On the Fitting Ideals in Free Resolutions

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Introduction

Throughout this paper, all rings are commutative with identity. If R is a ring and if $\phi: F \to G$ is a map of finitely generated free R-modules, then we define $I_i(\phi)$ $(i \geq 0)$ to be the ideal of R generated by the $i \times i$ minors of a matrix representing ϕ and the rank of ϕ, denoted by rank ϕ, to be the largest number t such that $I_t(\phi) \neq 0$. The ideals $I_i(\phi)$ are called the *Fitting ideals* of ϕ.

Let $(R, \mathfrak{m}, K)$ be a d-dimensional complete Noetherian local ring containing a field with maximal ideal $\mathfrak{m}$ and residue class field $K = R/\mathfrak{m}$. The purpose of this paper is to study a conjecture of C. Huneke concerning the behavior of Fitting ideals in free resolutions of finitely generated modules over R. In the meantime, a question about the annihilator ideal of the functor $\text{Ext}_R^{d+1}(-,-)$ is also considered. In order to present these questions, more definitions are needed.

Let R be as above. Then, by Cohen structure theorem,

$$R \cong K[|X_1, \dots, X_n|]/(f_1, \dots, f_t)$$

for some indeterminates $X_1, \dots, X_n$ and some power series

$$f_1, \dots, f_t \in K[|X_1, \dots, X_n|].$$

Therefore, from this representation, we may define the Jacobian ideal of R to be $I_h(\partial(f_1, \dots, f_t)/\partial(X_1, \dots, X_n))R$, that is, the ideal of R generated by the image of $h \times h$ minors of the Jacobian matrix $(\partial(f_1, \dots, f_t)/\partial(X_1, \dots, X_n))$, where $h = \text{height}(f_1, \dots, f_t)$. Furthermore, we denote by $I_s(R)$ the ideal defining the singular locus of R; that is, $I_s(R) = \bigcap_{P \notin \text{Reg}\, R} P$. If M is a finitely generated R-module then M is said to have a *well-defined rank r* if, for any $P \in \text{Ass}(R)$, M_P is free and $\mu_P(M) = r$. Finally, we denote by $(\mathbf{F}., \phi.)$ the following acyclic complex of finitely generated free R-modules:

$$\cdots F_d \xrightarrow{\phi_d} F_{d-1} \xrightarrow{\phi_{d-1}} \cdots \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0.$$

Let us state the questions as follows.

Conjecture 1. Let $(R, \mathfrak{m}, K)$ be a d-dimensional complete Noetherian local ring containing a field and let J be the Jacobian ideal of R. Let M be a

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finitely generated R-module and let $(\mathbf{F}_., \phi_.)$ be any finitely generated free resolution of M. Assume that M has a well-defined rank. Then

$$\begin{aligned} J &\subseteq I_1(\phi_j) \\ JI_1(\phi_j) &\subseteq I_2(\phi_j) \\ &\vdots \\ JI_{t_j-1}(\phi_j) &\subseteq I_{t_j}(\phi_j) \end{aligned}$$

for all $j \geq d+1$, where $t_j = \operatorname{rank}(\phi_j)$. In particular, $J^k \subseteq I_k(\phi_j)$ for all $k \leq t_j$.

QUESTION 2. Let $(R, \mathfrak{m}, K)$ be a d-dimensional complete Noetherian local ring containing a field, with J the Jacobian ideal of R. Then does it hold that $J \operatorname{Ext}_R^{d+1}(-,-) = 0$?

We would like to introduce several results related to the above questions, as they are helpful to our work. The following theorem [2, Thm. 1], due to Eisenbud and Green, was concerned with Fitting ideals and was initially conjectured by C. Huneke.

THEOREM 1. *Let R be a Noetherian ring containing* $\mathbf{Q}$ *and let M be a finitely generated R-module. Let* $I = \operatorname{ann}_R M$ *and let* $(\mathbf{F}_., \phi_.)$ *be a finitely generated free resolution of M. Assume that I contains a non-zero-divisor. Then*

$$II_i(\phi_j) \subset I_{i+1}(\phi_j) \quad \forall i = 0, \ldots, t_j - 1 \text{ and } \forall j \geq 1,$$

where $t_j = \operatorname{rank} \phi_j$.

On the other hand, according to Popescu and Roczen [5, Lemma 2.2], Question 2 has a weaker solution.

THEOREM 2. *Let* $(R, \mathfrak{m}, K)$ *be a d-dimensional reduced complete Cohen–Macaulay (CM) local ring containing a field and let* $I_s(R)$ *be the ideal defining the singular locus of R. Assume that K is perfect. Then there is a positive integer k such that* $I_s(R)^k \operatorname{Ext}_R^1(M, N) = 0$ *for any finitely generated R-modules M and N, with M a maximal CM module.*

Here, a finitely generated module M over a CM ring R is called a *maximal* CM module (MCM) if $\operatorname{depth} M = \dim R$.

We should remark that the proof of Theorem 2 given in [5] was not quite correct. However, the theorem remains valid and we shall give a complete proof in section 5.

The main results of this paper are as follows.

1. If R is a CM ring and J is the Jacobian ideal of R, then

$$J \operatorname{Ext}_R^{d+1}(M, N) = 0$$

for every pair of finitely generated R-modules M and N. Moreover, if M is a finitely generated R-module having a well-defined rank and $(\mathbf{F}_., \phi_.)$ is any finitely generated free resolution of M, then

$$(t_j - i) J I_i(\phi_j) \subseteq I_{i+1}(\phi_j) \quad \forall i = 0, \ldots, t_j - 1 \text{ and } \forall j \geq d+1,$$

where $t_j = \operatorname{rank} \phi_j$.

2. If R is equidimensional and either char $K = 0$ or K is perfect, then there exists an integer k such that:

(a) $J^k \operatorname{Ext}_R^{d+1}(M, N) = 0$ for every pair of finitely generated R-modules M and N; and

(b) if M is a finitely generated R-module having a well-defined rank and $(\mathbf{F}., \phi.)$ is any finitely generated free resolution of M, then

$$(t_j - i) J^k I_i(\phi_j) \subseteq I_{i+1}(\phi_j) \quad \forall i = 0, \ldots, t_j - 1 \text{ and } \forall j \geq d+1,$$

where $t_j = \operatorname{rank} \phi_j$.

1. Characterization of $\operatorname{ann}_R \operatorname{Ext}_R^1(M, -)$ and $\operatorname{ann}_R \operatorname{Tor}_1^R(M, -)$

Let R be a commutative ring. Then it is well known that projective modules are flat and that finitely presented flat modules are projective. In other words, for an R-module M, we have:

(1) if $\operatorname{Ext}_R^1(M, -) = 0$ then $\operatorname{Tor}_1^R(M, -) = 0$;

(2) if $\operatorname{Tor}_1^R(M, -) = 0$ and M is finitely presented, then $\operatorname{Ext}_R^1(M, -) = 0$.

In what follows, we shall generalize statements (1) and (2).

We begin this section by proving the following two lemmas.

Lemma 1.1. *Let R be a commutative ring and M an R-module with free presentation $F \xrightarrow{\phi} G \longrightarrow M \longrightarrow 0$. Let $x \in R$ be such that $x \operatorname{Ext}_R^1(M, -) = 0$. Then there is a R-homomorphism $\psi: G \to F$ such that $\phi \circ \psi \circ \phi = x\phi$.*

Proof. Let λ and i be the canonical homomorphisms in the following diagram:

$$\begin{array}{ccccccc} F & \xrightarrow{\phi} & G & \longrightarrow & M & \longrightarrow & 0 \\ & \lambda\searrow & & \nearrow i & & & \\ & & \operatorname{Im}\phi. & & & & \end{array}$$

Consider the short exact sequence

$$0 \longrightarrow \operatorname{Im}\phi \xrightarrow{i} G \longrightarrow M \longrightarrow 0.$$

Applying $* := \operatorname{Hom}_R(-, \operatorname{Im}\phi)$, we obtain the exact sequence

$$\operatorname{Hom}_R(G, \operatorname{Im}\phi) \xrightarrow{i^*} \operatorname{Hom}_R(\operatorname{Im}\phi, \operatorname{Im}\phi) \xrightarrow{\pi} \operatorname{Ext}_R^1(M, \operatorname{Im}\phi) \longrightarrow 0.$$

If $x \operatorname{Ext}_R^1(M, -) = 0$ then $\pi(x 1_{\operatorname{Im}\phi}) = 0$, and so there is a $j \in \operatorname{Hom}_R(G, \operatorname{Im}\phi)$ such that $j \circ i = x 1_{\operatorname{Im}\phi}$. Further, since G is free, there is a $\psi \in \operatorname{Hom}_R(G, F)$ such that $\lambda \circ \psi = j$. Consequently,

$$\phi \circ \psi \circ \phi = i \circ \lambda \circ \psi \circ i \circ \lambda = i \circ j \circ i \circ \lambda = x(i \circ \lambda) = x\phi. \qquad \square$$

LEMMA 1.2. *Let R be a commutative ring and M a finitely presented R-module with a finite free presentation $F \xrightarrow{\phi} G \xrightarrow{\pi} M$. Let $M' = \operatorname{coker} \phi^*$, where $* := \operatorname{Hom}_R(-, R)$. Then $\operatorname{Tor}_1^R(M, M') \cong \operatorname{Hom}_R(M, M)/\{f \in \operatorname{Hom}_R(M, M) \mid f$ factors through a finite free R-module$\}$.*

REMARK 1.3. Let K denote the set $\{f \in \operatorname{Hom}_R(M, M) \mid f$ factors through a finite free R-module$\}$ and K' the set $\{f \in K \mid f$ factors through $R\}$. Then it is easy to check that K is a submodule of $\operatorname{Hom}_R(M, M)$ and that K can be generated by K' as an R-module.

REMARK 1.4. If M_1, M_2 are R-modules, then for R-modules

$$\operatorname{Hom}_R(M_1, R) \otimes_R M_2 \quad \text{and} \quad \operatorname{Hom}_R(M_1, M_2)$$

there is a natural homomorphism $\theta\colon \operatorname{Hom}_R(M_1, R) \otimes_R M_2 \to \operatorname{Hom}_R(M_1, M_2)$ such that for $g \in \operatorname{Hom}_R(M_1, R)$, $x \in M_2$, and $y \in M_1$ we have $\theta(g \otimes x)(y) = g(y)x$. Moreover, if M_1 is a finite free R-module then θ is an isomorphism.

