The Topology of Minimal Surfaces in Seifert Fiber Spaces

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1. Introduction

A basic question in the theory of minimal surfaces in 3-dimensional manifolds is to decide which embeddings of surfaces can be realized by minimal surfaces. Fundamental results were obtained in the case of Riemannian metrics of positive curvature in [Fr], [L1], and [SY] for sectional, Ricci, and scalar curvatures, respectively. In [R1] a fairly complete description was obtained of the topology of embeddings of minimal surfaces in 3-manifolds of positive scalar curvature.

Seifert fiber spaces are an important class of examples of 3-dimensional manifolds that admit 1-dimensional foliations by circles. Thurston [Th] has proposed a geometrization program for classifying closed 3-manifolds by decomposing them into pieces that admit eight geometries. Six of the eight geometries occur on Seifert fiber spaces. Moreover, the natural metrics are compatible with the Seifert fiber structure, in the sense that (possibly after passing to a double cover) the isometry group of the metric has an $SO(2)$ component with orbits the circle fibers.

In [Ha], Hass studied the topology of π_1-injective minimal surfaces in Seifert fiber spaces. In Section 3 we obtain a topological classification of arbitrary embedded minimal surfaces in such 3-manifolds, extending [Ha]. Finally in Section 4, using the minimax technique developed in [Pi], [PR1], [PR2], and [HPR], examples of interesting minimal surfaces in Seifert fiber spaces are constructed. Note that, throughout this paper, the only restriction on the Riemannian metric is that the $SO(2)$ action of the previous paragraph be by isometries.

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2. Seifert Fiber Spaces

For details about Seifert fiber spaces, a good reference is Orlik [Or].

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Let D^2 denote the unit disk in the complex plane. A standard foliation of the solid torus $D^2 \times S^1$ by circles can be described by gluing together the ends of a cylinder $D^2 \times [0,1]$ using the map ϕ, $\phi(z,0) = (\exp(2\pi iq/p)z, 1)$, where $|z| \le 1$ and p, q are relative prime positive integers. The foliation of the cylinder by intervals $\{z\} \times [0,1]$ induces a foliation of the solid torus $D^2 \times [0,1]/\phi$ by circles.

A *Seifert fiber space* is a 3-manifold M that has a foliation by circles, where each circle has a closed regular neighborhood filled by leaves described by $D^2 \times [0,1]/\phi$. The *base space* (or leaf space) of M is the closed surface S, obtained by identifying each circle with a point, endowed with the quotient topology. There are a finite number of *exceptional fibers* where the solid torus neighborhood has $p > 1$. All other fibers are called *ordinary.*

A *vertical* surface is tangent everywhere to the ordinary fibers of M. Let C be a simple closed curve in S missing the exceptional points that are the images of the exceptional fibers. Then the fibers over C form a surface that is either a vertical torus or a vertical Klein bottle. Any vertical surface is the union of vertical tori and vertical Klein bottles. A *horizontal* surface is transverse everywhere to the fibers of M. A π_1-*injective* surface Σ is a closed surface embedded in M such that the map $\pi_1(\Sigma) \to \pi_1(M)$ induced by the embedding is one-to-one and Σ is not a 2-sphere or a projective plane. A principal result of [Ha] is that a π_1-injective minimal surface in a Seifert fiber space must be vertical or horizontal. (In fact, this is shown for immersions, but we only deal with embeddings here.) For future reference, note that if Σ is 2-sided, nonseparating, and horizontal, then the closure of $M - \Sigma$ is homeomorphic to $\Sigma \times [0,1]$, so M has the structure of a Σ-bundle over S^1. If Σ is 1-sided and horizontal, then M has the structure of two copies of $N(\Sigma)$, a regular neighborhood of Σ (i.e., a twisted line bundle over Σ), glued together along the boundaries $\partial N(\Sigma)$.

