

Let $\Omega \subset \mathbb{R}$ satisfy the condition $\Omega = 2\Omega$. A set $E \subset \Omega$ is said to be a 2-*dilation generator* for Ω if Ω is a disjoint union of the sets $\{2^n E\colon n \in \mathbb{Z}\}$. Let a and b be arbitrary positive numbers. The set $\{[-2b, -b) \cup [a, 2a)\} \cap \Omega$ is a 2-dilation generator for Ω. In particular, the set $\{[-4\pi, -2\pi) \cup [2\pi, 4\pi)\} \cap \Omega$ is a 2-dilation generator for Ω.

Let E be a measurable set in $\mathbb{R}$ with positive Lebesgue measure. A point x in $\mathbb{R}$ is called a *Lebesgue density point* of E if we have

$$\lim_{\rho\to 0} \frac{m(E \cap (x-\rho, x+\rho))}{2\rho} = 1.$$

It is known [12, p. 261] that almost all points in E are Lebesgue density points of E.

THEOREM 4.4. *Every nonzero closed reducing subspace of $\{D,T\}$ has an orthogonal wavelet.*

Proof. Let X be a nonzero closed reducing subspace of $\{D,T\}$. By Proposition 4.3,

$$\hat{X} = L^2(\mathbb{R})\cdot\chi_\Omega$$

where $\Omega = \bigcup_{n\in\mathbb{Z}} 2^n E$ for $E = \{[-4\pi, -2\pi) \cup [2\pi, 4\pi)\} \cap \Omega$ with $m(E) > 0$. We will show that there is a subset S of Ω such that

(i) S is a 2-dilation generator of Ω and
(ii) S is 2π-congruent to $[2\pi, 4\pi)$.

Let ψ_0 be a function in $L^2(\mathbb{R})$ given by $\hat{\psi}_0 := (1/\sqrt{2\pi})\chi_S$. Then property (ii) implies that $\{\hat{T}^l\hat{\psi}_0 : l \in \mathbb{Z}\} = \{e^{ils}\hat{\psi}_0(s) : l \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})\cdot\chi_S$. Then property (i) implies that $\{D^n e^{ils}\hat{\psi}_0(s) : n, l \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})\cdot\chi_\Omega$.

We have three cases:

(A) $m(\Omega \cap (-\infty, 0)) = 0$;
(B) $m(\Omega \cap (0, \infty)) = 0$;
(C) $m(\Omega \cap (-\infty, 0)) \neq 0$ and $m(\Omega \cap (0, \infty)) \neq 0$.

Proof of Case (A). Since $m(\Omega \cap (-\infty, 0)) = 0$, without loss of generality, we can assume that $\Omega \subseteq [0, \infty)$.

The set E satisfies condition (i). If $E = [2\pi, 4\pi)$ (modulo a null set), then it will satisfy condition (ii). In this case we take $S = E$ and we are done.

Assume that $F_0 := [2\pi, 4\pi)\backslash E$ is not a null set. Let $\{I_n : n \in \mathbb{N}\}$ be disjoint intervals in $[2\pi, 4\pi)$ with

$$\bigcup_{n\in\mathbb{N}} I_n = [2\pi, 4\pi) \quad \text{and} \quad m(I_n \cap E) > 0 \quad \text{for each } n \in \mathbb{N}.$$

Take a Lebesgue density point x_n from each $I_n \cap E$ such that x_n is not an endpoint of I_n. Since x_k is a Lebesgue density point that is not an endpoint of I_k, we can select a strictly increasing sequence $\{n_k\}$ in $\mathbb{N}$ (using induction if necessary) such that

(i) $J_k := \left(x_k - \dfrac{2\pi}{2^{n_k}}, x_k + \dfrac{2\pi}{2^{n_k}}\right) \subset I_k$;

(ii) $\dfrac{m(E \cap J_k)}{m(J_k)} > 1 - \dfrac{1}{2} \cdot \dfrac{1}{2^k}$.