The proof given below is similar to the one given in [7, Lemmas 3.8 & 3.9].

Proof of Lemma 1.2. Consider the following exact sequence induced by the presentation of M:

$$0 \longrightarrow \operatorname{Hom}_R(M, R) \xrightarrow{\pi^*} \operatorname{Hom}_R(G, R) \xrightarrow{\phi^*} \operatorname{Hom}_R(F, R) \longrightarrow M' \longrightarrow 0.$$

Then, applying $\otimes_R M$, we obtain a complex

$$\operatorname{Hom}_R(M, R) \otimes_R M \xrightarrow{\pi^* \otimes 1_M} \operatorname{Hom}_R(G, R) \otimes_R M$$
$$\xrightarrow{\phi^* \otimes 1_M} \operatorname{Hom}_R(F, R) \otimes_R M \longrightarrow M' \otimes_R M \longrightarrow 0.$$

Hence, by the definition of Tor_1^R, $\operatorname{Tor}_1^R(M, M') \cong \operatorname{Ker}(\phi^* \otimes 1_M)/\operatorname{Im}(\pi^* \otimes 1_M)$. Furthermore, by Remark 1.4 there are R-homomorphisms θ_i, $i = 1, 2, 3$, which make the following diagrams commute:

$$\begin{array}{ccccccc}
 & \operatorname{Hom}_R(M, R) \otimes_R M & \xrightarrow{\pi^* \otimes 1_M} & \operatorname{Hom}_R(G, R) \otimes_R M & \xrightarrow{\phi^* \otimes 1_M} & \operatorname{Hom}_R(F, R) \otimes_R M \\
 & \downarrow \theta_1 & & \downarrow \theta_2 & & \downarrow \theta_3 \\
0 \longrightarrow & \operatorname{Hom}_R(M, M) & \xrightarrow{\operatorname{Hom}_R(\pi, M)} & \operatorname{Hom}_R(G, M) & \xrightarrow{\operatorname{Hom}_R(\phi, M)} & \operatorname{Hom}_R(F, M)
\end{array}$$

As θ_2 and θ_3 are isomorphisms (since F and G are finite free R-modules) and the bottom row of the previous diagram is exact, it follows that

$$\begin{aligned}
\operatorname{Ker}(\phi^* \otimes 1_M)/\operatorname{Im}(\pi^* \otimes 1_M) &\cong \operatorname{Ker}(\operatorname{Hom}_R(\phi, M))/\operatorname{Im}(\theta_2 \circ (\pi^* \otimes 1_M)) \\
&\cong \operatorname{Ker}(\operatorname{Hom}_R(\phi, M))/\operatorname{Im}(\operatorname{Hom}_R(\pi, M) \circ \theta_1) \\
&\cong \operatorname{Hom}_R(M, M)/\operatorname{Im} \theta_1.
\end{aligned}$$

However, $\operatorname{Im} \theta_1$ is the submodule generated by the elements of K'. Therefore, by Remark 1.3, $\operatorname{Im} \theta_1 = K$ and the assertion follows. □

Using Lemma 1.1 and Lemma 1.2, we are able to show the generalization of the facts stated in the beginning of this section.

PROPOSITION 1.5. *Let R be a commutative ring, M an R-module, and $x \in R$. Then the following statements hold:*

(1) *if* $x\,\mathrm{Ext}^1_R(M,-) = 0$ *then* $x\,\mathrm{Tor}^R_1(M,-) = 0$;
(2) *if* $x\,\mathrm{Tor}^R_1(M,-) = 0$ *and M is finitely presented, then* $x\,\mathrm{Ext}^1_R(M,-) = 0$.

Proof. Let $(\mathbf{F}.,\phi.)$ be a free resolution of M with $\pi: F_0 \to M$ the augmentation map.

If $x\,\mathrm{Ext}^1_R(M,-) = 0$ then, by Lemma 1.1, there is a R-homomorphism ψ_1: $F_0 \to F_1$ such that $\phi_1 \circ \psi_1 \circ \phi_1 = x\phi_1$. Hence inductively, by using the projectivity of F_{i-1}, we can construct R-homomorphisms $\psi_i: F_{i-1} \to F_i$, $i = 1, 2, \ldots$, such that $x1_{F_i} = \psi_i \circ \phi_i + \phi_{i+1} \circ \psi_{i+1}$ for all $i \geq 1$. Thus $x\,\mathrm{Tor}^R_i(M,-) = 0$ for all $i \geq 1$.

Conversely, if M is finitely presented and $x\,\mathrm{Tor}^R_1(M,-) = 0$, then by Lemma 1.2 $x1_M$ can be factored through a finite free R-module. More precisely, there are free R-modules R^n and R-homomorphisms $\alpha: M \to R^n$ and β: $R^n \to M$ such that $\beta \circ \alpha = x1_M$. Moreover, since R^n is free and $F_0 \xrightarrow{\pi} M$ is onto, there is a R-homomorphism $\lambda: R^n \to F_0$ such that $\pi \circ \lambda = \beta$. Let $\eta = \lambda \circ \alpha$; then it is easy to see that $\pi \circ \eta = x1_M$ and $\pi \circ \eta \circ \pi = x\pi$, so that by the projectivity of F_{i-1} we may construct $\psi_i: F_{i-1} \to F_i$, $i = 1, 2, \ldots$, such that $x1_{F_0} = \eta \circ \pi + \phi_1 \circ \psi_1$ and $x1_{F_i} = \psi_i \circ \phi_i + \phi_{i+1} \circ \psi_{i+1}$ for all $i \geq 1$. Thus $x\,\mathrm{Ext}^i_R(M,-) = 0$ for all $i \geq 1$. □

In fact, the proof shows more.

COROLLARY 1.6. *Let R be a commutative Noetherian ring and M a finitely generated R-module. Then the following hold:*

(1) $\mathrm{ann}_R\,\mathrm{Ext}^i_R(M,-) \subseteq \mathrm{ann}_R\,\mathrm{Ext}^j_R(M,-)$ for all $j \geq i \geq 1$;
(2) *if* M_1 *is a first syzygy of M, then* $\mathrm{ann}_R\,\mathrm{Ext}^1_R(M,-) = \mathrm{ann}_R\,\mathrm{Ext}^1_R(M,M_1)$.
(3) *if M is finitely generated, then* $\mathrm{ann}_R\,\mathrm{Ext}^1_R(M,-) = \mathrm{ann}_R\,\mathrm{Tor}^R_1(M,-)$.

The next corollary is an immediate consequence of the previous proposition.

COROLLARY 1.7. *Let R be a Noetherian ring and M a finitely generated R-module. Let $x \in R$ be such that M_x is projective. Then, for some integer n,* $x^n\,\mathrm{Ext}^1_R(M,-) = 0$ *and the map* $M \xrightarrow{x^n} M$ *factors through a finitely generated free R-module.*

Proof. Let M_1 be a first syzygy R-module of M; then

$$(\mathrm{Ext}^1_R(M,M_1))_x \cong \mathrm{Ext}^1_{R_x}(M_x,(M_1)_x) = 0.$$

Hence there is an integer n such that $x^n\,\mathrm{Ext}^1_R(M,M_1) = 0$. The rest follows from Lemma 1.2 and Corollary 1.6. □

2. $\mathrm{ann}_R\,\mathrm{Ext}^1_R(M,-)$ and the Fitting Ideal of M

In this section, we will extend Theorem 1 by showing that if M has a well-defined rank then the conclusion remains valid for $I = \mathrm{ann}_R\,\mathrm{Ext}^1_R(M,-)$, and we will see at once that Question 2 is essentially stronger than Conjecture 1. However, before doing so, let us give more applications of Corollaries 1.6 and 1.7.

PROPOSITION 2.1. *Let $(R, \mathfrak{m})$ be a d-dimensional complete Noetherian local ring, $I_s(R)$ the ideal defining the singular locus of R, and M a finitely generated R-module. Then there exists an integer k such that*

$$I_s(R)^k \operatorname{Ext}_R^{d+1}(M, -) = 0.$$

Proof. We may assume that $I_s(R) \subseteq \mathfrak{m}$ and that d is positive; otherwise, the result is obvious. Let M_i denote an ith syzygy module of M. Then, for all $x \in I_s(R)$, $(M_{d-1})_x$ is projective since R_x is a regular ring of dimension $\leq d-1$. Hence, by Corollary 1.7, a certain power of x kills $\operatorname{Ext}_R^d(M, -) = \operatorname{Ext}_R^1(M_{d-1}, -)$; there is therefore an integer k such that $I_s(R)^k \operatorname{Ext}_R^d(M, -) = 0$. Thus, by (1) of Corollary 1.6, we have $I_s(R)^k \operatorname{Ext}_R^{d+1}(M, -) = 0$. □

REMARK 2.2. The proof in fact shows that if R is not regular then, for any finitely generated R-module M, there exists an integer k such that $I_s(R)^k \operatorname{Ext}_R^d(M, -) = 0$.

PROPOSITION 2.3. *Let R be a complete local CM ring, $I_s(R)$ the ideal defining the singular locus of R, and M a finitely generated maximal CM module. Then there exists an integer k such that $I_s(R)^k \operatorname{Ext}_R^1(M, -) = 0$.*

Proof. Let $x \in I_s(R)$; then R_P is regular for any $P \in \operatorname{Spec}(R_x)$, so that the MCM R_P-module M_P is free and hence M_x is projective. Therefore, by Corollary 1.7, a certain power of x kills $\operatorname{Ext}_R^1(M, -)$. Thus, since $I_s(R)$ is finitely generated, $I_s(R)^k \operatorname{Ext}_R^1(M, -) = 0$ for some k. □

The preceding proposition is indeed the same as [5, Lemma 2.1]. However, the original proof given in [5] did not touch the real point.

We now turn to our goal of this section.

