The Topological Whitehead Torsion of an Equivariant Fiber Homotopy Equivalence

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Introduction

In this paper we compute the topological equivariant torsion of an equivariant fiber homotopy equivalence between compact equivariant ANRs. Throughout this paper, G denotes a finite group. Let $p: E \to B$ and $p': E' \to B'$ be locally trivial G-fibrations between compact G-ANRs such that the fibers are equivariant compact ANRs. Let

$$\begin{array}{ccc} E' & \xrightarrow{h} & E \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ B' & \xrightarrow{f} & B \end{array} \qquad (*)$$

be a G-fiber homotopy equivalence over the G-homotopy equivalence f. Then we compute the topological torsion of h using the torsion of the pull-back of f and the fiberwise action of the equivariant Euler characteristic of B' to the torsion of the fibers (Theorem 6.5). This result generalizes the main theorems in [1], [2], and [7].

The structure of the paper is as follows: In Section 1 we summarize the properties of compact G-ANRs and G-CE maps between them. In Section 2 we recall the definition of the topological torsion of a G-homotopy equivalence between compact G-ANRs and we prove the composition and the sum formula. In Section 3 we summarize the theory of functorial additive invariants [6, Chap. IV; 11, §6]. As an application of this theory we prove the product formula for the topological torsion following the lines of proof of Theorem 6.11 in [11]. This product formula generalizes the one given in the cellular case in [8] and [13]. In Section 4 we define the pull-back map p^*: $\mathrm{Wh}_G^{\mathrm{Top}}(B) \to \mathrm{Wh}_G^{\mathrm{Top}}(E)$ determined by a locally trivial G-fibration $p: E \to B$ between compact G-ANRs. The definition and the properties of p^* are analogous to the ones proven in [2] for $G = 1$. In Section 5 we define the fiberwise product of the Euler characteristic of B' with the torsions of the homotopy equivalences of the fibers determined by $(*)$. In Section 6 we complete the computation of the topological torsion of h. We first give the proof for the special case that B is a finite G-complex. Using the fact that every compact

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G-ANR is finitely G-dominated by a finite G-complex and the methods of functorial additive invariants we extend the calculation to the general case.

Steinberger and West introduced the topological equivariant torsion in [15] for classifying G-h-cobordisms between locally linear G-manifolds (see [14] for the details). The formula for the torsion of a fiber G-homotopy equivalence and the product formula (which is a special case of this) are very important tools in computing the torsion in certain cases.

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1. Equivariant Absolute Neighborhood Retracts

We review the definitions and some of the facts about equivariant ANRs and equivariant cell-like maps. The proofs are either elementary or similar to the non-equivariant case.

By a G-ANR we will mean a metric G-space which has the neighborhood extension property in the category of metric G-spaces.

1.1. We summarize some of the properties of G-ANRs.

(i) X is a G-ANR if and only if, for each G-embedding $i: X \to Y$ into a G-metric space, there is a G-neighborhood of $i(X)$ which retracts to X.

(ii) If X is a G-ANR and H is a subgroup of G, then X is an H-ANR.

(iii) If X is a G-ANR and X' is a G'-ANR then $X \times X'$ is a $G \times G'$-ANR. Similarly, if X and X' are G-ANRs then $X \times X'$ is a G-ANR (with the diagonal action).

(iv) (Equivariant Hanner's Theorem) The following are equivalent for a G-space X:
  (1) X is a G-ANR;
  (2) each orbit G_x, $x \in X$, has an open G-neighborhood which is a G-ANR;
  (3) each point $x \in X$ with isotropy group G_x has a slice which is a G_x-ANR.

  In particular, locally linear G-manifolds are G-ANRs.

(v) If X is an H-ANR and H is a subgroup of G, then the balanced product $G \times_H X$ is a G-ANR.

(vi) Every finite G-CW complex is a compact G-ANR.

(vii) Every compact G-ANR is G-dominated by a finite G-complex.

A proper G-map $f: X \to Y$ between G-spaces is called a *G-cell-like* map (abbreviated G-CE map) if, for each $y \in Y$ with isotropy group G_y and for each G_y-neighborhood U of $f^{-1}(y)$ in X, the inclusion map $f^{-1}(y) \to U$ is G_y-nullhomotopic. It follows from the definition that if $f^{-1}(y)$ is a G_y-ANR for all $y \in Y$ then f is a G-CE map if and only if $f^{-1}(y)$ is G_y-contractible. The composition of two G-CE maps between G-ANRs is a G-CE map [14].

The following characterizes G-CE maps [14].

1.2. Let $f: X \to Y$ be a proper G-map between G-ANRs. Then the following are equivalent:

(i) f is a G-CE map;
(ii) for each open G-subset U of Y, f restricts to a G-homotopy equivalence $f^{-1}(U) \to U$;
(iii) f is an α-homotopy equivalence for each open G-cover α of Y.

There is very strong connection between G-ANRs and equivariant Hilbert cube manifolds. Steinberger and West [16] extended the work of Chapman [4] on Hilbert cube manifolds in the equivariant case. The G-Hilbert cube Q_G is the countable product of copies of the unit disc of the regular representation of G. A separable metric G-space M is called a *Q_G-manifold* if each orbit has a G-neighborhood which is G-homeomorphic to an open subspace of Q_G.

1.3. We now summarize the connections between Q_G-manifolds and G-ANRs.

(i) (Equivariant Edward's Theorem) A G-space X is a G-ANR if and only if $X \times Q_G$ is a Q_G-manifold.
(ii) If $f: X \to Y$ is a G-CE map between locally compact G-ANRs, then $f \times \mathrm{id}: X \times Q_G \to Y \times Q_G$ is G-homotopic to a G-homeomorphism.

2. Topological Equivariant Torsion

In this section we will prove the composition and the sum formula for the topological equivariant torsion. First we recall some facts about mapping cylinders of G-maps. In this section, by a *G-space* we will mean a compact G-ANR, even though some of the results are true for more general spaces. We start by recalling some facts from [12]. Let A, X, and Y be G-spaces such that $X \cap Y \supset A$. Then X and Y are called G-CE *equivalent* rel A, denoted $X \sim Y \operatorname{rel} A$, if there exists a G-space Z containing X and Y as well as G-CE maps $r: Z \to X$ and $s: Z \to Y$ such that, if $i: X \to Z$ and $j: Y \to Z$ are the inclusion maps,

$$ri = \mathrm{id}|_X, \quad sj = \mathrm{id}|_Y, \qquad ir \simeq_G \mathrm{id}|_Z \operatorname{rel} A, \quad js \simeq_G \mathrm{id}|_Z \operatorname{rel} A.$$

Notice that "$\sim$" rel A is an equivalence relation on the G-spaces containing A as a closed subspace.