For many purposes it is more convenient to work with an $SO(2)$ action rather than a foliation by circles. Next we describe the double cover $\tilde{M}$ of M that will be needed if M does not already admit such an action. Define a homomorphism $\pi_1(S) \to H_1(S) \to Z_2$ by assigning 0 to a closed curve C in S if the fibers over C form a torus and 1 if the surface is a Klein bottle. Unless all vertical surfaces in M are tori, this induces a double covering $\tilde{S}$ of S and $\tilde{M}$ of M, specified by the kernel of the homomorphism. In this case, $\tilde{M}$ has an $SO(2)$ action in which the orbits are fibers. In case there are only vertical tori, it is easy to see that M admits an $SO(2)$ action for which the orbits are fibers. In both cases we assume for the remainder of the paper that the $SO(2)$ action preserves the Riemannian metric on M, or the induced metric on $\tilde{M}$.

We now state a useful lemma about minimal surfaces in Seifert fiber spaces. We recall that a minimal surface is *stable* if its second variation of area is nonnegative; otherwise it is *unstable*. The *nullity* of a minimal surface is the dimension of the eigenspace belonging to zero of the second-variation elliptic differential operator.

LEMMA. *Let Σ be a closed connected 2-sided embedded minimal surface in a Seifert fiber space M that is endowed with an $SO(2)$ action. If Σ is horizontal, then Σ is stable with* $\text{nullity}(\Sigma)=1$. *If Σ is vertical, then Σ can be stable or unstable with nullity zero or nonzero in either case. If Σ is neither vertical nor horizontal, then Σ must be unstable.*

Proof. If Σ is horizontal, then the $SO(2)$ action on M foliates M by isometric translations of Σ, from which the conclusion follows. Easy examples of various minimal vertical tori, both stable and unstable, are shown in Section 4.

We assume now that Σ is neither horizontal nor vertical. Let E be the velocity vectorfield induced by the $SO(2)$ action on M. Near Σ we may write $E=N+T$, where $N|_\Sigma$ is normal and $T|_\Sigma$ is tangential to Σ. Rotation under the $SO(2)$ action is an isometry, so $0=\delta^2\Sigma(E)=\delta^2\Sigma(N+T)=\delta^2\Sigma(N)$. This last equality follows from a straightforward calculation much like that in [L2, Chap. 1, Sec. 4]. Since Σ is neither horizontal nor vertical, $N\neq 0$ on Σ and N must vanish for at least one point in Σ. This would not be possible if Σ were stable, since in that case $N|_\Sigma$ would have to be a nonzero multiple of the everywhere nonvanishing eigenvectorfield belonging to zero, a contradiction. □

3. Topology of Embeddings

In this section we give a topological classification of embeddings of minimal surfaces in Seifert fiber spaces.

Recall that a 2-sided surface Σ embedded in a 3-manifold M is called *compressible* if there is an embedded disk D in M, with $D\cap\Sigma=\partial D$ and ∂D a noncontractible loop in Σ. If no such *compressing disks* D can be found, then Σ is called *incompressible*. It follows from Dehn's lemma and the loop theorem (see [He]) that, for 2-sided surfaces, incompressibility is equivalent to being π_1-injective.

A *handlebody* is a compact 3-manifold W with connected boundary $\partial W=\Sigma$ having g handles ($g\geq 1$), so that there exist g disjoint properly embedded compressing disks $D_1, D_2, \dots, D_g$ for Σ in W and the closure of $W-(D_1\cup\cdots\cup D_g)$ is a 3-cell. Note that W is not necessarily orientable. If W is nonorientable, then Σ has $2g$ cross-caps.

A *hollow handlebody* is a compact 3-manifold W that has two or more boundary components $\partial W=\Sigma\cup\Sigma_1\cup\cdots\cup\Sigma_k$, none of which are spheres or projective planes. There are also disjoint properly embedded compressing disks $D_1,\dots,D_s$ with $s\geq 1$ and boundary curves on Σ, and the closure of $W-(D_1\cup\cdots\cup D_s)$ is a collection of products homeomorphic to $\Sigma_i\times[0,1]$ for $1\leq i\leq k$. One notes that hollow handlebodies may likewise be nonorientable.

We now state the main theorem, which gives a topological classification of embeddings of minimal surfaces in Seifert fiber spaces.