PROPOSITION 2.4. *Let R be a Noetherian ring, M a finitely generated R-module, and $x \in R$. Suppose that M has a well-defined rank and that $x \operatorname{Ext}_R^1(M, -) = 0$. Then, for any finitely generated free resolution $(\mathbf{F}., \phi.)$ of M, we have $(t_j - i)xI_i(\phi_j) \subseteq I_{i+1}(\phi_j)$ for all $i = 0, \ldots, t_j - 1$ and for all $j \geq 1$, where $t_j = \operatorname{rank} \phi_j$.*

For the proof we need the following lemma.

LEMMA 2.5. *Let R be a commutative ring and $x \in R$. Let $\phi_1 \in \operatorname{Hom}_R(R^m, R^k)$, $\phi_2 \in \operatorname{Hom}_R(R^n, R^m)$, $\psi_1 \in \operatorname{Hom}_R(R^k, R^m)$, and $\psi_2 \in \operatorname{Hom}_R(R^m, R^n)$ satisfy the following conditions:*

(1) $\phi_1 \circ \phi_2 = 0$;
(2) $\psi_1 \circ \phi_1 + \phi_2 \circ \psi_2 = x 1_{R^m}$; *and*
(3) $\operatorname{trace}(\phi_2 \circ \psi_2) = tx$ *for some integer t.*

Then $(t-i)xI_i(\phi_2) \subseteq I_{i+1}(\phi_2)$ for all $i = 0, \ldots, t-1$.

Proof. See [2, Theorem 1.1]. □

Proof of Proposition 2.4. Let $(\mathbf{F}., \phi.)$ be a finitely generated free resolution of M and let $F_0 \xrightarrow{\pi} M$ be the augmentation map. Let $x \in R$ be such that $x \operatorname{Ext}_R^1(M, -) = 0$; then, by (3) of Corollary 1.6, $x \operatorname{Tor}_1^R(M, -) = 0$, so that by the proof of Proposition 1.5 there exist $\psi_j: F_{j-1} \to F_j$ and $\eta: M \to F_0$ such that $\pi \circ \eta = x1_M$, $x1_{F_0} = \eta \circ \pi + \phi_1 \circ \psi_1$, and $x1_{F_j} = \psi_j \circ \phi_j + \phi_{j+1} \circ \psi_{j+1}$ for all $j \geq 1$. Therefore, by applying the above lemma, it remains to show that $\operatorname{trace}(\phi_j \circ \psi_j) = t_j x$ for all $j \geq 1$. However, M has a well-defined rank, and we see that, for all $j \geq 1$, $t_j + t_{j+1} = \operatorname{rank}(F_j)$ and

$$\operatorname{rank}(F_j)x = \operatorname{trace}(\phi_j \circ \psi_j) + \operatorname{trace}(\phi_{j+1} \circ \psi_{j+1});$$

thus it is sufficient to show

$$\operatorname{trace}(\phi_1 \circ \psi_1) = t_1 x.$$

To see this, let $W = R \backslash \bigcup_{P \in \operatorname{Ass}(R)} P$; then M_W is free and $\operatorname{trace}((\eta \circ \pi)_W) = \operatorname{rank}(M_W)x$ in R_W, so that $\operatorname{trace}((\phi_1 \circ \psi_1)_W) = \operatorname{rank}(F_0)x - \operatorname{trace}((\eta \circ \pi)_W) = \operatorname{rank}(F_0)x - \operatorname{rank}(M_W)x = (t_1 x)_W$ in R_W. Because W consists of non-zero-divisors, we conclude that $\operatorname{trace}(\phi_1 \circ \psi_1) = t_1 x$. □

COROLLARY 2.6. *Let R be a d-dimensional Noetherian local ring, $I_s(R)$ the ideal defining the singular locus of R, and M a finitely generated R-module. Suppose that M has a well-defined rank. Then there is an integer k such that, for all $x \in I_s(R)^k$ and for any finitely generated free resolution $(\mathbf{F}., \phi.)$ of M, $(t_j - i)xI_i(\phi_j) \subseteq I_{i+1}(\phi_j)$ for all $i = 0, \ldots, t_j - 1$ and for all $j \geq d+1$.*

Proof. From Proposition 2.1, we know that there exists an integer k such that $I_s(R)^k \operatorname{Ext}_R^{d+1}(M, -) = 0$. Therefore, by applying Proposition 2.4 to the d-syzygy R-module of M, we obtain the result. □

To end this section, we give the following example.

EXAMPLE 2.7. *Let K be a field, $A = K[|X_1, \ldots, X_n|]$, and $R = A/(f)$. Then $J \operatorname{Ext}_R^1(M, -) = 0$ for any MCM module M, where*

$$J = \left(\frac{\partial f}{\partial X_1}, \ldots, \frac{\partial f}{\partial X_n}\right)R.$$

Proof. Let M be a MCM module; then from [1] it is known that M has a free resolution of the form

$$\xrightarrow{C} R^t \xrightarrow{B} R^t \xrightarrow{C} R^t \longrightarrow M \longrightarrow 0$$

such that there are liftings $\tilde{B}$ and $\tilde{C}$ of B and C respectively in A with the property that $\tilde{B}\tilde{C} = \tilde{C}\tilde{B} = fI_t$, where I_t denotes the $t \times t$ identity matrix. Let $'$ denote $\partial/\partial X_i$; then $\tilde{B}'\tilde{C} + \tilde{B}\tilde{C}' = f'I_t$, so that in R we obtain a homotopy

$$\begin{array}{ccccccccc} \longrightarrow & R^t & \xrightarrow{B} & R^t & \xrightarrow{C} & R^t & \longrightarrow & M & \longrightarrow 0 \\ & \tilde{C}'\swarrow & & \downarrow f' & \tilde{B}'\swarrow & & & & \\ \longrightarrow & R^t & \xrightarrow{B} & R^t & \xrightarrow{C} & R^t & \longrightarrow & M & \longrightarrow 0; \end{array}$$

hence $f' \operatorname{Ext}_R^1(M, -) = 0$ and therefore $J \operatorname{Ext}_R^1(M, -) = 0$. □

3. One-Dimensional Case

Let $(R, \mathfrak{m})$ be a 1-dimensional complete Noetherian local domain, let $\bar{R}$ denote the integral closure of R, and let $\mathfrak{C} = \{r \in R \mid r\bar{R} \subseteq R\}$ the conductor of $\bar{R}$ into R. Then, by a theorem of Lipman and Sathaye [3, Thm. 2], it is known that if R contains rational numbers then the Jacobian ideal J of R is contained in $\mathfrak{C}$. Thus it seems appropriate to ask whether $\mathfrak{C}\,\mathrm{Ext}_R^2(-,-) = 0$. This indeed is true, even when R is reduced.

PROPOSITION 3.1. *Let $(R, \mathfrak{m})$ be a 1-dimensional reduced complete Noetherian local ring, $\bar{R}$ the integral closure of R in the total quotient ring of R, and $\mathfrak{C}$ the conductor of $\bar{R}$ into R. Then $\mathfrak{C}\,\mathrm{Ext}_R^1(M,-) = 0$ for any finitely generated MCM module M.*

Proof. We break the proof into two parts.

Step 1: Let M be a finitely generated R-module having a well-defined rank; we shall show that $\mathfrak{C}\,\mathrm{Ext}_R^2(M,-) = 0$. For this, let

$$\cdots \longrightarrow R^t \xrightarrow{B} R^n \xrightarrow{A} R^m \longrightarrow M \longrightarrow 0 \tag{1}$$

be a finitely generated free resolution of M, where A and B are matrices representing the corresponding boundary maps. Because $\otimes_R \bar{R}$ is right exact, we have the exact sequence

$$\bar{R}^n \xrightarrow{A} \bar{R}^m \longrightarrow M \otimes_R \bar{R} \longrightarrow 0.$$

Note that, for any maximal ideal P in $\bar{R}$, $\bar{R}_P$ is a discrete valuation ring and so $(\mathrm{Im}(A \otimes_R \bar{R}))_P$ is free. Moreover, since M has a well-defined rank and $\bar{R}$ is semilocal, it follows that $\mathrm{Im}(A \otimes_R \bar{R})$ is free. Hence there exist a free $\bar{R}$-module $\bar{R}^r$ and a matrix B' such that

$$0 \longrightarrow \bar{R}^r \xrightarrow{B'} \bar{R}^n \xrightarrow{A} \bar{R}^m \longrightarrow M \otimes_R \bar{R} \longrightarrow 0 \tag{2}$$

is exact, and it is then easy to see that $r = \operatorname{rank} B' = \operatorname{rank} B \le t$ and $I_r(B') = \bar{R}$.

Let $x \in \mathfrak{C}$. Then, to show $x\,\mathrm{Ext}_R^2(M,-) = 0$, it suffices to show that $x\Delta\,\mathrm{Ext}_R^2(M,-) = 0$ for any $r \times r$ minor Δ of B' as $I_r(B') = \bar{R}$; or (equivalently) we must show that for any $r \times r$ minor Δ of B' there exists a matrix D with entries in R such that $BDB = x\Delta B$ (see the proof of Proposition 1.5). Write $B' = \left(\frac{B_0}{B_1}\right)$, where B_0 is a matrix consisting of the first r rows of B'. Then, without loss of generality, we need only prove the case when Δ is the determinant of B_0.

Now, by the exactness of (2) and the fact that $AB = 0$, there is a matrix U' with entries in $\bar{R}$ such that $B = B'U'$. If we write $B = \left(\frac{U}{V}\right)$, where U is a matrix consisting of the first r rows of B, then $U = B_0 U'$ and we therefore obtain

$$x\Delta B = x\Delta B'U' = xB'\Delta U' = xB'(\operatorname{adj} B_0) B_0 U' = xB'(\operatorname{adj} B_0) U. \tag{3}$$

On the other hand, since the entries of $xB'(\operatorname{adj} B_0)$ are in R and since $AxB'(\operatorname{adj} B_0) = 0$, from the exactness of (1) there is a matrix Y with entries in R such that

$$xB'(\operatorname{adj} B_0) = BY. \tag{4}$$

Together (3) and (4) yield

$$x\Delta B = BYU = B(Y \mid O)\left(\tfrac{U}{V}\right) = B(Y \mid O)B.$$

Thus, by setting $D = (Y \mid O)$, we complete the proof of $x \operatorname{Ext}^2_R(M, -) = 0$.