An immediate consequence of the definition and [14, Cor. 4.7] is the following.

(M0) If $X \sim Y \operatorname{rel} A$ and $f: A \to B$ is a G-map, then $X \cup_f B \sim Y \cup_f B \operatorname{rel} B$.

The following property has been proved in [12].

(M1) Let $f_t: Y \to Y'$, $0 \le t \le 1$, be a G-homotopy such that $f_t|_X = f_0|_X$. Then $M(f_0) \sim M(f_1) \operatorname{rel}(M((f_0)|_X) \cup Y \cup Y')$.

Let X, Y, Z be G-spaces containing A as a closed subspace. Let $f: X \to Y$ and $g: Y \to Z$ be G-maps which are the identity on A. Define $M(f, g)$ to be the space obtained from $M(f) \amalg M(g)$ by identifying the base Y of the mapping cylinder $M(f)$ with the top $Y \times \{0\}$ of the mapping cylinder $M(g)$ by the identity map. Notice that the collapse map in $M(f)$ gives:

(M2) $M(f, g) \sim M(g) \operatorname{rel} M(g)$.

Also in [12] it was shown that:

(M3) $M(f, g)$ and $M(gf)$ are G-CE equivalent $\operatorname{rel}((A \times I) \cup X \cup Z)$.

LEMMA 2.1. *Let $f: X \to Y$ be a G-map and let A be a closed G-space contained in X. Then*

(i) $M(f|_A) \sim M(f) \operatorname{rel} Y$;
(ii) $M(f) \sim Y \operatorname{rel} Y$;
(iii) $(A \times I) \cup (X \times \{k\}) \sim X \times I \operatorname{rel}(X \times \{k\})$ *for* $k = 0, 1$;
(iv) $X \times \{k\} \sim X \times I \operatorname{rel}(X \times \{k\})$ *for* $k = 0, 1$.

Proof. (i) Let $i: A \to X$ be the inclusion map. Then, using (M2) and (M3), we have

$$M(f|_A) = M(fi) \sim M(i, f) \sim M(f) \operatorname{rel} Y.$$

(ii) The equivalence is given by the collapse map $M(f) \to Y$. (iii) and (iv) follow from (i). □

LEMMA 2.2. *Let $f, g: A \to B$ be two G-homotopic G-maps between G-spaces, and let A be a closed subset of a G-space X. Then*

$$B \cup_f X \sim B \cup_g X \operatorname{rel} B.$$

Proof. Let $h: A \times I \to B$ be the G-homotopy between f and g. Then

$$B \cup_f X = B \cup_h (X \cup_{A \times 0} A \times I) \sim B \cup_h (X \times I) \operatorname{rel} B;$$

$$B \cup_g X = B \cup_h (X \cup_{A \times 1} A \times I) \sim B \cup_h (X \times I) \operatorname{rel} B. \qquad \square$$

LEMMA 2.3. *Let $X \supset B \supset A$ be a triple of G-spaces so that the inclusions are G-homotopy equivalences. Let $r: B \to A$ be a strong G-deformation retraction. Then*

$$X \sim B \cup_A (A \cup_r X) \operatorname{rel} A.$$

Proof. Let $i: A \to B$ be the inclusion. Then $X = B \cup_{\mathrm{id}} X \sim B \cup_{ir} X \operatorname{rel} B$ by Lemma 2.2. Also $B \cup_{ir} X \sim B \cup_A (A \cup_r X) \operatorname{rel} A$. □

We now describe the construction of the group $\mathrm{Wh}_G^{\mathrm{Top}}(A)$ [14]. An element of $\mathrm{Wh}_G^{\mathrm{Top}}(A)$ is represented by a strong G-deformation pair (X, A). Two such pairs, (X, A) and (Y, A), represent the same element in $\mathrm{Wh}_G^{\mathrm{Top}}(A)$ if and only if there exists a G-space Z containing A as well as G-CE maps $r: Z \to X$ and $s: Z \to Y$ such that $r|_A = s|_A = \mathrm{id}_A$ and $fr \simeq_G gs \operatorname{rel} A$, where

$f: X \to A$ and $g: Y \to A$ are strong G-deformation retractions [14]. Notice that if $X \sim Y \operatorname{rel} A$ then $(X, A) = (Y, A) \in \mathrm{Wh}_G^{\mathrm{Top}}(A)$. $\mathrm{Wh}_G^{\mathrm{Top}}(-)$ is a functor from the category of compact G-ANRs and G-homotopy classes of G-maps to the category of abelian groups.

If $f: Y \to X$ is a G-homotopy equivalence then $(M(f), Y)$ is a strong G-deformation pair [11; 5]. Define $\tau(f) = f_*(M(f), Y) \in \mathrm{Wh}_G^{\mathrm{Top}}(X)$.

PROPOSITION 2.4 (Composition Formula). *Let $f: X \to Y$ and $g: Y \to Z$ be G-homotopy equivalences. Then*

$$\tau(gf) = \tau(g) + g_*\tau(f)$$

in $\mathrm{Wh}_G^{\mathrm{Top}}(Z)$.

Proof. Consider the triple $(M(f,g), M(f), X)$ and let $r: M(f) \to X$ be a G-retraction. Then, by Lemma 2.3,

$$M(f,g) \sim M(f) \cup_X (X \cup_r M(f,g)) \operatorname{rel} X,$$

which means that

$$\begin{aligned} &r_*(M(f,g), M(f)) + (M(f), X) = (M(f,g), X) \\ &\quad \Rightarrow r_*(M(f) \cup_Y M(g), M(f)) + (M(f), X) = (M(f,g), X). \end{aligned}$$

Let $j: Y \to M(f)$ be the inclusion map. Then

$$(M(f) \cup_Y M(g), M(f)) = j_*(M(g), Y).$$

Using (M3), we can write the above formula as follows:

$$r_* j_*(M(g), Y) + (M(f), X) = M(gf), X).$$

But $rjf \simeq_G \mathrm{id}_X \Rightarrow r_* j_* = (f_*)^{-1}$. Apply $g_* f_*$ to the above equation and get $\tau(gf) = \tau(g) + g_*\tau(f)$. □

Next we will give a proof of the sum formula. Let

$$\begin{array}{ccc} X_0 & \xrightarrow{i_2} & X_2 \\ {\scriptstyle i_1}\downarrow & & \downarrow \\ X_1 & \longrightarrow & X \end{array} \quad \text{and} \quad \begin{array}{ccc} Y_0 & \xrightarrow{k_2} & Y_2 \\ {\scriptstyle k_1}\downarrow & & \downarrow{\scriptstyle j_2} \\ Y_1 & \xrightarrow{j_1} & Y \end{array}$$

be G-push-out diagrams of G-spaces so that i_1 and k_1 are inclusion maps (or, more generally, G-cofibrations) (see [11] for general facts about G-push-out diagrams of G-spaces and G-cofibrations). Let $f_s: X_s \to Y_s$, $s = 0, 1, 2$, and $f: X \to Y$ be G-homotopy equivalences commuting with the maps in the diagrams.