THEOREM. *If Σ is a closed connected embedded minimal surface in a Seifert space M, then the embedding of Σ is one of three (mutually exclusive) types, as follows.*
(1) *Σ is a vertical torus or a vertical Klein bottle.*
(2) *Σ is horizontal and π_1-injective. As noted in* [Ha], *there are three distinct subcases.*
 (a) *Σ is 2-sided and nonseparating in M. The closure of $M-\Sigma$ is homeomorphic to a product of Σ and an interval.*
 (b) *Σ is 2-sided and separates M. M is the union of two twisted line bundles $N(\Lambda)$ over Λ and $\Sigma = \partial N(\Lambda)$ is a double cover of Λ.*
 (c) *Σ is 1-sided in M. The closure of $M-N(\Sigma)$ is another copy of $N(\Sigma)$. Here $N(\Sigma)$ is a small regular neighborhood of Σ in M.*
(3) *Σ is neither vertical nor horizontal. There is a (possibly empty) disjoint family $\mathfrak{Z}$ of vertical minimal tori and Klein bottles in $M-\Sigma$. The closure of $M-\Sigma-\bigcup\mathfrak{Z}$ has j components that are handlebodies or hollow handlebodies; the rest are Seifert fiber spaces with boundaries on $\bigcup\mathfrak{Z}$. Here $j=2$ if Σ is 2-sided in M and $j=1$ if Σ is 1-sided in M.*

Proof. The proof proceeds in six steps.

Step 1: Cases (1) and (2) are considered in [Ha]. Thus we assume throughout the proof that Σ is neither vertical nor horizontal. We assume for the time being that M has an isometric $SO(2)$ action compatible with the Seifert fiber space structure. We shall show how the general case follows from this at the end of the proof.

Step 2: Assume Σ is a 2-sided embedded minimal surface. Split M along Σ to form a compact 3-manifold with boundary M' such that $\partial M'$ consists of two copies of Σ. If Σ is separating then M' has two components, while if Σ does not separate then M' is connected. Consider a copy Σ' of Σ in $\partial M'$. By Dehn's lemma and the loop theorem, either this surface is π_1-injective (incompressible) in M' or there is a maximal family of disjoint nonparallel compressing disks $D_1, D_2, \ldots, D_k$, properly embedded in M', with boundaries on Σ'.

In this step we examine the case that if $N(D_i)$ are small disjoint regular neighborhoods of D_i in M' and $N(\Sigma')$ is a small neighborhood of Σ' in M', then $L = \partial(N(\Sigma') \cup N(D_1) \cup \cdots \cup N(D_k))$ consists of 2-spheres, projective planes, and Σ'.

If a Seifert fiber space M has a 2-sided projective plane or an essential embedded 2-sphere (i.e., which does not bound a 3-cell), it is easy to show that M is homeomorphic to one of $S^1 \times S^2$, $S^1 \times RP^2$, $RP^3 \# RP^3$, or $S^1 \tilde{\times} S^2$, where the latter is the nonorientable sphere bundle over S^1. In all cases, M is foliated by minimal 2-spheres and projective planes. By the maximum principle, no minimal surface Σ in M can be disjoint from such a projective plane or essential 2-sphere, unless Σ is itself a 2-sphere or projective plane and is part of a foliation as in (2) of the theorem. Otherwise we conclude that all the components of L are 2-spheres that bound 3-cells. If all these 3-cells lie in M', then clearly Σ' bounds a handlebody in M, as in the theorem.

Suppose a 2-sphere component J of L bounds a 3-ball containing Σ, but is essential in M'. In this case, by [MSY], J can be isotoped to obtain a minimal 2-sphere in M'. (This uses the maximum principle—note that Σ forms a barrier and stops J from shrinking to a point.) Denote this minimal 2-sphere by J again.