Step 2: Let M be a finitely generated MCM module; then there exists an element $y \in \mathfrak{m}$ such that y is a non-zero-divisor on M. Note that R is reduced and hence R_P is a field for all $P \in \operatorname{Min}(R)$, so M_y is locally free as a R_y-module and is therefore projective. Thus, by Corollary 1.7, there is an $n \in \mathbb{N}$ such that $y^n \operatorname{Ext}^1_R(M, -) = 0$.

We now consider the short exact sequence

$$0 \longrightarrow M \xrightarrow{y^n} M \longrightarrow M/y^nM \longrightarrow 0.$$

Let N be any R-module. Then, by applying $\operatorname{Hom}_R(-, N)$, we obtain a long exact sequence

$$\cdots \longrightarrow \operatorname{Ext}^1_R(M, N) \xrightarrow{y^n} \operatorname{Ext}^1_R(M, N) \longrightarrow \operatorname{Ext}^2_R(M/y^nM, N) \longrightarrow \cdots;$$

therefore, since the first map is 0, $\operatorname{Ext}^1_R(M, N)$ is isomorphic to a submodule of $\operatorname{Ext}^2_R(M/y^nM, N)$. Moreover M/y^nM has a well-defined rank 0, so by step 1 we get $\mathfrak{C} \operatorname{Ext}^2_R(M/y^nM, N) = 0$. Finally, as N is arbitrary, we conclude that $\mathfrak{C} \operatorname{Ext}^1_R(M, -) = 0$. □

COROLLARY 3.2. *Let $(R, \mathfrak{m})$ be a 1-dimensional reduced complete Noetherian local ring. Then $\mathfrak{C} \operatorname{Ext}^2_R(M, -) = 0$ for any finitely generated R-module M.*

Proof. Let M be a finitely generated R-module and M_1 a first syzygy R-module of M. Then, since R is CM, M_1 is a MCM module. It follows from Proposition 3.1 that $\mathfrak{C} \operatorname{Ext}^2_R(M, -) = \mathfrak{C} \operatorname{Ext}^1_R(M_1, -) = 0$. □

COROLLARY 3.3. *Let $(R, \mathfrak{m})$ be a 1-dimensional reduced complete Noetherian local ring and M a finitely generated R-module. Suppose M has a well-defined rank. Then, for any finitely generated free resolution* $(\mathbf{F}., \phi.)$ *of M,*

$$(t_j - i)\mathfrak{C} I_i(\phi_j) \subseteq I_{i+1}(\phi_j) \quad \forall i = 0, \ldots, t_{j-1} \text{ and } \forall j \geq 2,$$

where $t_j = \operatorname{rank} \phi_j$.

Proof. By Corollary 3.2, $\mathfrak{C} \operatorname{Ext}^2_R(M, -) = 0$. Therefore, by applying Proposition 2.4 to any first syzygy R-module of M, we obtain the desired result. □

The previous corollary, however, can be improved in the case when R is a domain.

PROPOSITION 3.4. *Let (R, m) be a 1-dimensional complete Noetherian local domain and M a finitely generated R-module. Then, for any finitely generated free resolution $(\mathbf{F}., \phi.)$ of M, $\mathfrak{C}I_i(\phi_j) \subseteq I_{i+1}(\phi_j)$ for all $i = 0, \ldots, t_j - 1$ and for all $j \geq 2$.*

LEMMA 3.5. *Let R be a commutative domain, I an ideal of R, and $B \in M_{n\times t}(R)$ of rank r. Suppose that for each $x \in I$ there are matrices $Y_x \in M_{t\times r}(R)$ and $W_x \in M_{(n-r)\times r}(I)$ such that $BY_x = \left(\frac{xI_r}{W_x}\right)$, where I_r denotes the $r \times r$ identity matrix. Then*

$$II_i(B) \subseteq I_{i+1}(B) \quad \forall i = 0, 1, \ldots, r-1.$$

Proof. Let us write $B = \left(\frac{U}{V}\right)$, with $U = (u_{ij}) \in M_{r\times t}(R)$ and with $V = (v_{ij}) \in M_{(n-r)\times t}(R)$, and set

$$B_x = \left(\begin{array}{c|c} U & xI_r \\ \hline V & W_x \end{array}\right).$$

Note that the columns of $\left(\frac{xI_r}{W_x}\right)$ are generated by those of $\left(\frac{U}{V}\right)$, so $I_i(B_x) = I_i(B)$ for all i and for all $x \in I$; in particular, $I \subseteq I_1(B)$. Moreover, since for each $x \in I$ and $1 \leq i \leq r-1$

$$x \det \begin{pmatrix} u_{11} & \ldots & u_{1i} \\ \vdots & \ddots & \vdots \\ u_{i1} & \ldots & u_{ii} \end{pmatrix} = \det \begin{pmatrix} u_{11} & \ldots & u_{1i} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ u_{i1} & \ldots & u_{ii} & 0 \\ u_{i+1,1} & \ldots & u_{i+1,i} & x \end{pmatrix} \in I_{i+1}(B_x) = I_{i+1}(B),$$

we have $II_i(U) \subseteq I_{i+1}(B)$.

If $r = 1$ then the assertion is obvious because $I \subseteq I_1(B)$. Hence we may assume $r \geq 2$. Let $1 \leq i \leq r-1$. Then to show the lemma it is enough to show that $I \det C \subseteq I_{i+1}(B)$ for any $i \times i$ matrix C of the form

$$C = \begin{pmatrix} u_{j_1k_1} & \ldots & u_{j_1k_i} \\ \vdots & \ddots & \vdots \\ u_{j_sk_1} & \ldots & u_{j_sk_i} \\ v_{m_1k_1} & \ldots & v_{m_1k_i} \\ \vdots & \ddots & \vdots \\ v_{m_lk_1} & \ldots & v_{m_lk_i} \end{pmatrix},$$

where $s + l = i$. We shall proceed by induction on l. If $l = 0$ then $I \det C \subseteq II_i(U) \subset I_{i+1}(B)$, from the above discussion. If $l \geq 1$, we let $x \in I$ and assume for simplicity that

$$C = \begin{pmatrix} u_{11} & \ldots & u_{1i} \\ \vdots & \ddots & \vdots \\ u_{s1} & \ldots & u_{si} \\ v_{11} & \ldots & v_{1i} \\ \vdots & \ddots & \vdots \\ v_{l1} & \ldots & v_{li} \end{pmatrix};$$

we further set

$$D=\begin{pmatrix} u_{11} & \cdots & u_{1i} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ u_{s1} & \cdots & u_{si} & 0 \\ u_{s+1,1} & \cdots & u_{s+1,i} & x \\ v_{11} & \cdots & v_{1i} & w_{1,s+1} \\ \vdots & \ddots & \vdots & \vdots \\ v_{l1} & \cdots & v_{li} & w_{l,s+1} \end{pmatrix}.$$

Then D is a $(i+1)\times(i+1)$ submatrix of B_x, hence $\det D\in I_{i+1}(B_x)=I_{i+1}(B)$. Furthermore, let $\Delta_1, \ldots, \Delta_l$ be the $i\times i$ minors of D corresponding to $w_{1,s+1}, \ldots, w_{l,s+1}$. Then, since $w_{1,s+1}, \ldots, w_{l,s+1}\in I$, by induction $w_{j,s+1}\Delta_j\in I_{i+1}(B)$ for all $j=1, \ldots, l$, and consequently we have $x\det C\in I_{i+1}(B)$. □

Proof of Proposition 3.4. Adopting the proof of Proposition 3.1, we know that $I_r(B')=\bar{R}$. Since $\bar{R}$ is local, some of the $r\times r$ minors of B' are units; thus we may assume that B' is of the form $\binom{B_0}{B_1}$ with $B_0=I_r$. It follows that $B=B'U$, and for each element $x\in\mathfrak{C}$ there is a $t\times r$ matrix Y_x such that $xB'=BY_x$. If we set $W_x=xB_1$, then $W_x\in M_{(n-r)\times r}(\mathfrak{C})$ and the condition $Y_x=\binom{xI_r}{W_x}$ in the lemma is satisfied, so that the assertion follows. □

4. Jacobian Ideals and Jacobian Criteria

In view of Example 2.7 and Proposition 3.1, we realize the importance of having a regular local ring (RLR) related to a given ring, because every module over a RLR has finite projective dimension. In this paper, as we are concerned with complete Noetherian local rings, the candidates of RLRs are obvious.

DEFINITION 4.1. Let $(R,\mathfrak{m},K)$ be a d-dimensional complete Noetherian local ring containing a field. A RLR A of the form $K[|X_1, \ldots, X_d|]$ is called a (Noether) normalization of R if $A\subseteq R$ and R is finite over A.

By the Cohen structure theorem, if $x_1, \ldots, x_d$ is a system of parameters (s.o.p.) of R then $K[|x_1, \ldots, x_d|]$ is a normalization of R; in fact, every normalization of R can be constructed in this way.

In order to establish our main results, we are obliged to develop in this section some properties about Jacobian ideals, especially those which are related to normalizations. To attain this aim, we first study the relation between the Jacobian ideals and the following ideals.

DEFINITION 4.2. Let A be a Noetherian ring and R a finitely generated A-algebra. Let $R=A[X_1, \ldots, X_n]/(f_1, \ldots, f_t)$ be a presentation of R over A. Then the ideal in R generated by the $n\times n$ minors of the Jacobian matrix $(\partial f_i/\partial X_j)$ is called the *Jacobian ideal* of R over A, denoted $J_{R/A}$.