PROPOSITION 2.5. *With the preceding notation,*

$$\tau(f) = j_{1*}\tau(f_1) + j_{2*}\tau(f_2) - j_{0*}\tau(f_0), \tag{1}$$

where $j_0 = j_1 k_1 = j_2 k_2$.

Proof. We first reduce the problem to the case that all the maps in the diagrams are inclusion maps. Let

$$\begin{array}{ccc} X_0' & \xrightarrow{i_2'} & X_2' \\ {\scriptstyle i_1'}\downarrow & & \downarrow \\ X_1' & \longrightarrow & X' \end{array} \quad \text{and} \quad \begin{array}{ccc} Y_0' & \xrightarrow{k_2'} & Y_2' \\ {\scriptstyle k_1'}\downarrow & & \downarrow{\scriptstyle j_2'} \\ Y_1' & \xrightarrow{j_1'} & Y' \end{array}$$

be G-push-out diagrams, where $X_s' = X_s$ and $Y_s' = Y_s$ for $s = 0, 1$, and where $X_2' = M(i_2)$, $Y_2' = M(k_2)$, and all the maps are the inclusion maps. Let c: $X_2' \to X_2$, $d: Y_2' \to Y_2$, $c_X: X' \to X$, and $c_Y: Y' \to Y$ be the G-CE maps induced by the collapse maps in the mapping cylinders. They induce G-homotopy equivalences $f_s': X_s' \to Y_s'$, $s = 0, 1, 2$, and $f': X' \to Y'$ which commute with the maps in the diagrams.

Claim: The formula (1) in the proposition follows from the formula

$$\tau(f') = j_{1*}'\tau(f_1') + j_{2*}'\tau(f_2') - j_{0*}'\tau(f_0'), \tag{2}$$

where $j_0' = j_1'k_1' = j_2'k_2'$ is the inclusion map.

Proof of Claim: Notice that $f = c_Y f'$, $j_s = c_Y j_s'$, and $f_s = f_s'$ $(s = 0, 1)$; f_2' is given as the composition

$$X_2' \xrightarrow{c} X_2 \xrightarrow{f_2} Y_2 \xrightarrow{\iota} Y_2'$$

where ι is the inclusion map and $j_2 = c_Y j_2' \iota$. Since the maps c_Y and c are G-CE maps and ι is the inverse of the G-CE map d, their torsion is zero. So the composition formula gives:

$$(c_Y)_*\tau(f') = \tau(c_Y f') = \tau(f);$$

$$(c_Y)_* j_{s*}'\tau(f_s') = (c_Y j_s')_*\tau(f_s) = (j_s)_*\tau(f_s) \quad \text{for } s = 0, 1;$$

$$(c_Y)_* j_{2*}'\tau(f_2') = (c_Y)_*(j_2')_*\iota_*\tau(f_2) = (j_2)_*\tau(f_2).$$

Applying $(c_Y)_*$ to both sides of (2), we get (1).

Thus we have reduced the proof of the proposition to the following case.

Special Case: Formula (1) is true if all the maps in the diagrams are assumed to be inclusion maps of closed subsets.

Proof of Special Case: We use the approach in [5, 23.1]. Let

$$D: M(f_0) \cup X \to X$$

be a strong G-deformation retraction. Then, by the composition formula,

$$\tau(M(f), X) = D_*\tau(M(f), M(f_0) \cup X) + \tau(M(f_0) \cup X, X).$$

Since $M(f) = (M(f_1) \cup X) \cup_{M(f_0) \cup X} (M(f_2) \cup X)$,

$$\tau(M(f), M(f_0) \cup X) = \tau(M(f_1) \cup X, M(f_0) \cup X) + \tau(M(f_2) \cup X, M(f_0) \cup X).$$

The composition formula implies

$$\tau(M(f_s)\cup X, X) = D_*\tau(M(f_s)\cup X, M(f_0)\cup X) + \tau(M(f_0)\cup X, X), \quad s=1,2.$$

Putting all these together, we get

$$\tau(M(f), X) = \tau(M(f_1)\cup X, X) + \tau(M(f_2)\cup X, X) - \tau(M(f_0)\cup X, X).$$

The proof follows by applying f_* to the above formula and the definitions (the details are in [5, 23.1]).

The proof of the Special Case completes the proof of the proposition. □

3. Product Formula

In this chapter we prove a product formula for the equivariant topological torsion. This formula has been proven in [9] for the non-equivariant case and in [13; 7; 11] for the cellular equivariant case. The proof of the formula in the equivariant topological case generalizes the methods in [11].

We use the theory of functorial additive invariants developed in [11, Chap. I, §6]. We recall some of the definitions and the basic facts about functorial additive invariants. Let $\mathcal{C}$ be a category with cofibrations and weak equivalences in the sense of Waldhausen [17; 11]. A functorial additive invariant (A, a) for $\mathcal{C}$ consists of a functor $A: \mathcal{C} \to$ (Abelian groups) and a function associating to an object X of $\mathcal{C}$ an element $a(X) \in A(X)$ such that the following hold.

(1) *Homotopy Invariance:* If $f: X \to Y$ is a weak equivalence then

$$A(f)(a(X)) = a(Y).$$

(2) *Additivity:* Let

$$\begin{array}{ccc} X_0 & \longrightarrow & X_2 \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j_2} \\ X_1 & \xrightarrow{j_1} & X \end{array}$$

be a push-out diagram with i a cofibration. Then

$$a(X) = A(j_1)(a(X_1)) + A(j_2)(a(X_2)) - A(j_0)(a(X_0)),$$

where $j_0 = j_1 i$.

(3) *Normalization:* $a(\emptyset) = 0$, where $\emptyset$ is the distinguished initial object in $\mathcal{C}$.