The length of the enlarged interval $2^{n_k}J_k$ is 4π, so it contains an interval $[2(m_k+1)\pi, 2(m_k+2)\pi)$ for some $m_k \in \mathbb{N}$. (The number m_k is uniquely decided by J_k, since J_k is open.) Thus we have

$$[2\pi, 4\pi) + 2m_k\pi \subset 2^{n_k}J_k, \tag{1}$$

$$m(2^{n_k}E \cap 2^{n_k}J_k) > 4\pi - \frac{1}{2^k}\cdot 2\pi. \tag{2}$$

Define

$$\Delta_1 := 2^{n_1}E \cap (F_0 + 2m_1\pi);$$

$$S_1 := (I_1 \cap E)\Big\backslash \frac{1}{2^{n_1}}\Delta_1;$$

$$F_1 := \frac{1}{2^{n_1}}\Delta_1 \cup \{F_0 \backslash (\Delta_1 - 2m_1\pi)\}.$$

Assume that we have defined Δ_t, S_t, and F_t for all $t < k$. Define

$$\Delta_k := 2^{n_k} E \cap (F_{k-1} + 2m_k\pi);$$
$$S_k := (I_k \cap E) \Big\backslash \frac{1}{2^{n_k}} \Delta_k;$$
$$F_k := \frac{1}{2^{n_k}} \Delta_k \cup \{F_{k-1} \backslash (\Delta_k - 2m_k\pi)\}.$$

By definitions of Δ_k, J_k, I_k and (1), we have

$$\frac{1}{2^{n_k}} \Delta_k \subseteq E \cap J_k \subseteq I_k \cap E \quad \text{for } k \in \mathbb{N}.$$

Hence we have

$$E = \left(\dot{\bigcup_{j=1}^{\infty}} S_k \right) \dot{\cup} \left(\dot{\bigcup_{j=1}^{\infty}} \frac{1}{2^{n_k}} \Delta_k \right). \tag{3}$$

Since $n_k > 1$ and is strictly increasing, the intervals $[2(m_k+1)\pi, 2(m_k+2)\pi)$, $k \in \mathbb{N}$, are disjoint. Since $\Delta_k \subseteq [2(m_k+1)\pi, 2(m_k+2)\pi)$ for $k \in \mathbb{N}$, it follows that Δ_k and Δ_m are disjoint for $k \neq m$. Since $S_k \subseteq [2\pi, 4\pi)$ and $\Delta_n \subseteq [2(m_n+1)\pi, 2(m_n+2)\pi)$, Δ_n and S_k are disjoint for each pair (n, k). Define

$$S := \left(\dot{\bigcup_{j=1}^{\infty}} S_k \right) \dot{\cup} \left(\dot{\bigcup_{j=1}^{\infty}} \Delta_k \right). \tag{4}$$

We will prove that the set S is what we need in case (A).

Let $\Omega_k = \bigcup_{j \in \mathbb{Z}} 2^j (I_k \cap E)$. It is clear that $\Omega = \dot{\bigcup}_{j=1}^{\infty} \Omega_j$. Note that

$$I_k \cap E = S_k \dot{\cup} \left(\frac{1}{2^{n_k}} \Delta_k \right).$$

The set $I_k \cap E$ is a 2-dilation generator for Ω_k, so the set $S_k \dot{\cup} \Delta_k$ is also a 2-dilation generator for Ω_k. It is easy to verify that the set $S = \dot{\bigcup}_{j=1}^{\infty} (S_k \dot{\cup} \Delta_k)$ is a 2-dilation generator for Ω.

Next, in Lemmas 4.5, 4.6 and 4.7, we will prove that the set S is 2π-congruent to $[2\pi, 4\pi)$.

LEMMA 4.5.

(i) *The collection* $\{S_k, (\Delta_k - 2m_k\pi) : k \in \mathbb{N}\}$ *is a family of mutually disjoint subsets in* $[2\pi, 4\pi)$.

(ii) $m(F_k) \leq (1/2^{n_k}) \cdot 2\pi + (1/2^k) \cdot 2\pi$.