LEMMA 4.3. *Let* $(R, \mathfrak{m}, K)$ *be a d-dimensional complete Noetherian local ring containing a field, and let J be the Jacobian ideal of R. Then* $J = \sum_A J_{R/A}$, *where the sum is over all normalizations of R.*

Proof. To show $J \supseteq J_{R/A}$, let $A = K[|Y_1, ..., Y_d|]$ be a normalization of R and $R = A[X_1, ..., X_n]/(f_1, ..., f_t)$. Then $R = K[|Y_1, ..., Y_d, X_1, ..., X_n|]/(f_1, ..., f_t)$. Since the height of $(f_1, ..., f_t)$ in $K[|Y_1, ..., Y_d, X_1, ..., X_n|]$ is n,

$$J_{R/A} = I_n\left(\frac{\partial f_i}{\partial X_j}\right)R \subseteq I_n\left(\frac{\partial(f_1, ..., f_t)}{\partial(X_1, ..., X_n, Y_1, ..., Y_d)}\right)R = J.$$

Conversely, let $n = \mu(m) - d$; then by prime avoidance, we can choose a minimal set of generators $x_1, ..., x_{d+n}$ such that $(x_{i_1}, ..., x_{i_d})$ is a s.o.p. of R whenever $1 \le i_1 < i_2 < \cdots < i_d \le n+d$. Let $R = K[|X_1, ..., X_{n+d}|]/(f_1, ..., f_t)$ be a presentation of R such that the image of X_i in R is x_i for all i, and let $A_{i_1, ..., i_d} = K[|X_{i_1}, ..., X_{i_d}|]$. Then it is easy to check that

$$J = \textstyle\sum_{1 \le i_1 < \cdots < i_d \le n+d} J_{R/A_{i_1, ..., i_d}}. \qquad \square$$

If in the previous lemma, $\sqrt{J}$ happens to be the defining ideal of the singular locus of R, then the conclusion simply says that if P is a regular prime ideal then there exists a normalization A of R such that $J_{R/A} \not\subset P$. However, in order to obtain the main result about non-CM rings, we must find for each regular prime P a normalization A of R satisfying not only $J_{R/A} \not\subset P$ but also that $R_{P \cap A}$ is CM. Fortunately, this can be done when R contains rational numbers.

PROPOSITION 4.4. *Let* $(R, \mathfrak{m}, K)$ *be a d-dimensional complete Noetherian local ring containing* $\mathbf{Q}$ *and let J be the Jacobian ideal of R. Let* $P \in \operatorname{Spec}(R) - \{\mathfrak{m}\}$ *be such that* R_P *is regular. Then there is a normalization A of R such that* (1) $J_{R/A} \not\subset P$ *and* (2) $R_{P \cap A}$ *is CM.*

Proof. Let $P \in \operatorname{Spec}(R) - \{\mathfrak{m}\}$ be such that R_P is regular, and let I_0 be the ideal defining the non-CM locus of R; that is, $I_0 = \bigcap_{\{Q \in \operatorname{Spec}(R) \mid R_Q \text{ is not CM}\}} Q$. Then I_0 is not contained in P or in any minimal prime ideal, and we can choose $x_0 \in I_0 \setminus \bigcup_{Q \in \operatorname{Min}(R)} Q \cup P$. Assume that $\operatorname{ht} P = h$. Then by prime avoidance we can further choose $x_1, ..., x_h \in P$ so that

(i) $\operatorname{ht}(x_0, ..., x_i) = i+1$ for all $i \le h$ and
(ii) $(x_1, ..., x_h)R_P = PR_P$.

This is possible because for each i we can choose $x_i \in P$ which is neither in any minimum prime over $(x_0, ..., x_{i-1})R$ nor in the ideal $P^{(2)} + (x_1, ..., x_{i-1})R$. (One should notice that, by the choice of x_0, P is not in the union of the minimum primes of $(x_0, ..., x_{h-1})R$.) Now extend $x_0, ..., x_h$ to a s.o.p. $x_0, ..., x_{d-1}$ and let $A = K[|x_0, ..., x_{d-1}|]$; then we are done if we can show that A satisfies (1) and (2). Condition (2) is immediate, since $x_0 \notin P \cap A$ and $x_0 \in I_0$. As for (1), let $R = A[Y_1, ..., Y_n]/(f_1, ..., f_t)$ be a presentation of R over A and $R' = R_q/qR_q$, where $q = P \cap A$. Then $R' = \kappa(q)[Y_1, ..., Y_n]/(f_1, ..., f_t)$. Since R' is finite over $\kappa(q)$, $\dim R' = 0$, so the height of $(f_1, \ldots, f_t)$ in $\kappa(q)[Y_1, ..., Y_n]$ is n. Furthermore, by (ii) we have

$$(x_1, \ldots, x_h)R_P \subseteq qR_P \subseteq PR_P = (x_1, \ldots, x_h)R_P,$$

hence $R'_P = R_P/qR_P = R_P/PR_P = \kappa(P)$ is a field and therefore 0-smooth over $\kappa(q)$ because $\operatorname{char} R = 0$. Thus, by [4, Thm. 30.3], $I_n(\partial f_i/\partial X_j)R' \not\subset PR'$, which is equivalent to saying that $J_{R/A} \not\subset P$. □

If one applies the above proof to the case when $\operatorname{char} R = p$ then the only apparent problem is that $\kappa(P)$ is not smooth over $\kappa(q)$. To conquer this difficulty, we add some mild conditions on R.

PROPOSITION 4.5. *Let $(R, \mathfrak{m}, K)$ be a d-dimensional complete Noetherian local ring containing a field. Assume that R is equidimensional,* $\operatorname{char} K = p$, *and K is perfect. Let $P \in \operatorname{Spec}(R) - \{\mathfrak{m}\}$ such that R_P is regular. Then there is a normalization A of R such that* (1) *$J_{R/A} \not\subset P$ and* (2) *$R_{P \cap A}$ is CM.*

As the proof requires knowledge of universal-finite modules, we should postpone it for the moment. At present we would like to give the definition of universal-finite module and list some properties related to it. For some of the proofs, we refer to [6].

DEFINITION 4.6. Let K be a ring and R a K-algebra. A K-derivation $R \to M$ is called *finite* if M is a finitely generated R-module; a finite K-derivation $d_{R/K}: R \to D_K(R)$ is called *universal-finite* if for any finite K-derivation δ: $R \to M$ there exists a R-homomorphism $h: D_K(R) \to M$ such that $\delta = h \circ d_{R/K}$. If $d_{R/K}$ exists then we call $d_{R/K}$ the *universal-finite derivation* of R over K and $D_K(R)$ the *universal-finite module* of R over K.

REMARK 4.7. If $d_{R/K}$ exists then it is unique up to isomorphism, and $D_K(R) = Rd_{R/K}(R)$.

PROPOSITION 4.8. *Let K be a valuation field and R a local analytic K-algebra. Then the universal-finite derivation of R over K exists.*

REMARK 4.9. Here, an analytic algebra R over a valuation field K is defined to be a finite algebra over a convergent power series ring (see [6]). In particular, any complete Noetherian local ring containing a field is an analytic algebra with trivial valuation.

PROPOSITION 4.10. *Let $(R, \mathfrak{m})$ be a Noetherian local K-algebra, $P \in \operatorname{Spec}(R)$, and $S = R/P$. Assume that $d_{R/K}$ exists. Then $d_{S/K}$ exists and $D_K(S) = D_K(R)/(Rd_{R/K}(P) + PD_K(R))$.*

Proof. Let $M = D_K(R)/(Rd_{R/K}(P) + PD_K(R))$. Then M is a finitely generated R/P-module and there is a natural K-derivation $d: S \to M$ which sends $a + P$ to $d_{R/K}a + Rd_{R/K}(P) + PD_K(R)$ for all $a \in R$. Therefore we have the following commutative diagram:

$$\begin{array}{ccc} R & \xrightarrow{\pi} & S \\ {\scriptstyle d_{R/K}}\downarrow & & \downarrow{\scriptstyle d} \\ D_K(R) & \xrightarrow{\pi'} & M, \end{array}$$

where π and π' are the canonical surjective maps.

Let $\delta: S \to N$ be any finite K-derivation. Since N is finite as a R-module, $\delta \circ \pi: R \to N$ is a finite K-derivation and so, by the definition of $d_{R/K}$, there is a R-homomorphism $h: D_K(R) \to N$ such that $\delta \circ \pi = h \circ d_{R/K}$. Note that $h(PD_K(R)) \subseteq PN = 0$ and $h(d_{R/K}(P)) = \delta \circ \pi(P) = 0$; hence h induces a S-homomorphism $h': M \to N$ such that $h = h' \circ \pi'$. It follows that

$$(h' \circ d - \delta) \circ \pi = h' \circ d \circ \pi - \delta \circ \pi = h' \circ \pi' \circ d_{R/K} - \delta \circ \pi = h \circ d_{R/K} - \delta \circ \pi = 0.$$

Since π is surjective, $h' \circ d = \delta$. Thus we conclude that $d: S \to M$ is the universal-finite derivation of S over K. □

COROLLARY 4.11. *Let $(R, \mathfrak{m})$ be a reduced Noetherian local K-algebra. Assume that $d_{R/K}$ exists. Then, for all $P \in \mathrm{Min}(R)$,*

$$D_K(R/P)_P = D_K(R)_P.$$

Proof. Let $P \in \mathrm{Min}(R)$. Then, from Proposition 4.10, to show the assertion it suffices to show that $(PD_K(R))_P = (Rd_{R/K}(P))_P = 0$. Since R is reduced, $PR_P = 0$, so $(PD_K(R))_P = 0$. On the other hand, let $a \notin P$ be such that $aP = 0$; then, for any $x \in P$, $ad_{R/K}x + xd_{R/K}a = 0$, so that $d_{R/K}(x)R_P = 0$ as $xd_{R/K}(a)R_P = 0$ and $aR_P = R_P$. Therefore $(Rd_{R/K}(P))_P = 0$. □

DEFINITION 4.12. Let K be a valuation field and R a d-dimensional local analytic K-algebra. Assume R is reduced and equidimensional. Then a s.o.p. $x_1, \ldots, x_d$ of R is called *separable* if the total quotient ring of R is separable over the quotient field of $K[|x_1, \ldots, x_d|]$.