A functorial additive invariant (U, u) is called *universal* if, for any functorial additive invariant (A, a), there is exactly one natural transformation Φ: $U \to A$ such that $\Phi(X)(u(X)) = a(X)$ for all objects X.

Proposition 3.1. *There is a universal functorial additive invariant* (U, u) *unique up to natural equivalence.*

Proof. Chapter I, Theorem 6.1(a) in [11]. □

Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be categories with cofibrations and weak equivalences. Let F: $\mathcal{C}\times\mathcal{D} \to \mathcal{E}$ be a functor such that $F(X, -): \mathcal{D} \to \mathcal{E}$ and $F(-, Y): \mathcal{C} \to \mathcal{E}$ are functors between categories, with cofibrations and weak equivalences for all X in $\mathcal{C}$ and for all Y in $\mathcal{D}$. Let $(U_{\mathcal{C}}, u_{\mathcal{C}})$ and $(U_{\mathcal{D}}, u_{\mathcal{D}})$ be the universal functorial additive invariants for $\mathcal{C}$ and $\mathcal{D}$ and let (A, a) be an arbitrary functorial additive invariant for $\mathcal{E}$.

PROPOSITION 3.2. (i) *With the previous notation, there is exactly one natural pairing*

$$P(X,Y)\colon U_{\mathcal{C}}(X)\otimes U_{\mathcal{D}}(Y)\to V$$

such that $P(X,Y)(u_{\mathcal{C}}(X)\otimes u_{\mathcal{D}}(Y)) = V(X\times Y)$ *for all* X *in* $\mathcal{C}$ *and for all* Y *in* $\mathcal{D}$.

(ii) *Assume that* $\mathcal{C}$ *has an internal product such that* $(-,Y)\colon \mathcal{C}\to\mathcal{C}$ *is a functor between categories with cofibrations and weak equivalences. Then there is a natural pairing*

$$P(X,Y)\colon U_{\mathcal{C}}(X)\otimes U_{\mathcal{C}}(Y)\to U_{\mathcal{C}}(X\times Y)$$

uniquely determined by the property that

$$P(X,Y)(u_{\mathcal{C}}(X)\otimes u_{\mathcal{C}}(Y)) = u_{\mathcal{C}}(X\times Y).$$

Proof. Lemma 6.3 and Corollary 6.4 in [11, Chap. I]. □

Let G be a finite group and X an arbitrary G-space. In [11, Chap. I, §5], a category $\prod_0(G,X)$ is defined with objects G-maps $x\colon G/H\to X$, where H is a subgroup of G. A morphism σ from $x\colon G/H\to X$ to $y\colon G/K\to X$ is a G-map $\sigma\colon G/H\to G/K$ such that $y\sigma \simeq_G x$. Let $U^G(X)$ be the free abelian group generated by $\operatorname{Iso}\prod_0(G,X)$, the set of isomorphism classes of objects in $\prod_0(G,X)$. Then $U^G(-)$ becomes a functor from the category of G-spaces to the category of abelian groups.

REMARK. In [11, Chap. I, §5], $\prod_0(G,X)$ is defined for any group G.

For any subgroup H of G, set $W(H) = N(H)/H$. Then there is an action of $W(H)$ on X^H which induces an action of $W(H)$ on $\pi_0(X^H)$. Similarly, there is an action of G on $\coprod \pi_0(X^H)$, where the disjoint sum is taken over all the subgroups of G. Notice that there are bijections

$$\operatorname{Iso}\textstyle\prod_0(G,X)\to\coprod \pi_0(X^H)/W(H)\to\coprod\pi_0(X^H)/G, \tag{3.2.1}$$

where the second disjoint sum is taken over the conjugacy classes of subgroups of G.

Let X be a compact G-ANR. Since X is finitely G-dominated, its equivariant Euler characteristic is defined by $\chi_G(X)\in U^G(X)$ [11, 5.3]. We summarize the properties of the Euler characteristic of a compact G-ANR [11, Thm. 5.4] as follows.

3.3.

(a) If $f, g: X \to Y$ are G-homotopic, then $f_\# = g_\#: U^G(X) \to U^G(Y)$.

(b) If $f: X \to Y$ is a G-homotopy equivalence then $f_\#(\chi_G(X)) = \chi_G(Y)$.

(c) If

$$\begin{array}{ccc} X_0 & \xrightarrow{i_2} & X_2 \\ {\scriptstyle i_1}\downarrow & & \downarrow{\scriptstyle j_2} \\ X_1 & \xrightarrow{j_1} & X \end{array} \tag{$*$}$$

is a G-push-out diagram with i_1 a G-cofibration, then

$$\chi_G(X) = (j_1)_\#(\chi_G(X_1)) + (j_2)_\#(\chi_G(X_2)) - (j_0)_\#(\chi_G(X_0)),$$

where $j_0 = j_1 i_1 = j_2 i_2$.

Consider the category $\mathcal{C}$ with objects G-homotopy equivalences $f: X \to Y$ between compact G-ANRs and morphisms commutative diagrams

$$\begin{array}{ccc} X_0 & \xrightarrow{f_0} & Y_0 \\ {\scriptstyle g_0}\downarrow & & \downarrow{\scriptstyle g_1} \\ X_1 & \xrightarrow{f_1} & Y_1. \end{array} \tag{$*$}$$

A morphism is a *cofibration* if g_0, g_1 are inclusion maps of closed subspaces and a *weak equivalence* if g_0, g_1 are G-homotopy equivalences with topological torsion zero. Then $\mathcal{C}$ becomes a category with cofibrations and weak equivalences.

The proof of the following proposition is the same as the proof of Theorem 6.11 in [11, Chap. I].

PROPOSITION 3.4. *The universal functorial additive invariant for $\mathcal{C}$ is $(\mathrm{Wh}_G^{\mathrm{Top}} \oplus U^G, (\tau, \chi_G))$. That is, this functor sends the object $(X \xrightarrow{f} Y)$ to $(\tau(f), \chi_G(Y)) \in \mathrm{Wh}_G^{\mathrm{Top}}(Y) \oplus U^G(Y)$ and the morphism (g_0, g_1) in $(*)$ to*

$$\mathrm{Wh}_G^{\mathrm{Top}}(g_1) \oplus U^G(g_1): \mathrm{Wh}_G^{\mathrm{Top}}(Y_0) \oplus U^G(Y_0) \to \mathrm{Wh}_G^{\mathrm{Top}}(Y_1) \oplus U^G(Y_1).$$

Proof. It is an additive invariant because $\mathrm{Wh}_G^{\mathrm{Top}} \oplus U^G$ is a G-homotopy functor (using the composition formula for the torsion component and [11, Chap. I, Thm. 5.4] for the Euler characteristic component), and it is an additive functor because of Proposition 3.5 and [11, Chap. I, Thm. 5.4]. The proof that it is universal is the same as the proof of Theorem 6.11 in [11, Chap. I]. □

The next theorem is the product formula for the topological equivariant torsion; it follows from Propositions 3.2 and 3.4 [11, Chap. I, Thm. 7.1].