Proof. (i) It is clear that $\{S_k : k \in \mathbb{N}\}$ is a family of mutually disjoint sets. We will prove that

(a) $S_k \cap \{\Delta_t - 2m_t\pi\} = \emptyset$ for $t, k \in \mathbb{N}$ and

(b) $\{\Delta_t - 2m_t\pi\} \cap \{\Delta_k - 2m_k\pi\} = \emptyset$ for $t \neq k$.

Let $t, k \in \mathbb{N}$, $t < k$. By definition of Δ_k and F_k, we have

$$\Delta_t - 2m_t\pi \subseteq F_{t-1} \subseteq F_0 \cup \left(\bigcup_{i=1}^{t-1} \frac{1}{2^{n_i}} \Delta_i \right) \tag{5}$$

$$\Delta_k - 2m_k\pi \subseteq F_{k-1} \subseteq F_t \cup \left(\bigcup_{i=t}^{k-1} \frac{1}{2^{n_i}} \Delta_i \right). \tag{6}$$

(a) Let $J_{t,k}=S_k\cap\{\Delta_t-2m_t\pi\}$, $t,k\in\mathbb{N}$. By (5) we have

$$\Delta_t-2m_t\pi\subseteq F_0\cup\left(\bigcup_{j=1}^{\infty}\frac{1}{2^{n_j}}\Delta_j\right).$$

Since $S_k\subseteq E$ and $E\cap F_0=\emptyset$, we have $J_{t,k}\subseteq\bigcup_{j=1}^{\infty}(1/2^{n_j})\Delta_j$. Let $s\in J_{t,k}$. Then $x\in(1/2^{n_j})\Delta_j$ for some $j\in\mathbb{N}$. Because $S_k\cap(1/2^{n_k})\Delta_k=\emptyset$ for $k\in\mathbb{N}$, $x\notin(1/2^{n_k})\Delta_k$. If $j\neq k$ then the set $(1/2^{n_j})\Delta_j\subseteq I_j\cap E$ is disjoint from $I_k\cap E$, which contains S_k. Hence $J_{t,k}=\emptyset$.

(b) Let $I_{t,k}=\{\Delta_t-2m_t\pi\}\cap\{\Delta_k-2m_k\pi\}$. Since $(1/2^{n_k})\Delta_k\subset I_k\cap E$ for $k\in\mathbb{N}$, and since $F_0\cap E=\emptyset$, $\{F_0,(1/2^{n_i})\Delta_i\colon i\in\mathbb{N}\}$ is a family of disjoint sets. By (5) and (6), the only possible common elements of $\Delta_t-2m_t\pi$ and $\Delta_s-2m_s\pi$ would be in F_t. Thus we have

$$\begin{aligned}I_{t,k}&\subseteq\{\Delta_t-2m_t\pi\}\cap F_t\\&\subseteq\{\Delta_t-2m_t\pi\}\cap\left(\frac{1}{2^{n_t}}\Delta_t\cup\{F_{t-1}\setminus\{\Delta_t-2m_t\pi\}\}\right)\\&\subseteq\{\Delta_t-2m_t\pi\}\cap\left(\frac{1}{2^{n_t}}\Delta_t\right).\end{aligned}$$

Since $\Delta_t\subseteq F_{t-1}+2m_t\pi$, we have $I_{t,k}\subseteq F_{t-1}\cap(1/2^{n_t})\Delta_t$. By (5) we have

$$I_{t,k}\subseteq\left(F_0\cup\left(\bigcup_{i=1}^{t-1}\frac{1}{2^{n_i}}\Delta_i\right)\right)\cap\frac{1}{2^{n_t}}\Delta_t.$$

This is an empty set.