PROPOSITION 4.13. *Let K be a valuation field and R a d-dimensional local analytic K-algebra. Assume that R is a domain with quotient field L and that $x_1, \ldots, x_d$ is a s.o.p. of R. Then $x_1, \ldots, x_d$ is separable if and only if*

$$(D_K(R)/(Rd_{R/K}x_1 + \cdots + Rd_{R/K}x_d)) \otimes_R L = 0.$$

THEOREM 4.14. *Let K be a valuation field and R a d-dimensional local analytic K-algebra. Assume R is reduced and equidimensional. Then:*

(1) *R has a separable s.o.p.*
(2) *Let $x_1, \ldots, x_d$ be a s.o.p. of R; then $x_1, \ldots, x_d$ is separable if and only if $(D_K(R)/(Rd_{R/K}x_1 + \cdots + Rd_{R/K}x_d))_P = 0$ for every minimal prime ideal P.*

Proof. (1) follows from [6, Lemma 7.2]. As for (2), let F be the quotient field of $K[|x_1, \ldots, x_d|]$, $\mathrm{Min}(R) = \{P_1, \ldots, P_t\}$, and $R_{P_i} = K_i$. Then, by Definition 4.12, we know that $x_1, \ldots, x_d$ is separable if and only if K_i is separable over F for all $i = 1, \ldots, t$. Further, let $R_i = R/P_i$ and $h_i: R \to R_i$ be the canonical maps; then (by Definition 4.12 again) we see that K_i is separable over F and $h_i(x_1), \ldots, h_i(x_d)$ is a separable s.o.p. of R_i are equivalent. Furthermore, by Proposition 4.13, the latter is equivalent to saying that

$$(D_K(R_i)/(R_i d_{R_i/K}(h_i(x_1)) + \cdots + R_i d_{R_i/K}(h_i(x_d))))_{P_i} = 0.$$

However $D_K(R_i)_{P_i} = D_K(R)_{P_i}$ by Corollary 4.11 and we have the following commutative diagram:

$$\begin{array}{ccccc} R & \xrightarrow{h_i} & R/P_i & \longrightarrow & R_{P_i} \\ {\scriptstyle d_{R/K}}\downarrow & & \downarrow{\scriptstyle d_{R_i/K}} & & \downarrow \\ D_K(R) & \longrightarrow & D_K(R/P_i) & \longrightarrow & D_K(R)_P. \end{array}$$

Thus we conclude that $x_1, \ldots, x_d$ is separable if and only if

$$(D_K(R)/(Rd_{R/K}x_1 + \cdots + Rd_{R/K}x_d))_{P_i} = 0 \quad \forall i = 1, \ldots, t. \qquad \square$$

Now, we are ready.

Proof of Proposition 4.5. Let $P \in \operatorname{Spec}(R) - \{\mathfrak{m}\}$ and $\operatorname{ht} P = h$; then, according to the proof of Proposition 4.4, we can choose $x_0, x_1, \ldots, x_h$ so that:

(i) $\operatorname{ht}(x_0, \ldots, x_h) = h+1$;

(ii) $(x_1, \ldots, x_h)R_P = PR_P$; and

(iii) $x_0 \in I_0 \cap \mathfrak{m}$, where I_0 is the ideal defining the non-CM locus of R.

Let $\operatorname{Min}(R/(x_1, \ldots, x_h)R) = \{Q_1, \ldots, Q_s\}$ and $R_1 = R/(Q_1 \cap \cdots \cap Q_s)$. Then, since R is equidimensional and catenary, and since $x_1, \ldots, x_h$ is part of a s.o.p. of R, R_1 is equidimensional and reduced; thus, by Theorem 4.14, R_1 has a separable s.o.p. $y_1, \ldots, y_{d-h}$ such that

$$(D_K(R_1)/(R_1 d_{R_1/K}y_1 + \cdots + R_1 d_{R_1/K}y_{d-h}))_{Q_i} = 0 \quad \forall i = 1, \ldots, s.$$

Let x be the image of x_0 in R_1; then, by condition (i), x is a non-zero-divisor on R_1. We claim that there is a separable s.o.p. $y_1', \ldots, y_{d-h}'$ of R_1 with $y_1' = x^p y_1$. To see this, let $y_1' = x^p y_1$ and choose y_2' as follows: Let $P_1, \ldots, P_i, P_{i+1}, \ldots, P_l$ be minimal primes over y_1' such that $y_2 \in P_1 \cap \cdots \cap P_i$ and $y_2 \notin P_{i+1} \cup \cdots \cup P_l$; then we may choose $y_2' = y_2 + z^p$, where $z \in \mathfrak{m} \cap P_{i+1} \cap \cdots \cap P_l \backslash P_1 \cup \cdots \cup P_i$. It is obvious that $(y_1', y_2')R'$ is a height-2 ideal and that $d_{R_1/K}y_2' = d_{R_1/K}y_2$. Similarly, we can construct $y_2', \ldots, y_{d-h}'$ such that $d_{R_1/K}(y_i') = d_{R_1/K}y_i$ for all $i = 2, \ldots, d-h$ and $y_1', \ldots, y_{d-h}'$ is a s.o.p. of R_1. Finally, since $d_{R_1/K}(x^p y_1) = x^p d_{R_1/K}y_1$ and x^p is a non-zero-divisor on R_1, $(R_1 d_{R_1/K}y_1')_{Q_i} = (R_1 d_{R_1/K}y_1)_{Q_i}$ for every i; hence

$$(D_K(R_1)/(R_1 d_{R_1/K}y_1' + \cdots + Rd_{R_1/K}y_{d-h}'))_{Q_i} = 0 \quad \forall i = 1, \ldots, s,$$

and it follows from Theorem 4.14 that $y_1', \ldots, y_{d-h}'$ is a separable s.o.p. of R_1.

Now we may lift $y_1', \ldots, y_{d-h}'$ to $z_1, \ldots, z_{d-h}$ in R such that $z_1 \in I_0$; let $A = K[|x_1, \ldots, x_h, z_1, \ldots, z_{d-h}|]$. It remains to show that A satisfies (1) and (2). Let $q = P \cap A$; then $q = (x_1, \ldots, x_h)A$. This is because R/P is finite over A/q and

$$h \le \operatorname{ht} q = d - \dim A/q = d - \dim R/P = \operatorname{ht} P = h.$$

Therefore (2) is obvious, as $z_1 \in I_0$ and $z_1 \notin q$. As for (1), let $R_2 = R/(x_1, \ldots, x_h)$; then, by condition (ii), $(R_2)_P = (R_1)_P = \kappa(P)$ which is, by the definition of separable s.o.p., separable over $\kappa(q)$. On the other hand, let

$$R = A[U_1, \ldots, U_n]/(f_1, \ldots, f_t)$$

be a presentation of R over A. Then

$$R_2 = K[|z_1, \ldots, z_{d-h}|][U_1, \ldots, U_n]/(f_1, \ldots, f_t)$$

and

$$(R_2)_q = \kappa(q)[U_1, \ldots, U_n]/(f_1, \ldots, f_t);$$

hence, by [4, Thm. 30.3] and the fact that $(R_2)_P$ is separable over $\kappa(q)$, we get $I_n(\partial f_i/\partial U_j)(R_2)_q \not\subset P(R_2)_q$. Thus $I_n(\partial f_i/\partial U_j)R_2 \not\subset PR_2$, and assertion (I) follows. □

5. Main Theory

We shall prove our main results of this paper and in so doing will see that the conjecture holds when R is CM of characteristic 0.

THEOREM 5.1. *Let $(R, \mathfrak{m}, K)$ be a d-dimensional complete CM local ring containing a field, let J be the Jacobian ideal of R, and let M be a finitely generated R-module. Suppose M has a well-defined rank. Then, for any finitely generated free resolution* $(\mathbf{F}., \phi.)$ *of M,*

$$(t_j - i)JI_i(\phi_j) \subseteq I_{i+1}(\phi_j) \quad \forall i = 0, \ldots, t_j - 1 \textit{ and } \forall j \geq d+1,$$

where $t_j = \operatorname{rank} \phi_j$.

THEOREM 5.2. *Let $(R, \mathfrak{m}, K)$ be a d-dimensional equidimensional complete Noetherian local ring containing a field, and let J be the Jacobian ideal of R. Assume that either* $\operatorname{char} K = 0$ *or K is perfect. Then there exists an integer k such that, for any finitely generated R-module M having a well-defined rank and for any finitely generated free resolution* $(\mathbf{F}., \phi.)$ *of M,*

$$(t_j - i)J^kI_i(\phi_j) \subseteq I_{i+1}(\phi_j) \quad \forall i = 0, \ldots, t_j - 1 \textit{ and } \forall j \geq d+1,$$

where $t_j = \operatorname{rank} \phi_j$.

According to Proposition 2.4, to show the above two theorems it suffices to show the following theorems.

THEOREM 5.3. *Let $(R, \mathfrak{m}, K)$ be a d-dimensional complete CM local ring containing a field, and let J be the Jacobian ideal of R. Then* $J\operatorname{Ext}_R^{d+1}(M, -) = 0$ *for any finitely generated R-module M.*

THEOREM 5.4. *Let $(R, \mathfrak{m}, K)$ be a d-dimensional equidimensional complete Noetherian local ring containing a field, with J the Jacobian ideal of R. Assume that either* $\operatorname{char} K = 0$ *or K is perfect. Then there exists an integer k such that* $J^k \operatorname{Ext}_R^{d+1}(M, -) = 0$ *for any finitely generated R-module M.*

First we give a definition.

DEFINITION 5.5. *Let A be a commutative ring and R an A-algebra. Let R^e denote the envelope algebra $R \otimes_A R$ and let $\mu\colon R \otimes_A R \to R$ be the augmented* *map defined by $\mu(x \otimes y) = xy$ for $x, y \in R$; let I be the kernel of μ. Then the Noetherian different ideal of R over A, denoted $\mathfrak{N}_A^R$, is the ideal $\mu(\operatorname{ann}_{R^e} I)$.*

We now prove some useful lemmas about $\mathfrak{N}_A^R$.

LEMMA 5.6. *Let S be a ring, I an ideal of S, and $R = S/I$. Then*

$$(\operatorname{ann}_S I) \operatorname{Ext}_S^1(R, -) = 0.$$

Proof. Let $\pi: S \to R$ be the canonical surjective map. Then, from the short exact sequence

$$0 \longrightarrow I \longrightarrow S \xrightarrow{\pi} R \longrightarrow 0,$$

we know that as an S-module I is a first syzygy module of R. Hence, by Corollary 1.6,

$$\operatorname{ann}_S \operatorname{Ext}_S^1(R, -) = \operatorname{ann}_S \operatorname{Ext}_S^1(R, I) \supseteq \operatorname{ann}_S I. \qquad \square$$

By applying this lemma to the case when $S = R^e$, we obtain the following corollary.