THEOREM 3.5. *Let G and H be finite groups.*

(a) *There is a natural pairing*

$$P(X, Y): (\mathrm{Wh}_G^{\mathrm{Top}}(X) \oplus U^G(X)) \oplus (\mathrm{Wh}_H^{\mathrm{Top}}(Y) \oplus U^H(Y)) \\ \to \mathrm{Wh}_{G\times H}^{\mathrm{Top}}(X \times Y) \oplus U^{G\times H}(X \times Y)$$

uniquely determined by the property that

$$P(X,Y)((\tau(f),\chi_G(X))\otimes(\tau(g),\chi_H(Y)))=((\tau(f\times g),\chi_{G\times H}(X\times Y))$$

for a G- (respectively H-) homotopy equivalence $f\colon X'\to X$ (resp. $g\colon Y'\to Y$) between compact G- (resp. H-) ANRs.

(b) (Product Formula) *Let* $\otimes$: $\mathrm{Wh}_G^{\mathrm{Top}}(X)\oplus U^H(Y)\to \mathrm{Wh}_{G\times H}^{\mathrm{Top}}(X\times Y)$ *be the pairing sending (u,v) to the component of $P(X,Y)((u,\chi_G(X)),(0,v))$ in* $\mathrm{Wh}_{G\times H}^{\mathrm{Top}}(X\times Y)$, *and define analogously*

$$\otimes\colon U^G(X)\oplus \mathrm{Wh}_H^{\mathrm{Top}}(Y)\to \mathrm{Wh}_{G\times H}^{\mathrm{Top}}(X\times Y).$$

Then

$$\tau(f\times g)=\chi_G(X)\otimes\tau(g)+\tau(f)\otimes\chi_H(Y)\in \mathrm{Wh}_{G\times H}^{\mathrm{Top}}(X\times Y).$$

Proof. Part (a) follows from Propositions 3.2 and 3.4. Part (b) follows from part (a), the homotopy invariance of the torsion, and the composition formula. □

REMARK 3.6. We derive some immediate consequences from the product formula just described.

(a) Let $f\colon X'\to X$ be a G-homotopy equivalence between compact G-ANRs and let Y be a compact H-ANR such that $\chi(Y_\alpha^K)=0$ for each component of the fixed point set of any subgroup K of H. Then $\tau(f\times \mathrm{id}\colon X'\times Y\to X\times Y)=0\in \mathrm{Wh}_{G\times H}^{\mathrm{Top}}(X\times Y)$. (For the PL-case, see [8, Thm. A]).

(b) Let $f\colon X'\to X$ be a G-homotopy equivalence between compact G-ANRs, and let V be a unitary representation of H. Then

$$\tau(f\times \mathrm{id}\colon X'\times S(V)\to X\times S(V))=0\in \mathrm{Wh}_{G\times H}^{\mathrm{Top}}(X\times S(V)).$$

Let G be a finite group and let H be a subgroup. If X is a compact G-ANR, then the H-space $\operatorname{res} X$ is a compact H-ANR.

LEMMA 3.7. *There is a functorial additive invariant on the foregoing category $\mathcal{C}$ sending an object $(X\xrightarrow{f}Y)$ to $\tau(\operatorname{res} f)\in \mathrm{Wh}_H^{\mathrm{Top}}(Y)$.*

Proof. Obvious from the definition. □

REMARK 3.7.1. Using Proposition 3.2(a) for any X a compact G-ANR, we get a natural pairing

$$\operatorname{res} X\colon \mathrm{Wh}_G^{\mathrm{Top}}(Y)\otimes U^G(Y)\to \mathrm{Wh}_H^{\mathrm{Top}}(Y)\otimes U^H(Y) \qquad (**)$$

sending $(\tau(f),\chi_G(Y))$ to $(\tau(\operatorname{res} f),\chi_H(Y))$ for any G-homotopy equivalence $f\colon X\to Y$ of compact G-ANRs.

Let G be a finite group. Set $H=G$, the diagonal subgroup of $G\times G$. Applying the previous theory we obtain the next theorem.

THEOREM 3.8. (a) *There is a natural pairing*

$$\begin{aligned}P(X,Y)\colon (\mathrm{Wh}_G^{\mathrm{Top}}(X)\oplus U^G(X))\otimes(\mathrm{Wh}_G^{\mathrm{Top}}(Y)\oplus U^G(Y))\\ \to \mathrm{Wh}_G^{\mathrm{Top}}(X\times Y)\oplus U^G(X\times Y),\end{aligned}$$

uniquely determined by the property that

$$P(X,Y)((\tau(f),\chi_G(X))\otimes(\tau(g),\chi_G(Y)))=((\tau(f\times g),\chi_G(X\times Y))$$

for G-homotopy equivalences $f: X'\to X$ *(resp.* $g: Y'\to Y$*) between compact G-ANRs.*

(b) (Diagonal Product Formula) *Let*

$$\otimes: \mathrm{Wh}_G^{\mathrm{Top}}(X)\oplus U^G(Y)\to \mathrm{Wh}_G^{\mathrm{Top}}(X\times Y)$$

be the pairing sending (u, v) *to the component of* $P(X,Y)((u,\chi_G(X)),(0,v))$ *in* $\mathrm{Wh}_G^{\mathrm{Top}}(X\times Y)$ *and analogously* $\otimes: U^G(X)\oplus \mathrm{Wh}_G^{\mathrm{Top}}(Y)\to \mathrm{Wh}_G^{\mathrm{Top}}(X\times Y)$. *Then*

$$\tau(f\times g)=\chi_G(X)\otimes\tau(g)+\tau(f)\otimes\chi_G(Y)\in \mathrm{Wh}_G^{\mathrm{Top}}(X\times Y).$$

REMARK 3.9. The analogs of Remark 3.6 are in this case as follows.

(a) Let $f: X'\to X$ be a G-homotopy equivalence between compact G-ANRs, and let Y be a compact G-ANR such that $\chi(Y_\alpha^K)=0$ for each component of the fixed point set of any subgroup K of G. Then

$$\tau(f\times \mathrm{id}: X'\times Y\to X\times Y)=0\in \mathrm{Wh}_G^{\mathrm{Top}}(X\times Y),$$

where $X\times Y$ supports the diagonal G-action.