(ii) It is clear that $\{F_{k-1}+2m_k\pi\}\subseteq[2\pi,4\pi)+2m_k\pi\subseteq 2^{n_k}J_k$. We have

$$\begin{aligned}2m_k\pi+F_{k-1}\setminus\{\Delta_k-2m_k\pi\}&=(F_{k-1}+2m_k\pi)\setminus\Delta_k\\&=(F_{k-1}+2m_k\pi)\setminus\{2^{n_k}E\cap(F_{k-1}+2m_k\pi)\}\\&\subseteq 2^{n_k}J_k\setminus(2^{n_k}E\cap 2^{n_k}J_k).\end{aligned}$$

By (2), we have

$$\begin{aligned}m(F_{k-1}\setminus\{\Delta_k-2m_k\pi\})&=m(2^{n_k}J_k\setminus(2^{n_k}E\cap 2^kJ_k))\\&\leq m(2^{n_k}J_k)-m((2^{n_k}E\cap 2^kJ_k))\\&<\frac{1}{2^k}\cdot 2\pi.\end{aligned}$$

Hence $m(F_{k-1}\setminus\{\Delta_k-2m_k\pi\})<(1/2^k)\cdot 2\pi$. Since

$$\Delta_k\subseteq[2(m_k+1)\pi,2(m_k+2)\pi),$$

we have

$$m\left(\frac{1}{2^{n_k}}\Delta_k\right)\leq\frac{1}{2^{n_k}}\cdot 2\pi.$$

We therefore have

$$m(F_k) = m\left(\frac{1}{2^{n_k}}\Delta_k \cup (F_{k-1} \setminus \{\Delta_k - 2m_k\pi\})\right)$$
$$< \frac{1}{2^k} \cdot 2\pi + \frac{1}{2^{n_k}} \cdot 2\pi. \qquad \square$$

LEMMA 4.6.

$$F_m \cup \left(\bigcup_{k=1}^{m} S_k\right) \cup \left(\bigcup_{k=1}^{m} (\Delta_k - 2m_k\pi)\right) \cup \left(\bigcup_{k>m} (I_k \cap E)\right) = [2\pi, 4\pi).$$

Proof. We prove this formula by induction on m. By definition we have $\Delta_k = 2^{n_k}E \cap (F_{k-1} + 2m_k\pi)$. Hence we have $\Delta_k - 2m_k\pi \subseteq F_{k-1}$ or (equivalently)

$$F_{k-1} = \{F_{k-1} \setminus (\Delta_k - 2m_k\pi)\} \cup (\Delta_k - 2m_k\pi). \tag{7}$$

Because $(1/2^{n_k})\Delta_k \subseteq E \cap I_k$, we have

$$I_k \cap E = \left(\frac{1}{2^{n_k}}\Delta_k\right) \cup \left((I_k \cap E) \Big\backslash \frac{1}{2^{n_k}}\Delta_k\right). \tag{8}$$

As a result,

$$\begin{aligned}[2\pi, 4\pi) &= F_0 \cup E = F_0 \cup (I_1 \cap E) \cup \left(\bigcup_{k>1} I_k \cap E\right)\\ &= \left(\frac{1}{2^{n_1}}\Delta_1\right) \cup (F_0 \setminus (\Delta_1 - 2m_1\pi))\\ &\quad \cup (\Delta_1 - 2m_1\pi) \cup \left((I_1 \cap E) \Big\backslash \frac{1}{2^{n_1}}\Delta_1\right) \cup \left(\bigcup_{k>1} I_k \cap E\right)\\ &= F_1 \cup S_1 \cup (\Delta_1 - 2m_1\pi) \cup \left(\bigcup_{k>1} I_k \cap E\right).\end{aligned}$$

Hence the formula is true for $m = 1$. Using (7) and (8) and by similar computation, we can prove the formula by induction. We leave the details to the reader. $\square$

LEMMA 4.7. *S is 2π-congruent to $[2\pi, 4\pi)$.*

Proof. We need to show that $\{S_i, \Delta_i - 2m_i\pi : i \in \mathbb{N}\}$ is a partition of $[2\pi, 4\pi)$ (modulo null sets). By Lemma 4.5(i), the above sets are mutually disjoint and $\sum_{i=1}^{\infty} m(S_i) + \sum_{i=1}^{\infty} m(\Delta_i - 2m_i\pi) \leq 2\pi$. It suffices to show that the equality actually holds.