COROLLARY 5.7. *With the same notation as in Definition 5.5,*

$$(\operatorname{ann}_{R^e} I) \operatorname{Ext}_{R^e}^1(R, -) = 0.$$

LEMMA 5.8. *Let A be a Noetherian ring and R a finitely generated A-algebra. Then $J_{R/A} \subseteq \mathfrak{N}_A^R$.*

Proof. Let $R = A[X_1, \dots, X_n]/(f_1, \dots, f_t)$ be a presentation of R over A. Then the module of differentials $\Omega_{R/A} \cong R^n/\langle \partial f_i/\partial X_j \rangle_{i=1,\dots,t,\, j=1,\dots,n}$, so that $\Omega_{R/A}$ has the following presentation:

$$R^t \xrightarrow{(\partial f_i/\partial X_j)} R^n \longrightarrow \Omega_{R/A} \longrightarrow 0.$$

Let I be the kernel of the augmented map μ. Then I is generated by $1 \otimes x_i - x_i \otimes 1$ as a R^e-module, so that I has a presentation of the form

$$(R^e)^s \xrightarrow{(g_{ij})} (R^e)^n \longrightarrow I \longrightarrow 0.$$

Since $\Omega_{R/A} \cong I/I^2$, by $\otimes_{R^e} R$ we get another presentation of $\Omega_{R/A}$:

$$R^s \xrightarrow{(\mu(g_{ij}))} R^n \longrightarrow \Omega_{R/A} \longrightarrow 0.$$

Therefore, by the invariant property of Fitting ideals, we get $I_n(\mu(g_{ij}))R = I_n(\partial f_i/\partial X_j) = J_{R/A}$. Because $I_n(g_{ij})I = 0$, $J_{R/A} = \mu(I_n(g_{ij})) \subseteq \mu(\operatorname{ann}_{R^e} I) = \mathfrak{N}_A^R$. $\square$

PROPOSITION 5.9. *Let A be a Noetherian ring and R a finitely generated A-algebra. Then, for any finitely generated R-modules M and N,*

$$\mathfrak{N}_A^R \operatorname{ann}_A \operatorname{Ext}_A^1(M, N) \subseteq \operatorname{ann}_R \operatorname{Ext}_R^1(M, N).$$

We now need a couple of lemmas.

LEMMA 5.10. *Let S be a Noetherian ring, I an ideal of S, and $R = S/I$. Let $M_1 \xrightarrow{\phi} M_2 \xrightarrow{\psi} M_3$ be a complex of S-modules and $_* := \mathrm{Hom}_S(R, \text{-})$. Then there are two short exact sequences,*

$$0 \longrightarrow C_1 \longrightarrow \frac{\mathrm{Ker}\,\psi_*}{\mathrm{Im}\,\phi_*} \longrightarrow \frac{\mathrm{Hom}_S(R, \mathrm{Ker}\,\psi)}{\mathrm{Hom}_S(R, \mathrm{Im}\,\phi)} \longrightarrow 0$$

and

$$0 \longrightarrow \frac{\mathrm{Hom}_S(R, \mathrm{Ker}\,\psi)}{\mathrm{Hom}_S(R, \mathrm{Im}\,\phi)} \longrightarrow \mathrm{Hom}_S\left(R, \frac{\mathrm{Ker}\,\psi}{\mathrm{Im}\,\phi}\right) \longrightarrow C_2 \longrightarrow 0,$$

such that C_1 and C_2 are both killed by $\mathrm{ann}_S I$ as S-modules.

Proof. Let I be the kernel of the canonical map $\pi: S \to R$. Notice that, for any S-module M,

$$\mathrm{Hom}_S(R, M) = \{x \in M \mid xI = 0\}$$

is a R-module; the S-module structure that comes from being a submodule of M is the same as the one via π. Also, for any S-homomorphism f, $f_* = \mathrm{Hom}_S(R, f)$ is a R-homomorphism; hence, in particular, ϕ_* and ψ_* are R-homomorphisms. Now, factoring the complex in the assumption, we get

$$\begin{array}{ccccccc} \mathrm{Ker}\,\phi & \longrightarrow & M_1 & \xrightarrow{\phi} & M_2 & \xrightarrow{\psi} & M_3 \\ & & {\scriptstyle\lambda}\downarrow & & \uparrow & & \\ & & \mathrm{Im}\,\phi & \longrightarrow & \mathrm{Ker}\,\psi. & & \end{array}$$

Then, by the left exactness of $\mathrm{Hom}_S(R, \text{-})$,

$$\mathrm{Im}\,\phi_* \subseteq \mathrm{Hom}_S(R, \mathrm{Ker}\,\psi) = \mathrm{Ker}\,\psi_* \subseteq \mathrm{Hom}_S(R, M_2)$$

and $\mathrm{Im}\,\phi_* = \mathrm{Im}\,\lambda_*$. Let C_1 denote the cokernel of λ_*; that is,

$$C_1 \cong \frac{\mathrm{Hom}_S(R, \mathrm{Im}\,\phi)}{\mathrm{Im}\,\lambda_*}.$$

Then

$$0 \longrightarrow C_1 \longrightarrow \frac{\mathrm{Ker}\,\psi_*}{\mathrm{Im}\,\phi_*} \longrightarrow \frac{\mathrm{Hom}_S(R, \mathrm{Ker}\,\psi)}{\mathrm{Hom}_S(R, \mathrm{Im}\,\phi)} \longrightarrow 0$$

is exact. Moreover, since $M_1 \xrightarrow{\lambda} \mathrm{Im}\,\phi$ is onto, C_1 can be embedded into $\mathrm{Ext}^1_S(R, \mathrm{Ker}\,\lambda)$ as S-modules, so that by Lemma 5.6 we obtain

$$(\mathrm{ann}_S I)\,\mathrm{Ext}^1_S(R, \text{-}) = 0$$

and hence $(\mathrm{ann}_S I) C_1 = 0$. On the other hand, the short exact sequence

$$0 \longrightarrow \mathrm{Im}\,\phi \longrightarrow \mathrm{Ker}\,\psi \xrightarrow{\pi'} \frac{\mathrm{Ker}\,\psi}{\mathrm{Im}\,\phi} \longrightarrow 0$$

induces a long exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}_S(R, \mathrm{Im}\,\phi) &\longrightarrow \mathrm{Hom}_S(R, \mathrm{Ker}\,\psi) \\ &\xrightarrow{\pi'_*} \mathrm{Hom}_S\left(R, \frac{\mathrm{Ker}\,\psi}{\mathrm{Im}\,\phi}\right) \longrightarrow \mathrm{Ext}^1_S(R, \mathrm{Im}\,\phi) \longrightarrow \cdots. \end{aligned}$$

By setting $C_2 = \operatorname{coker}(\pi'_*)$ we get the second short exact sequence, and it is easy to see from the above long exact sequence that C_2 can be embedded into $\operatorname{Ext}^1_S(R, \operatorname{Im}\phi)$ as S-modules and that $\operatorname{ann}_S I$ kills C_2. ❐

COROLLARY 5.11. *Let A be a Noetherian ring and R a finitely generated A-algebra. Let $M_1 \xrightarrow{\phi} M_2 \xrightarrow{\psi} M_3$ be a complex of R^e-modules and $_* := \operatorname{Hom}_{R^e}(R, -)$. Then there are two short exact sequences,*

$$0 \longrightarrow C_1 \longrightarrow \frac{\operatorname{Ker}\psi_*}{\operatorname{Im}\phi_*} \longrightarrow \frac{\operatorname{Hom}_{R^e}(R, \operatorname{Ker}\psi)}{\operatorname{Hom}_{R^e}(R, \operatorname{Im}\phi)} \longrightarrow 0$$

and

$$0 \longrightarrow \frac{\operatorname{Hom}_{R^e}(R, \operatorname{Ker}\psi)}{\operatorname{Hom}_{R^e}(R, \operatorname{Im}\phi)} \longrightarrow \operatorname{Hom}_{R^e}\left(R, \frac{\operatorname{Ker}\psi}{\operatorname{Im}\phi}\right) \longrightarrow C_2 \longrightarrow 0,$$

such that C_1 and C_2 are both killed by $\mathfrak{N}^R_A$ as R-modules.

LEMMA 5.12. *Let A be a Noetherian ring and R a finitely generated A-algebra. Let M be a finitely generated R-module and N a R-module. Let $I.: I_0 \to I_1 \to I_2 \to \cdots$ be an injective R-resolution of N, with H the homology of the complex*

$$\operatorname{Hom}_A(M, I_0) \xrightarrow{\phi} \operatorname{Hom}_A(M, I_1) \xrightarrow{\psi} \operatorname{Hom}_A(M, I_2).$$

Then $(\operatorname{ann}_R \operatorname{Hom}_{R^e}(R, H))\mathfrak{N}^R_A \operatorname{Ext}^1_R(M, N) = 0$, *and H can be embedded into* $\operatorname{Ext}^1_A(M, N)$ *as A-modules.*

Proof. Notice that, for any two R-modules M_1 and M_2, $\operatorname{Hom}_A(M_1, M_2)$ is a R^e-module. The R^e-module structure is given by $[\phi(x \otimes y)](m) = [\phi(xm)]y$ for any $\phi \in \operatorname{Hom}_A(M_1, M_2)$, $x, y \in R$ and $m \in M_1$, so it follows that $\operatorname{Hom}_{R^e}(R, \operatorname{Hom}_A(M_1, M_2)) = \operatorname{Hom}_R(M_1, M_2)$ is a R-module. Hence, by applying $_* := \operatorname{Hom}_{R^e}(R, -)$ to the above complex, we get a complex of R-modules

$$\operatorname{Hom}_R(M, I_0) \xrightarrow{\phi_*} \operatorname{Hom}_R(M, I_1) \xrightarrow{\psi_*} \operatorname{Hom}_R(M, I_2).$$