(b) Let $f: X'\to X$ be a G-homotopy equivalence between compact G-ANRs, and let V be a unitary representation of G. Then

$$\tau(f\times \mathrm{id}: X'\times S(V)\to X\times S(V))=0\in \mathrm{Wh}_G^{\mathrm{Top}}(X\times S(V)),$$

where $S(V)$ is the sphere of V and G acts diagonally on $X\times S(V)$ [8, Cor. B].

4. The Transfer Map of a Locally Trivial G-Fibration

Let $p: E\to B$ be a locally trivial G-fibration [3; 10] between compact G-ANRs such that, for each $b\in B$ with isotropy group G_b, $p^{-1}(b)$ is a compact G_b-ANR. We define a homomorphism $p^*: \mathrm{Wh}_G^{\mathrm{Top}}(B)\to \mathrm{Wh}_G^{\mathrm{Top}}(E)$ as follows.

Let $f: B'\to B$ be a G-homotopy equivalence between compact G-ANRs representing an element $\tau\in \mathrm{Wh}_G^{\mathrm{Top}}(B)$. Form the pull-back diagram

$$\begin{array}{ccc} f^*E & \xrightarrow{\bar f} & E \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ B' & \xrightarrow{f} & B. \end{array}$$

By the Equivariant Hanner's Theorem, f^*E is a compact G-ANR and $\bar f$ is a G-homotopy equivalence. Define $p^*(\tau)=\tau(\bar f)\in \mathrm{Wh}_G^{\mathrm{Top}}(E)$.

(i) *p' is a well-defined map* [2, §2]: First notice that if f is a G-CE map then $f\times \mathrm{id}: B'\times Q_G\to B\times Q_G$ is G-homotopic to a G-homeomorphism. Since p satisfies the G-homotopy lifting property, $\bar f\times \mathrm{id}: f^*E\times Q_G\to E\times Q_G$ is

G-homotopic to a G-homeomorphism and, by [14], $\tau(\bar{f}) = 0 \in \mathrm{Wh}_G^{\mathrm{Top}}(E)$. Therefore $p^*(\tau(f)) = 0$ if f is a G-CE map.

Let $f_1: B_1 \to B$ and $f_2: B_2 \to B$ be strong G-deformation retractions which represent the same element in $\mathrm{Wh}_G^{\mathrm{Top}}(B)$. Then there is a compact G-ANR Z containing B, as well as G-CE maps $r: Z \to B_1$ and $s: Z \to B_2$ such that $r|_B = s|_B = \mathrm{id}_B$ and $f_1 r \simeq_G f_2 s \operatorname{rel} B$. Define a map $k = f_1 r: Z \to B$. Let $\bar{k} = \bar{f}_1 \bar{r}$: $k^*E \to E$ be the pull-back. Then, from the foregoing remark, $\tau(\bar{k}) = \tau(\bar{f}_1) = p^*(\tau(f_1))$. Also, $\bar{k} \simeq_G \bar{f}_2 \bar{s}$ and $\tau(\bar{k}) = \tau(\bar{f}_2) = p^*(\tau(f_2)) = p^*(\tau(f_1))$. Hence p^* is well-defined.

(ii) *p^* is a group homomorphism:* The proof is similar to that given in [2, §2].

5. Fiberwise Products

From this point on, by a G-fibration we shall mean a locally trivial G-fibration $p: E \to B$ such that E and B are compact G-ANRs and, for each $b \in B$ with isotropy subgroup G_b, $p^{-1}(b)$ is a compact G_b-ANR. Let

$$\begin{array}{ccc} E' & \xrightarrow{h} & E \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ B' & \xrightarrow{f} & B \end{array} \qquad (*)$$

be a fiber G-homotopy equivalence between two G-fibrations. We shall define a "fiberwise" product $\chi_G(B')\tau(h_b) \in \mathrm{Wh}_G^{\mathrm{Top}}(E)$ where $h_b: E'_b \to E_{f(b)}$ is the G_b-equivalence induced by h on the fibers. This product generalizes the product defined in [7, §5].

We first need some preliminary observations. Let $b \in B'^H$. Then the fiber over b is a compact H-ANR, $h_b: E'_b \to E_{f(b)}$ is the H-equivalence, and $\tau(h_b) \in \mathrm{Wh}_H^{\mathrm{Top}}(E_{f(b)})$. Write Ind_b^G for the composition

$$\mathrm{Wh}_H^{\mathrm{Top}}(E_{f(b)}) \to \mathrm{Wh}_G^{\mathrm{Top}}(G \times_H E_{f(b)}) \to \mathrm{Wh}_G^{\mathrm{Top}}(E_{Gf(b)}) \to \mathrm{Wh}_G^{\mathrm{Top}}(E),$$

where the first map is the induction map, the second map is induced by the G-map $G \times_H E_{f(b)} \to GE_{f(b)} = E_{Gf(b)}$, and the last map is induced by the inclusion.

LEMMA 5.1. (i) *Let b and c belong to the same path component of B'^H. Then* $\mathrm{Ind}_b^G(\tau(h_b)) = \mathrm{Ind}_c^G(\tau(h_c))$ *in* $\mathrm{Wh}_G^{\mathrm{Top}}(E)$.

(ii) *If $b \in B'^H$ then* $\mathrm{Ind}_b^G(\tau(h_b)) = \mathrm{Ind}_{gb}^G(\tau(h_{gb}))$ *in* $\mathrm{Wh}_G^{\mathrm{Top}}(E)$ *for all $g \in G$.*

Proof. (i) follows from the fact that there are H-homeomorphisms $\phi: E'_b \to E'_c$ and $\phi': E_{f(b)} \to E_{f(c)}$ such that $\phi' h_b = h_c \phi$.

(ii) Set $K = gHg^{-1}$. Then $gb \in B'^K$, and there are natural G-homeomorphisms $\phi: G \times_H E'_b \to G \times_K E'_{gb}$ and $\phi': G \times_H E_{f(b)} \to G \times_K E_{f(gb)}$ such that $\phi'(\mathrm{id}_G \times h_b) = (\mathrm{id}_G \times h_{gb})\phi$. □

By (3.2.1), $U^G(B')$ is the free abelian group generated by the set $A = (\coprod \pi_0(B'^H))/G$. Thus, any element $\chi \in U^G(B')$ can be represented by a function $\chi\colon A \to \mathbf{Z}$.