By Lemma 4.5(ii) we have $\lim_{k\to\infty} m(F_k) = 0$. It is clear that we also have $\lim_{k\to\infty} \sum_{i=k+1}^{\infty} m(I_i \cap E) = 0$. By Lemma 4.6 we have

$$\sum_{i=1}^{k} m(S_i) + \sum_{i=1}^{k} m(\Delta_i - 2m_i\pi) + m(F_k) + \sum_{i=k+1}^{\infty} m(I_i \cap E) \geq 2\pi.$$

Let $k \to \infty$. Then

$$\sum_{i=1}^{\infty} m(S_i)+\sum_{i=1}^{\infty} m(\Delta_i-2m_i\pi)\geq 2\pi.$$

Lemma 4.7 is proven. □

Proof of Theorem 4.4 (continuation). Case (B) is similar to Case (A).

Case (C). Let $E_-:=\Omega\cap[-2\pi,-\pi)$ and $E_+:=\Omega\cap[\pi,2\pi)$. As in Case (A), we can construct sets S_- and S_+ with the following properties.

(i) S_- is a 2-dilation generator for $\Omega\cap(-\infty,0)$, and is 2π-congruent to $[-2\pi,-\pi)+4\pi$ (modulo null sets).

(ii) S_+ is a 2-dilation generator for $\Omega\cap(0,\infty)$, and is 2π-congruent to $[\pi,2\pi)+2\pi$ (modulo null sets).

The set $S:=S_-\cup S_+$ is a 2-dilation generator for Ω and is 2π-congruent to $[2\pi,4\pi)$. We leave the details to the reader. Theorem 4.4 is now proven. □

5. Examples

In this last section we will give examples of closed subspaces which are not reducing subspaces of D and T and which have orthogonal wavelets with regularity properties.

The following lemma is a weak version of Lemma 4.1 in [4].

LEMMA 5.1. *Let f be in $L^2(\mathbb{R})$ with support K_0. Assume that K_0 is a 2-dilation generator for some set Ω with $2\Omega=\Omega$. Let X be a closed subspace of $L^2(\mathbb{R})$ such that $\hat{X}=L^2(\mathbb{R})\cdot\chi_\Omega$. Assume there is a measurable subset $I_0\subseteq K_0$ with positive measure such that $I_0+2n_0\pi\subseteq K_0$ for some $n_0\in\mathbb{Z}$. Then the function $\mathfrak{F}^{-1}f$ is not an orthogonal wavelet for X.*

Proof. The function $\mathfrak{F}^{-1}f$ is an orthogonal wavelet for X if and only if $\{D^nT^l(\mathfrak{F}^{-1}f): n,l\in\mathbb{Z}\}$ is an orthonormal basis for X if and only if $\{D^{-n}(e^{-ils}f): n,l\in\mathbb{Z}\}$ is an orthonormal basis for $\hat{X}$. By assumption, $\operatorname{supp}(e^{ils}f)=K_0$, so $\operatorname{supp}(D^{-n}(e^{-ils}f))=2^nK_0$ for $n\in\mathbb{Z}$. Since K_0 is a 2-dilation generator for Ω, the sets $2^{-n}K_0$, $n\in\mathbb{Z}$, form a partition for Ω. Hence $\mathfrak{F}^{-1}f$ is an orthogonal wavelet for X if and only if $\{e^{-ils}f: l\in\mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})\cdot\chi_{K_0}$. Assume that it is an orthonormal basis.