Hence, by Corollary 5.11, we obtain two short exact sequences,

$$0 \longrightarrow C_1 \longrightarrow \operatorname{Ext}^1_R(M, N) \longrightarrow \frac{\operatorname{Hom}_{R^e}(R, \operatorname{Ker}\psi)}{\operatorname{Hom}_{R^e}(R, \operatorname{Im}\phi)} \longrightarrow 0$$

and

$$0 \longrightarrow \frac{\operatorname{Hom}_{R^e}(R, \operatorname{Ker}\psi)}{\operatorname{Hom}_{R^e}(R, \operatorname{Im}\phi)} \longrightarrow \operatorname{Hom}_{R^e}(R, H) \longrightarrow C_2 \longrightarrow 0,$$

with C_1 and C_2 being killed by $\mathfrak{N}^R_A$ as R-modules. Therefore,

$$\operatorname{ann}_R(\operatorname{Hom}_{R^e}(R, H))\mathfrak{N}^R_A \operatorname{Ext}^1_R(M, N) = 0.$$

Moreover, from the factorization

$$\begin{array}{ccccc} I_0 & \longrightarrow & I_1 & \longrightarrow & I_2 \\ & \searrow & & \nearrow & \\ & & I_0/N, & & \end{array}$$

we know that $\operatorname{Ker}\psi = \operatorname{Hom}_A(M, I_0/N)$ and $\operatorname{Im}\phi = \operatorname{Im}(\operatorname{Hom}_A(M, I_0) \to \operatorname{Hom}_A(M, I_0/N))$. Thus, from the long exact sequence

$$0 \longrightarrow \operatorname{Hom}_A(M, N) \longrightarrow \operatorname{Hom}_A(M, I_0) \longrightarrow \operatorname{Hom}_A(M, I_0/N) \longrightarrow \operatorname{Ext}^1_A(M, N) \longrightarrow \cdots,$$

we conclude that H can be embedded into $\operatorname{Ext}^1_A(M, N)$ as A-modules. □

Proof of Proposition 5.9. Let $x \in \operatorname{ann}_A \operatorname{Ext}^1_A(M, N)$; then by Lemma 5.12 we have $xH = 0$, so that $x \in \operatorname{ann}_R(\operatorname{Hom}_{R^e}(R, H))$. Hence (by Lemma 5.12 again) $x\mathfrak{N}^R_A \operatorname{Ext}^1_R(M, N) = 0$, and therefore

$$\mathfrak{N}^R_A \operatorname{ann}_A \operatorname{Ext}^1_A(M, N) \subseteq \operatorname{ann}_R \operatorname{Ext}^1_R(M, N). \qquad \square$$

Now we are able to prove Theorem 5.3.

Proof of Theorem 5.3. Since R is CM, every dth syzygy module is a MCM module; thus it is sufficient to show that $J \operatorname{Ext}^1_R(M, -) = 0$ for any MCM module M. Let M be such a module. Then, for any normalization A of R, M is finitely generated free as an A-module and so $\operatorname{Ext}^1_A(M, N) = 0$ for any R-module N. Hence, by Lemma 5.8 and Proposition 5.9, $J_{R/A} \subseteq \mathfrak{N}^R_A \subseteq \operatorname{ann}_R \operatorname{Ext}^1_R(M, -)$. It follows that $J \subseteq \operatorname{ann}_R \operatorname{Ext}^1_R(M, -)$. □

COROLLARY 5.13. *Let $(R, \mathfrak{m})$ be a complete CM local ring containing a field. Then $\mathfrak{N}^R \operatorname{Ext}^1_R(M, -) = 0$ for all MCM modules M, where $\mathfrak{N}^R = \sum_A \mathfrak{N}^R_A$ and the sum is over all normalizations of R.*

For non-CM rings, the previous proposition allows us to study $\operatorname{ann}_A \operatorname{Ext}^1_A(-, -)$ instead of $\operatorname{ann}_R \operatorname{Ext}^1_R(-, -)$. In fact, when A is a RLR, we have the following uniform property on $\operatorname{ann}_A \operatorname{Ext}^1_A(-, -)$. In the sequel, let M_d denote any dth syzygy R-module of M.

LEMMA 5.14. *Let R be a d-dimensional complete Noetherian local ring containing a field, and A a normalization of R. Let $x \in A$ be such that $x \operatorname{Ext}^1_A(R, -) = 0$. Then $x^d \operatorname{Ext}^1_A(M_d, -) = 0$ for any finitely generated R-module M.*

Proof. Let $(\mathbf{F}., \phi.)$ be a resolution of M such that $\operatorname{Ker}\phi_{d-1} = M_d$; let $M_i = \operatorname{Ker}\phi_{i-1}$, $i = 1, \ldots, d$. Notice that we have the following exact sequences:

$$\begin{array}{llllr}
\cdots \longrightarrow & \operatorname{Ext}^d_A(F_0, -) \longrightarrow & \operatorname{Ext}^d_A(M_1, -) \longrightarrow & \operatorname{Ext}^{d+1}_A(M_0, -) \longrightarrow & \cdots (1) \\
\cdots \longrightarrow & \operatorname{Ext}^{d-1}_A(F_1, -) \longrightarrow & \operatorname{Ext}^{d-1}_A(M_2, -) \longrightarrow & \operatorname{Ext}^d_A(M_1, -) \longrightarrow & \cdots (2) \\
& \vdots & \vdots & \vdots & \vdots \\
\cdots \longrightarrow & \operatorname{Ext}^1_A(F_{d-1}, -) \longrightarrow & \operatorname{Ext}^1_A(M_d, -) \longrightarrow & \operatorname{Ext}^2_A(M_{d-1}, -) \longrightarrow & \cdots (d).
\end{array}$$

By assumption, $x \operatorname{Ext}^1_A(R, -) = 0$; hence, by Corollary 1.6(1), $x \operatorname{Ext}^d_A(F_0, -) = x \operatorname{Ext}^d_A(R, -) = 0$. Since A is a d-dimensional RLR, $\operatorname{Ext}^{d+1}_A(M_0, -) = 0$; therefore, from (1) we know that $x \operatorname{Ext}^d_A(M_1, -) = 0$. Similarly, from (2) we get

$x^2 \operatorname{Ext}_A^{d-1}(M_2, -) = 0$ as $x \operatorname{Ext}_A^d(M_1, -) = x \operatorname{Ext}_A^{d-1}(F_1, -) = 0$. Inductively, we obtain that $x^d \operatorname{Ext}_A^1(M_d, -) = 0$. □

Proof of Theorem 5.4. We first show the following claim: If $P \in \operatorname{Spec}(R)$ is such that $J \not\subset P$, then there exists an element $x \in J \setminus P$ such that $x \operatorname{Ext}_R^1(M_d, -) = 0$ for any finitely generated R-module M. To prove this claim, note that by Propositions 4.4 and 4.5 we know there is a normalization A of R such that (1) $J_{R/A} \not\subset P$ and (2) $R_{P \cap A}$ is CM. If $q = P \cap A$ and if Q is any prime of R lying over q, then R/Q is finite over A/q, so that $\dim R/Q = \dim A/q$. Moreover, since R is equidimensional and catenary,

$$\operatorname{ht} q = d - \dim A/q = d - \dim R/Q = \operatorname{ht} Q;$$

therefore, any s.o.p. of A_q is a s.o.p. of $R_Q = (R_q)_Q$. However, R_Q is a CM ring by (2), hence any s.o.p. of A_q forms a regular sequence on R_q. It follows that R_q is free as an A_q-module, and so $(\operatorname{Ext}_A^1(R, -))_q = 0$. Following the same argument of the proof of Corollary 1.7, we see that there exists $y \in A \setminus q$ such that $y \operatorname{Ext}_A^1(R, -) = 0$; thus, by Lemma 5.14, we obtain $y^d \operatorname{Ext}_A^1(M_d, -) = 0$ for any finitely generated R-module M. Finally, by (1) we can choose an element $z \in J_{R/A} \setminus P$ and set $x = y^d z$. Then $x \in J \setminus P$, and by Lemma 5.8 and Proposition 5.9, $x \operatorname{Ext}_R^1(M_d, -) = 0$.

Let $J_0 = \bigcap_M \operatorname{ann}_R \operatorname{Ext}_R^{d+1}(M, -)$, where the intersection is over all finitely generated R-modules M. Then, by the claim, for any prime $P \not\supset J$ there is an element $x \notin P$ such that $x \operatorname{Ext}_R^{d+1}(M, -) = x \operatorname{Ext}_R^1(M_d, -) = 0$ for any finitely generated module M, which means $x \in J_0$ and hence $P \not\supset J_0$. It follows that $J \subset \sqrt{J_0}$, or $J^k \subset J_0$ for some integer k, and then $J^k \operatorname{Ext}_R^{d+1}(M, -) = 0$ for any finitely generated module M. □

From Propositions 4.4 and 4.5 we know that, under the assumptions of Theorem 5.4, $\sqrt{J} = I_s(R)$. We therefore have the following corollary.

COROLLARY 5.15. *Let $(R, \mathfrak{m}, K)$ be a d-dimensional equidimensional complete Noetherian local ring containing a field. Assume that either* $\operatorname{char} K = 0$ *or K is perfect. Then there exists an integer k such that $I_s(R)^k \operatorname{Ext}_R^{d+1}(M, -) = 0$ for any finitely generated R-module M.*

A Noetherian local ring R is called *generalized CM* if R_P is CM for all $P \in \operatorname{Spec}(R) - \{\mathfrak{m}\}$.

COROLLARY 5.16. *Let $(R, \mathfrak{m}, K)$ be a d-dimensional complete Noetherian local ring containing a field, and let J be the Jacobian ideal of R. Assume that R is an equidimensional generalized CM ring. Then there exists an integer k such that $J^k \operatorname{Ext}_R^{d+1}(M, -) = 0$ for any finitely generated R-module M.*

Proof. Let $P \in \operatorname{Spec}(R)$ be such that $J \not\subset P$; then $P \neq \mathfrak{m}$. In view of the proof of Theorem 5.4 and since R is equidimensional, we know it is enough to show that there exists a normalization A of R such that (1) $J_{R/A} \not\subset P$ and

(2) $R_{P\cap A}$ is CM. But condition (2) is redundant, as $P \neq \mathfrak{m}$ guarantees it; hence the assertion follows from Lemma 4.3. □

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