First we shall define the fiberwise product $\chi\tau(h_b)$ under the condition that every subgroup of G is an isotropy group for the G action on B'. Let $a \in A$ and $\alpha \in \pi_0(B'^H)$ represent a. Choose $b \in \alpha$ so that $G_b = H$. By Lemma 5.1(a), $\mathrm{Ind}_b^G(\tau(h_b))$ does not depend on the particular point b but only on the component α. By Lemma 5.1(b), $\mathrm{Ind}_b^G(\tau(h_b))$ depends only on a and not on the representative α. We write $\mathrm{Ind}_b^G = \mathrm{Ind}_a^G$. Define

$$\chi\tau(h_b) = \sum \chi(a)\,\mathrm{Ind}_a^G\,\tau(h_b),$$

where b belongs to a representative of a and the sum is taken over all $a \in A$.

In the general case, we cross the G-fiber homotopy equivalence (*) by the unit disc D of the regular representation $\mathbf{C}[G]$. Since D is G-contractible, we can identify $U^G(B')$ with $U^G(B' \times D)$, and crossing with D does not change the torsion data. Also, since D has every subgroup of G as isotropic group, we can define the fiberwise product as in the first case.

6. The Torsion of a Fiber Homotopy Equivalence

In this section we prove a formula that calculates the torsion of a G-fiber homotopy equivalence between locally trivial G-fibrations between compact G-ANRs.

First we observe that the proof given in [7, §6] generalizes to the topological case, provided that the base space is a finite G-CW complex.

PROPOSITION 6.1. *Let*

$$\begin{array}{ccc} E' & \xrightarrow{h} & E \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{\mathrm{id}} & B \end{array}$$

be a fiber G-homotopy equivalence, and let B be a finite G-CW complex. Then

$$\tau(h) = \chi_G(B)\,\tau(h_b) \in \mathrm{Wh}_G^{\mathrm{Top}}(E).$$

Proof. The proof is the same as the proof of Theorem 6.2 in [7]. We consider three cases.

Case 1: Let $B = G/H \times D$, where D is a contractible space with the trivial G-action. Consider the G-subspace G/H of B. Let $h^0\colon E_0' \to E_0$ be the restriction of h over G/H, and let h_x^0 be the restriction of h over $x = H/H$. In this case the natural maps $G \times_H E_x' \to E_{Gx}'$ and $G \times_H E_x \to E_{Gx}$ are G-homeomorphisms, and h^0 is given as the composition

$$E_{Gx}' \approx G \times_H E_x' \to G \times_H E_x \approx E_{Gx}.$$

By definition,

$$\tau(h^0) = \chi_G(G/H)\tau(h_x^0). \tag{1}$$

Notice that the inclusion maps $i'\colon E'_0 \to E'$ and $i\colon E_0 \to E$ have torsion zero in the corresponding Whitehead groups since they are pull-backs of $G/H \to B$, which has zero torsion because it is the inverse of a G-CE map. Then $hi' = ih^0$ and, by the composition formula, $\tau(h) = i_*\tau(h^0)$. Since $\chi_G(G/H)\tau(h_x^0) = \chi_G(B)\tau(h_x^0)$, it follows from the definition and (1) that $\tau(h) = \chi_G(B)\tau(h_x^0)$.

Case 2: Let $B = G/H \times S^n$, where G acts trivially on S^n. We will show the formula by induction on n. Write $S^n = D^n_+ \cup_{S^{n-1}} D^n_-$, and let the inclusion maps be $j_\pm\colon G/H \times D^n_\pm \to B$ and $j\colon G/H \times S^{n-1} \to B$. Also, let $h_\pm$ and h_0 be the restriction of h over $G/H \times D^n_\pm$ and $G/H \times S^{n-1}$, respectively. Then, by the sum formula,

$$\tau(h) = (j_+)_*\tau(h_+) + (j_-)_*\tau(h_-) - (j_0)_*\tau(h_0).$$

The result then follows from Case 1, the induction hypotheses, and the sum formula for the Euler characteristic [7, Thm. 6.2].

Case 3: Let B a finite G-CW complex. We use induction on the number of G-cells of B. The details are the same as in [7]. □

Let $\mathcal{C}$ be the category with objects fiber G-homotopy equivalences of the form

$$\begin{array}{ccc} E' & \xrightarrow{h} & E \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{\mathrm{id}} & B. \end{array}$$

We write $[h\colon E' \to E; B]$ for an object in $\mathcal{C}$. A morphism $\phi\colon [h\colon E' \to E; B] \to [h_0\colon E'_0 \to E_0;\ B_0]$ is a triple (k', k, s), where $k'\colon E' \to E'_0$, $k\colon E \to E_0$, and $s\colon B \to B_0$ are G-maps making the corresponding diagrams commutative. We call a morphism $\phi = (k', k, s)$

(co) a *cofibration* if k', k, and s are inclusions;

(we) a *weak equivalence* if:

(we1) the maps k' and k are fiber G-homotopy equivalences and s is a G-homotopy equivalence; and

(we2) k', k, s, and the equivariant homotopy equivalences induced on the fibers by k' and k have zero torsion on the corresponding Whitehead groups.

Then $\mathcal{C}$ becomes a category with cofibrations and weak equivalences.

Define a functor A from $\mathcal{C}$ to the category of abelian groups by

$$A[h\colon E' \to E; B] = \mathrm{Wh}_G^{\mathrm{Top}}(E) \oplus U^G(E)$$

on objects and $A(\phi) = (k_*, k_\#)$, where $\phi = (k', k, s)$ as before and $k_*, k_\#$ are the maps induced by k. Also define, for each object $[h\colon E' \to E; B]$, elements

$$a_1([h\colon E' \to E;\ B]) = (\tau(h), \chi_G(E))$$
$$a_2([h\colon E' \to E;\ B]) = (\chi_G(B)\tau(h_b), \chi_G(E))$$

of $A[h\colon E' \to E]$.

PROPOSITION 6.2. *The pairs (A, a_1) and (A, a_2) are universal functorial additive invariants on the category $\mathcal{C}$.*

Proof. From the properties of the Whitehead torsion, (A, a_2) is an additive functorial invariant. The additive property of (A, a_1) follows from the additivity of the Euler characteristic and the naturality properties of the fiberwise product. The homotopy invariance follows from the composition formula for the Whitehead torsion.