Let g be a function on $\mathbb{R}$ defined as follows:

$$g(s)=\begin{cases}-1 & \text{if } s\in I_0,\\ 1 & \text{if } s\in I_0+2n_0\pi,\\ 0 & \text{otherwise.}\end{cases}$$

The function $g\cdot f$ is in $L^2(\mathbb{R})\cdot\chi_{K_0}$. Then

$$g\cdot f=\sum_{n\in\mathbb{Z}}\alpha_n e^{ins}\cdot f$$

for some $(\alpha_n)\in\ell^2(\mathbb{Z})$. Let h be the 2π-periodic function given by the sum $\sum_{n\in\mathbb{Z}}\alpha_n e^{-ins}$, where convergence is in $L^2[0,2\pi]$ with 2π-periodic extension to $\mathbb{R}$. It follows that

$$g(s)\cdot f(s) = h(s)\cdot f(s) \text{ a.e. on } \mathbb{R}.$$

Since $f(s) \neq 0$ a.e. on K_0, we must have $g(s) = h(s)$ a.e. on K_0. We have

$$-1 = g(s) = h(s) = h(s+2n_0\pi) = g(s+2n_0\pi) = 1$$

for $s \in I_0$ (a.e.), a contradiction to the definition of g. Lemma 5.1 is proven. □

The Meyer's wavelet ψ_{Me} is defined as follows

$$\hat{\psi}_{Me}(\xi) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{i\xi/2} \sin\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}|\xi|-1\right)\right] & \text{if } \frac{2\pi}{3} \leq |\xi| \leq \frac{4\pi}{3}, \\ \frac{1}{\sqrt{2\pi}} e^{i\xi/2} \cos\left[\frac{\pi}{2}\nu\left(\frac{3}{4\pi}|\xi|-1\right)\right] & \text{if } \frac{4\pi}{3} \leq |\xi| \leq \frac{8\pi}{3}, \\ 0 & \text{otherwise.} \end{cases}$$

Here ν is a C^k or C^∞ function satisfying

$$\nu(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ 1 & \text{if } x \geq 1, \end{cases}$$

with the additional property that

$$\nu(x) + \nu(1-x) = 1.$$

Let $K_0 = [-8\pi/3, -2\pi/3] \cup [2\pi/3, 8\pi/3]$. The set K_0 is *not* a 2-dilation generator for any set. It is clear that $\text{supp}(\hat{\psi}_{Me}) = K_0$ (modulo a null set). Let $J_0 = [-8\pi/3, -4\pi/3] \subset K_0$ and let J_0 satisfy the condition $J_0 + 4\pi \subset K_0$. This ψ_{Me} is an orthogonal wavelet for $L^2(\mathbb{R})$.

EXAMPLE 5.2. Define a function $\psi_{2\pi}$ by

$$\hat{\psi}_{2\pi}(\xi) = \begin{cases} \hat{\psi}_{Me}(\xi - 2\pi) & \text{if } \xi \geq 2\pi, \\ \hat{\psi}_{Me}(\xi + 2\pi) & \text{if } \xi \leq -2\pi, \\ 0 & \text{if } \xi \in (-2\pi, 2\pi). \end{cases}$$

The support of $\hat{\psi}_{2\pi}(s)$ is $K_0 = [-14\pi/3, -8\pi/3) \cup [8\pi/3, 14\pi/3)$. Let $K = [-16\pi/3, -8\pi/3) \cup [8\pi/3, 16\pi/3)$. K is a 2-dilation generator for $\mathbb{R}$. K_0 is a proper subset of K and is a 2-dilation generator for the set $\Omega := \bigcup_{j=1}^{\infty} 2^j K_0$. Let X be the closed subspace such that $\hat{X} = L^2(\mathbb{R}) \cdot \chi_\Omega$. By Proposition 4.3, X is a reducing subspace of $\{D, T\}$.

Consider the set $\mathfrak{B}_3 = \{D^n T^l \psi_{2\pi} : n, l \in \mathbb{Z}\}$. Let $Y = \overline{\text{span}}\{\mathfrak{B}_3\}$. Let $I_0 = [-14\pi/3, -10\pi/3)$. Then $I_0 \subset K_0$ and $I_0 + 8\pi \subset K_0$. By Lemma 5.1, $\mathfrak{B}_3$ is not an orthonormal basis for X, so Y is a proper subspace of X.