Let $p: M \to S$ denote the projection to the base or orbit space of the Seifert fibering. If we apply the $SO(2)$ action to J, by the maximum principle it follows that all the translates of J are disjoint from Σ. Hence $p(J) \cap p(\Sigma) = \emptyset$. But then $F \cap J = \emptyset$ whenever F is any fiber projecting to a point in $p(\Sigma)$, so F lies in the 3-cell bounded by J. But no fiber of a Seifert fibering is contractible, unless the space is the 3-sphere (see [Or]), in which case J bounds a 3-cell disjoint from Σ as required. This completes the argument that if L consists of 2-spheres and projective planes then Σ' bounds a handlebody.

Step 3: We continue studying L, the result of compressing Σ as much as possible in the complement of Σ. Assume L has a component L_0 that is not a 2-sphere or a projective plane, but has genus less than L'. As in Step 2, since L_0 is incompressible in M', by [MSY] and the maximum principle L_0 can be isotoped to a minimal surface in M', which will again be denoted by L_0. Applying the $SO(2)$ action and the maximum principle, it follows that $p(L_0)$ is disjoint from $p(\Sigma)$.

Recall that each fiber F of M has a neighborhood of the form $D \times [0,1]/\phi$, where $\phi(z, 0) = (\exp(2\pi i q/p)z, 1)$. The positive integer p is called the *multiplicity* of F. There are finitely many fibers F for which $p \geq 2$; such fibers are called *exceptional* (see [Or]).

Now choose a small regular neighborhood $N(p(\Sigma))$ of $p(\Sigma)$ in S, so that $N(p(\Sigma))$ is disjoint from $p(L_0)$. Let S_0 be the component of $S - \mathrm{Int}(N(p(\Sigma)))$ that contains $p(L_0)$, and suppose S_0 is a disk with the projection of at most one exceptional fiber. Then $p^{-1}(S_0)$ is a solid torus containing L_0. But L_0 is a π_1-injective surface in M'. This is a contradiction, as $p^{-1}(S_0)$ is contained in M'. We conclude that either S_0 is not a disk or else S_0 is a disk containing at least two exceptional fibers.

Let C be a boundary curve of S_0. Then the torus T of fibers over C is π_1-injective in M', since if not then T would bound a solid torus in M'. Consequently, by [MSY] and the maximum principle, there is a minimal vertical torus isotopic to T in the complement of Σ, which we denote by T_1. Note that, by construction, either (a) T_1 is incompressible in M or (b) T_1 bounds a solid torus $D^2 \times S^1$ containing Σ and T_1 is incompressible in $M - \mathrm{Int}(D^2 \times S^1)$. In both cases, if we split M along Σ and choose a boundary component Σ' of M', then all the compressing disks for Σ' can be selected to be disjoint from T_1. In other words, if the Seifert fiber space M is split along T_1, we can work in the component M_1 containing Σ. M_1 is a Seifert fiber space with minimal vertical tori boundary.

The procedure can be iterated; if a component of L is not a 2-sphere bounding a 3-cell or parallel to a boundary torus of M_1, then we can find another minimal vertical torus T_2 disjoint from Σ and not isotopic to T_1. The procedure clearly stops after a finite number of steps. In fact, each time the Seifert fiber space is split along such a vertical torus, either the orbit surface has a disk with at least two exceptional fibers removed, or a handle is cut open along a nonseparating simple closed curve, or an annulus with at least one exceptional fiber is cut out.

In conclusion, there is a family $\mathfrak{T}$ of disjoint minimal vertical tori such that the result L of compressing Σ' is a collection of 2-spheres bounding 3-cells and copies of tori in $\mathfrak{T}$. Hence Σ' together with $\mathfrak{T}$ bounds a hollow handlebody.

Step 4: Suppose Σ' is incompressible in M'. According to the lemma, Σ is unstable and can be isotoped to a surface $\bar{\Sigma}$ of strictly less area lying entirely on one side of Σ. We can now minimize area among surfaces in the isotopy class of $\bar{\Sigma}$ in $\operatorname{Int} M'$. By [MSY], this yields a new minimal surface Σ^* isotopic to Σ' in $\operatorname{Int} M'$.