Now we prove that (A, a_1) and (A, a_2) are universal. We follow the proof of Theorem 6.11 in [11]. Let (D, d) be a functorial additive invariant. Let $[h\colon E' \to E;\ B]$ be an object of $\mathcal{C}$, and let $r\colon X \to E$ be a strong G-deformation retraction representing an element $\tau \in \mathrm{Wh}_G^{\mathrm{Top}}(E)$. This defines an object $[r\colon X \to E;\ *]$ in $\mathcal{C}$. Notice also that the fiberwise product defined in $[r\colon X \to E;\ *]$ is simply τ. Since the map $h_\#\colon U^G(E') \to U^G(E)$ is a bijection, any element of $U^G(E')$ can be written as $h_\#(\eta)$ for some $\eta \in U^G(E')$. Let $x\colon G/H \to E'$ represent an element of $U^G(E')$. Then x induces a morphism

$$\hat{x} = (x, hx, p'x)\colon [\mathrm{id}\colon G/H \to G/H; G/H] \to [h\colon E' \to E;\ B]$$

where in $[\mathrm{id}\colon G/H \to G/H;\ G/H]$ all the maps are the identity. Define a natural transformation $F\colon \mathrm{Wh}_G^{\mathrm{Top}} \oplus U^G \to D$ by the rule

$$\begin{aligned} &F([h\colon E' \to E;\ B])(\tau, h_\#(\eta)) \\ &\quad = \textstyle\sum(\eta(x) - \chi_G(x)) D(\hat{x})(d([\mathrm{id}\colon G/H \to G/H; G/H])) \\ &\quad\quad + d([r\colon X \to E;\ *]) \end{aligned}$$

[11, Thm. 6.11]. F is a natural transformation for both (A, a_1) and (A, a_2) because $a_1([r\colon X \to E;\ *]) = a_2([r\colon X \to E;\ *]) = \tau$. Hence (A, a_1) and (A, a_2) are universal functorial additive invariants. □

COROLLARY 6.3. (i) *For each object $[h\colon E' \to E; B]$ in $\mathcal{C}$, there is a natural isomorphism*

$$\phi([h\colon E' \to E;\ B])\colon \mathrm{Wh}_G^{\mathrm{Top}}(E) \oplus U^G(E) \to \mathrm{Wh}_G^{\mathrm{Top}}(E) \oplus U^G(E)$$

such that $\phi([h\colon E' \to E; B])(a_1([h\colon E' \to E; B])) = a_2([h\colon E' \to E; B])$.

(ii) *If B is a finite G-CW complex then $\phi([h\colon E' \to E;\ B])$ is the identity.*

Proof. Part (i) follows from the universal properties of (A, a_1) and (A, a_2). Part (ii) follows from Proposition 6.1. □

If B is a compact G-ANR, then B is finitely G-dominated; that is, there is a finite G-CW complex K and G-maps $B \xrightarrow{d} K \xrightarrow{r} B$ such that $rd \simeq_G \mathrm{id}_B$. Using this observation, we can generalize Proposition 6.1 to any compact G-ANR.

THEOREM 6.4. *Let*

$$\begin{array}{ccc} E' & \xrightarrow{h} & E \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{\text{id}} & B \end{array}$$

be an object of $\mathcal{C}$. *Then* $\tau(h) = \chi_G(B)\tau(h_b) \in \mathrm{Wh}_G^{\mathrm{Top}}(E)$.

Proof. The object $[h: E' \to E;\ B]$ and the finite domination of B as above determine an object $[r^*h: r^*E' \to r^*E; K]$ of $\mathcal{C}$ by pull-back. Also, since $rd \simeq_G \mathrm{id}_B$, there is a morphism in $\mathcal{C}$,

$$\phi: [h: E' \to E; B] \to [r^*h: r^*E' \to r^*E; K].$$

Then one component of ϕ is a G-map $D: E \to r^*E$ such that $\bar{r}D \simeq_G \mathrm{id}_E$. Thus the map $(D_*, D_\#): \mathrm{Wh}_G^{\mathrm{Top}}(E) \oplus U^G(E) \to \mathrm{Wh}_G^{\mathrm{Top}}(r^*E) \oplus U^G(r^*E)$ is a monomorphism. By Corollary 6.3(i), there is a natural isomorphism

$$\phi([h: E' \to E; B]): \mathrm{Wh}_G^{\mathrm{Top}}(E) \oplus U^G(E) \to \mathrm{Wh}_G^{\mathrm{Top}}(E) \oplus U^G(E)$$

such that $\phi([h: E' \to E;\ B])(\chi_G(B)\tau(h_b), \chi_G(E)) = (\tau(h), \chi_G(E))$. By naturality, $\phi([r^*h: r^*E' \to r^*E; K])(D_*, D_\#) = (D_*, D_\#)\phi([h: E' \to E; B])$. But by Corollary 6.3(ii), $\phi([r^*h: r^*E' \to r^*E; K])$ is the identity, and since $(D_*, D_\#)$ is a monomorphism it follows that $\phi[(h: E' \to E;\ B])$ is the identity. Therefore $\chi_G(B)\tau(h_b) = \tau(h)$. □

Now we can prove the general theorem which computes the torsion of a fiber G-homotopy equivalence.

THEOREM 6.5. *Let*

$$\begin{array}{ccc} E' & \xrightarrow{h} & E \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ B' & \xrightarrow{f} & B \end{array}$$

be a fiber G-homotopy equivalence. Then

$$\tau(h) = \chi_G(B')\tau(h_b) + p^*(\tau(f)).$$

Proof. By the universal property of the pull-back h, we have the diagram

$$\begin{array}{ccccc} E' & \xrightarrow{h'} & f^*E & \xrightarrow{\bar{f}} & E \\ {\scriptstyle p'}\downarrow & & {\scriptstyle p_0}\downarrow & & \downarrow{\scriptstyle p} \\ B' & \xrightarrow{\text{id}} & B' & \xrightarrow{f} & B. \end{array}$$

Then $\tau(h) = \tau(\bar{f}) + \bar{f}_*\tau(h') = p^*(\tau(f)) + \bar{f}_*\tau(h')$. Using Theorem 6.4, we obtain

$$\begin{aligned} \bar{f}_*\tau(h') &= \bar{f}_*(\chi_G(B')\tau(h'_b)) = \chi_G(B')((\bar{f}_b)_*\tau(h'_b)) \\ &= \chi_G(B')\tau(\bar{f}_b h'_b) = \chi_G(B')\tau(h_b) \end{aligned}$$

by the naturality properties of the fiberwise product [7, Thm. 6.1]. □

REMARK. The product formula (Theorems 3.5 and 3.8) can be derived as a special case of Theorem 6.5.

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