We will show that $\mathfrak{B}_3$ is an orthonormal set. For $l, l' \in \mathbb{Z}$, we have

$$\begin{aligned} \langle T^l \psi_{2\pi}, T^{l'} \psi_{2\pi} \rangle &= \langle \mathfrak{F} T^l \psi_{2\pi}, \mathfrak{F} T^{l'} \psi_{2\pi} \rangle \\ &= \langle \hat{T}^l \hat{\psi}_{2\pi}, \hat{T}^{l'} \hat{\psi}_{2\pi} \rangle \\ &= \langle e^{-ils} \hat{\psi}_{2\pi}, e^{-il's} \hat{\psi}_{2\pi} \rangle \end{aligned}$$

$$= \int_{\mathbb{R}} e^{-ils}\hat{\psi}_{2\pi}(s) \cdot \overline{e^{-il's}\hat{\psi}_{2\pi}(s)}\, ds$$
$$= \int_{2\pi}^{\infty} e^{-ils}\hat{\psi}_{Me}(s-2\pi) \cdot \overline{e^{-il's}\hat{\psi}_{Me}(s-2\pi)}\, ds$$
$$+ \int_{-\infty}^{-2\pi} e^{-ils}\hat{\psi}_{Me}(s-2\pi) \cdot \overline{e^{-il's}\hat{\psi}_{Me}(s-2\pi)}\, ds$$
$$= \int_{\mathbb{R}} e^{-ils}\hat{\psi}_{Me}(s) \cdot \overline{e^{-il's}\hat{\psi}_{Me}(s)}\, ds$$
$$= \langle e^{-ils}\hat{\psi}_{Me}(s), e^{-il's}\hat{\psi}_{Me}(s)\rangle$$
$$= \langle \hat{T}^l\hat{\psi}_{Me}, \hat{T}^{l'}\hat{\psi}_{Me}\rangle$$
$$= \langle T^l\psi_{Me}, T^{l'}\psi_{Me}\rangle$$
$$= \delta_{l,l'}.$$

Therefore $\{T^l\psi_{2\pi}: l \in \mathbb{Z}\}$ is an orthonormal set. Since the operator D is unitary, and since supports for the functions $D^{-n}e^{-ils}\hat{\psi}_{2\pi}$ and $D^{-n'}e^{-il's}\hat{\psi}_{2\pi}$ are disjoint for different $n, n' \in \mathbb{Z}$, the set $\{D^nT^l\psi_{2\pi}: n, l \in \mathbb{Z}\}$ is an orthonormal set. Hence $\psi_{2\pi}$ is an orthogonal wavelet for the space Y.

Because K_0 is the support of $\hat{\psi}_{2\pi}$ and is a 2-dilation generator for Ω, X is the smallest reducing subspace of $\{D, T\}$ containing $\psi_{2\pi}$. Thus, the space Y is *not* a reducing subspace of $\{D, T\}$ that has an orthogonal wavelet $\psi_{2\pi}$. By Lemma 4.1, the space Y has *no* multiresolution analysis. It is left to the reader to check that the function $\psi_{2\pi}$ satisfies the same regularity properties as Meyer's.

EXAMPLE 5.3. Define a function η by

$$\hat{\eta}(s) := \hat{\psi}_{Me}(s-8\pi).$$

The support of $\hat{\eta}$ is $K_0 = [16\pi/3, 22\pi/3) \cup [26\pi/3, 32\pi/3)$, which is a proper subset of $K := [16\pi/3, 32\pi/3)$. K is a 2-dilation generator for $\mathbb{R}$, and K_0 is a 2-dilation generator for $\Omega = \bigcup_{j=1}^{\infty} 2^j K_0$. Let X be a closed subspace such that $\hat{X} = L^2(\mathbb{R}) \cdot \chi_\Omega$. This X is a proper subspace of the Hardy space $\mathcal{H}^2$. Let

$$Y = \overline{\text{span}}\{D^nT^l\eta: n, l \in \mathbb{Z}\}.$$

Then η is an orthogonal wavelet for Y and Y is not a reducing subspace of $\{D, T\}$. The function η satisfies the same regularity properties as ψ_{Me} does.

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