The same technique as in Step 2 and Step 3 can be applied to show that either $p(\Sigma) \cap p(\Sigma^*) = \emptyset$ or there is an isometry in $SO(2)$ taking Σ to Σ^*. In the latter case, it is easy to see that either M is foliated by parallel disjoint copies of Σ, or M is a union of two copies of twisted line bundles $N(\Lambda)$, where Σ is a horizontal incompressible surface that double covers Λ. In the former case, by choosing a small regular neighborhood $N(p(\Sigma))$ disjoint from $p(\Sigma^*)$ in S we find that $p^{-1}(\partial N(p(\Sigma)))$ is a collection of vertical tori separating Σ^* from Σ. By standard 3-manifold theory (see e.g. [He]), since Σ' and Σ^* are incompressible and parallel in M', it follows that Σ', Σ^*, and Σ are vertical tori.

Step 5: Suppose that Σ is a 1-sided embedded minimal surface. Then there is a natural choice of double covering $\hat{M}$ of M for which Σ lifts to a connected double covering minimal surface $\hat{\Sigma}$. In fact, define a homomorphism $\pi_1(M) \to H_1(M) \to Z_2$ by taking intersection number modulo 2 of closed curves in M with Σ. The kernel of this homomorphism then specifies $\hat{M}$. The previous discussion (Steps 1–4) now applies to the 2-sided minimal surface $\hat{\Sigma}$ in $\hat{M}$.

If $\hat{\Sigma}$ is a vertical torus, then clearly Σ is a vertical torus or a Klein bottle. If $\hat{\Sigma}$ is nonseparating, horizontal, and π_1-injective, then Σ is also horizontal and π_1-injective. Note that here M is formed by gluing together two copies of $N(\Sigma)$, a twisted line bundle over Σ. This case actually does not occur for our choice of double covering $\hat{\Sigma}$. If $\hat{\Sigma}$ separates M into two twisted line bundles over Λ, then Λ is homeomorphic to Σ and again M is formed by gluing together two copies of $N(\Sigma)$. Suppose $\hat{\Sigma}$ together with a family $\mathfrak{T}$ of vertical tori decomposes M into two hollow handlebodies and a collection of Seifert fibered pieces. The two sides of $\hat{\Sigma}$ are flipped by the covering transformation for the projection of $\hat{M} \to M$. Consequently we can replace the decomposition of one side with the image of the splitting induced on the other side via the covering transformation. Then $\mathfrak{J}$ is converted into an equivariant family $\hat{\mathfrak{J}}$ of vertical tori that projects to a family (which can be denoted again by $\mathfrak{J}$) of vertical minimal tori in M. The complement of a regular neighborhood of Σ in M is decomposed along $\mathfrak{J}$ into a hollow handlebody and Seifert fibered pieces.

Step 6: It remains to justify the assertion in Step 1. We know that the theorem holds for minimal surfaces in $\tilde{M}$, and we wish to prove it in M. The argument is very similar to Step 5 and is left to the reader. Note that it is easy to check that a hollow handlebody glued to some Seifert fibered pieces along incompressible boundary tori can only double cover another hollow handlebody plus Seifert fibered pieces. (This follows either by Dehn's lemma and the loop theorem, or by the characteristic variety theorem in the quotient manifold; see e.g. [Ja].) This completes the proof of the theorem. □

COROLLARY. *Suppose that Σ is a closed connected embedded minimal surface in a Seifert fiber space M, and that there is a disjoint family $\mathfrak{J}$ of vertical minimal tori and Klein bottles, as in part (3) of the theorem. Then all the minimal surfaces in $\mathfrak{J}$ can be chosen to be π_1-injective, unless Σ is a 2-sided Heegaard surface in a vertical solid torus.*

Proof. Note that, in the proof of the theorem, if any torus T bounds a solid torus in M', then we can add this solid torus to fill in part of the hollow handlebody bounded by Σ'. On the other hand, if T bounds a solid torus $D^2 \times S^1$ not in M', then Σ is contained in $D^2 \times S^1$. It is easy to check that the proof of the theorem establishes that Σ is a 2-sided Heegaard surface for this solid torus; that is, Σ splits the solid torus into a handlebody and a hollow handlebody with boundary $\Sigma \cup T$. □

4. Examples

In this section we show methods for constructing minimal surfaces as specified in the theorem.

First of all, vertical or horizontal incompressible surfaces can be isotoped to stable minimal surfaces by [MSY]. A compressible vertical 2-sided torus clearly bounds a solid torus. It is easy to choose an $SO(2)$-invariant metric on $\tilde{M}$ for which T can be realized by a stable minimal surface also. If S is the orbit surface and C is a contractible simple closed curve with T lying over C, then one chooses the induced metric on S so that C is a "neck". On the other hand, for the standard geometries on Seifert fiber spaces, if T is a minimal torus bounding a solid torus, then T must be a Heegaard torus and $\tilde{M}$ is a lens space or S^3. The reason is that for geometric metrics, the induced structure on S is hyperbolic, spherical, or Euclidean, with cone points at the exceptional fibers (cf. [Sc]). If the metric is hyperbolic or Euclidean, then it does not admit simple closed contractible geodesics C, except if C bounds a disk D containing at least two cone points. But then $p^{-1}(D)$ is not a solid torus. If the metric is spherical, then a geodesic C must bound two disks with at most one cone point. Hence T is a Heegaard torus, separating $\tilde{M}$ into two solid tori.

We now turn to the construction of unstable minimal surfaces Σ as in part (3) of the theorem. We use methods similar to those of [HPR]. Assume for simplicity that M admits an $SO(2)$ action with orbit surface S that is orientable of genus ≥ 1. Choose a simple closed noncontractible loop C on S that pulls back to a vertical minimal torus T. Suppose first that C is nonseparating. Choose a dual simple closed curve C' on S that crosses C transversely in a single point. Let C'' be the loop obtained by Dehn twisting C twice about C'. There are two choices of twisting, giving C'' homologous to $C \pm 2C'$. We can similarly find a minimal vertical torus T'' over C'' by isotoping C'' to a geodesic.

We define a sweep-out between T and T'' using a nonorientable surface Σ' of genus 4. One attaches a tube running from one side of T to the other, which projects onto a small regular neighborhood of C'. Starting with T (the tube having been pinched out), one expands the tube until it compresses to become two vertical annuli (cf. [PR2, Fig. 2]). This is really a compression of Σ' along a disk with two boundary arcs on the tube and two on T. By applying the minimax technique of [Pi; PR1; HPR] to this sweep-out, one obtains a minimal surface that is either a nonorientable surface Σ of genus 4 or a torus T^*. Note that a Klein bottle cannot occur, since we are assuming that T and T'' are tori. It is then easy to show that any compression of Σ' can be carried out in a neighborhood of $T \cup T''$, which does not contain Klein bottles. Furthermore, the case of a torus T^* can be excluded if we use a geometry of type $[SL(2,R)]^{\sim}$ or $H^2 \times R$ (see [Sc]), since in these cases it is easy to show that any minimal torus is vertical and strictly stable. The minimax surface, on the other hand, must have either nonzero index or nonzero nullity.

One notes that if $C_0 = \partial N(C \cup C')$ is a simple closed curve in the boundary of a small regular neighborhood of $C \cup C'$, then there are two possibilities, depending on whether or not C_0 is contractible. If the genus of S is 1 and C_0 is contractible with at most one exceptional fiber in the disk bounded by C_0, then it is easy to see that Σ is a 1-sided Heegaard surface (cf. [R2]) and so the complement of $N(\Sigma)$, a small regular neighborhood, has closure a handlebody. If the genus of S is greater than 1 and either C_0 is noncontractible or C_0 bounds a disk with at least two exceptional fibers, then deforming C_0 to a geodesic gives a vertical minimal torus T_0 over C_0. We can choose the sweep-out in the complement of T_0; then in the theorem $\mathfrak{J} = \{T_0\}$ and $\partial N(\Sigma) \cup T_0$ bounds a hollow handlebody.

This example can be perturbed to give an orientable 2-sided minimal surface $\hat{\Sigma}$ by taking the natural double cover $\hat{M}$ of M for which Σ lifts to $\hat{\Sigma}$, which has genus 3. If Σ is a 1-sided Heegaard surface, then $\hat{\Sigma}$ is an ordinary Heegaard surface. On the other hand, if the genus of S is greater than 2 then it is easy to see that T_0 lifts to a pair of disjoint tori $T_0' \cup T_0''$ and that both $\hat{\Sigma} \cup T_0'$ and $\hat{\Sigma} \cup T_0''$ bound hollow handlebodies.

Suppose next that C is a separating curve. One chooses a nonseparating simple closed curve C' that is disjoint from C. Now run a tube in M from T back to itself so that, if $N(C)$ is a small regular neighborhood of C, then the projection of the tube together with $N(C)$ has boundary curves homologous to C, $C''=C+C'$, and C'. As in the previous paragraph, one constructs a sweep-out from T to two disjoint tori T' and T'' over C' and C'', respectively. By the same technique as in [HPR], assuming that M has a geometric metric, we obtain that the minimax surface Σ is unstable and orientable of genus 2. Note that in this case $T \cup \Sigma$ and $T' \cup T'' \cup \Sigma$ bound hollow handlebodies.

Finally, if the orbit surface S is a 2-sphere and if there are at least five exceptional fibers, then one can choose two simple closed geodesics C and C' that are disjoint and bound an annulus with exactly one exceptional fiber. It is easy to construct a sweep-out from T to T', the tori over C and C', by a surface Σ' of genus 2. Applying the minimax construction, one again obtains an unstable orientable minimal surface Σ of genus 2, with $\Sigma \cup T$ and $\Sigma \cup T'$ bounding hollow handlebodies.

Take a product of the circle and a nonorientable surface of genus 3. Consider an orientation-preserving nonseparating simple closed curve C on the surface. There is a torus lying over C, and if a small open regular neighborhood of C is removed from the surface then the result is a Möbius band with an open disk removed. Consider a tube running from one side of the vertical torus over C to the other and projecting to a strip joining the two boundary curves of the punctured Möbius band. This gives a nonorientable surface of genus 4 in M. By expanding the tube, the surface can be collapsed onto another vertical torus lying over a curve representing the result of Dehn twisting C twice along a dual simple closed curve C' meeting C once. The minimax procedure of [HPR] can be applied to obtain a nonorientable minimal surface of genus 4. Lifting to the orientable double covering manifold $\hat{M}$, Σ lifts to $\hat{\Sigma}$, which is orientable of genus 3 and isotopic to a surface that is a pair of nonseparating vertical tori joined by two tubes. It is easy to see that $\hat{\Sigma}$ is nonseparating, but together with a vertical torus bounds two hollow handlebodies.

There is also an interesting class of examples related to the corollary. Note that, by the discussion at the beginning of this section, for geometric structures on Seifert fiber spaces we cannot find minimal surfaces inside vertical solid tori unless the manifold is a lens space and the minimal surface is a Heegaard surface for the lens space.

Consider the class of minimal surfaces Σ_i in the 3-sphere with its standard metric (described in [PR3]) with the property that, as $i \to \infty$, Σ_i converges in the varifold metric to a copy of the Clifford torus with multiplicity 2. Of course there is an $SO(2)$ action on S^3 for which the Clifford torus is a vertical minimal surface. Let F be a great circle in S^3 at maximal distance $\pi/4$ away from the Clifford torus. Let $N(F)$ be a small $SO(2)$-invariant neighborhood of F. Remove $N(F)$ and glue in a Seifert fiber space $\bar{M}$ with a boundary torus $\bar{T}$. The metrics on $S^3 - \mathrm{Int}(N(F))$ and $\bar{M}$ can be chosen so that $\bar{T}$ is parallel to a stable minimal vertical torus T. (Of course, we require the metric on $\bar{M}$ to be $SO(2)$-invariant so that the $SO(2)$ actions on $\bar{T}$ and $\partial N(F)$ match up.) Now, for i large enough, it can easily be shown that $\Sigma_i \cap N(F) = \emptyset$. Thus Σ_i gives a minimal surface inside the vertical solid torus $S^3 - \mathrm{Int}(N(F))$.

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