On Composition Sequences in the Unit Disk

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1. General Composition Sequences

In complex iteration theory, the case of analytic self-maps of the unit disk **D** is well understood (see e.g. [Va; P1; BP]). Its dynamics is simpler than that of rational functions [Be]. We want to generalize some results on sequences of iterates to composition sequences.

We consider the analytic functions

$$\varphi_n : \mathbf{D} \to \mathbf{D} \quad (n = 1, 2, \ldots) \tag{1.1}$$

and form the forward composition sequence (f_n) defined by $f_0(z) \equiv z$ and $f_n = \varphi_n \circ f_{n-1}$, that is,

$$f_n = \varphi_n \circ \cdots \circ \varphi_2 \circ \varphi_1 \quad (n = 1, 2, \ldots). \tag{1.2}$$

This is the same process as for iteration but with different functions. In other contexts the backward compositions $\varphi_1 \circ \cdots \circ \varphi_n$ are more important—for example, for continued fractions [JT] and for branching processes [AN; P2].

We write $z_n = f_n(0)$. The normalized functions

$$g_n(z) = \frac{f_n(z) - z_n}{1 - \bar{z}_n f_n(z)} = \frac{f_n(z) - f_n(0)}{1 - \overline{f_n(0)} f_n(z)} \tag{1.3}$$

again map the unit disk **D** into itself but such that $g_n(0) = 0$. It follows from (1.2) and (1.3) that

$$g_n'(0) = \frac{f_n'(0)}{1 - |z_n|^2} = \prod_{k=1}^{n} \frac{1 - |z_{k-1}|^2}{1 - |z_k|^2} \varphi_k'(z_{k-1}). \tag{1.4}$$

The factors are at most 1 in absolute value.

THEOREM. *If*

$$\alpha_n \equiv \frac{1 - |z_{n-1}|^2}{1 - |z_n|^2} |\varphi_n'(z_{n-1})| \geq \alpha > 0 \quad (z \in \mathbf{D}) \tag{1.5}$$

for $n = 1, 2, \ldots$, *then the limit*

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$$g(z)=\lim_{n\to\infty}\frac{g_n(z)}{g_n'(0)}=z+\cdots \tag{1.6}$$

exists locally uniformly in **D** *and g is univalent in*

$$\{|z|<\rho\},\quad \rho=\alpha/(1+\sqrt{1+\alpha^2}). \tag{1.7}$$

Moreover, if $|z|\le r<1$ *and* $m<n$ *then*

$$\left|\frac{g_m(z)}{g_m'(0)}-\frac{g_n(z)}{g_n'(0)}\right|\le M(r,\alpha)(|g_m'(0)|-|g_n'(0)|), \tag{1.8}$$

where $M(r,\alpha)$ *depends only on r and* α.

It follows from (1.4) and (2.8) below that

$$|g_n'(0)|\le|g_{n-1}'(0)|,\qquad |g_n(z)|\le|g_{n-1}(z)|\quad (z\in\mathbf{D}). \tag{1.9}$$

Hence $|g_n'(0)|$ and $|g_n(z)|$ converge as $n\to\infty$. There are two distinct cases:

(i) If $\lim_{n\to\infty}|g_n(z)|\equiv 0$ then $g_n'(0)\to 0$ as $n\to\infty$ and

$$f_n(z)-z_n=\frac{1-|z_n|^2}{1+\bar{z}_n g_n(z)}g_n(z)\sim(1-|z_n|^2)g_n(z).$$

Hence we conclude from (1.3), (1.4), and (1.6) that

$$g(z)=\lim_{n\to\infty}\frac{f_n(z)-f_n(0)}{f_n'(0)}\quad \text{locally uniformly in } \mathbf{D}. \tag{1.10}$$

(ii) If $\lim_{n\to\infty}|g_n(z)|\not\equiv 0$ then $|g_n'(0)|\to b\ne 0$ by (1.6) and therefore $|g(z)|<1/b<\infty$ for $z\in\mathbf{D}$.

In the iteration case we have $\varphi_n=f$ for all n so that f_n is the nth iterate. There is an important additional feature: We also have $f_n=f_{n-1}\circ f$, and it follows from (1.3) that there is a functional equation

$$g\circ f=\tau\circ g\quad \text{where } \tau \text{ is a Möbius transformation.}$$

The case (i) occurs if and only if

$$(z_n-z_{n-1})/(1-\bar{z}_{n-1}z_n)\to 0\quad \text{as } n\to\infty$$

[Va; BP], in particular if f has an attractive fixed point in **D**. In case (ii) the normalizations in [Va] and [P1] are somewhat different from that in (1.3).

In Section 3 we shall study the case where $\varphi_n(0)=0$ for all n. We shall see that the awkward condition (1.5) cannot be deleted and that the convergence is non-uniform (in the function data) if $|\varphi_n'(0)|$ is near to 1, even in the iteration case.

2. Proof of the Theorem

First we state some well-known facts that we need for the proof of the theorem. Let φ be analytic in **C** and let $\varphi(\mathbf{D})\subset\mathbf{D}$. If $\zeta\in\mathbf{D}$ we can write

$$\frac{\varphi(z)-\varphi(\zeta)}{1-\overline{\varphi(\zeta)}\varphi(z)} \equiv \psi(s) \equiv s\frac{a+\chi(s)}{1+\bar{a}\chi(s)}, \quad z=\frac{\zeta+s}{1+\bar{\zeta}s} \tag{2.1}$$

for $s \in \mathbf{D}$, where χ is analytic and

$$a=\frac{1-|\zeta|^2}{1-|\varphi(\zeta)|^2}\varphi'(\zeta), \qquad |\chi(s)| \leq |s| \quad (s \in \mathbf{D}). \tag{2.2}$$

It follows that, for $s \in \mathbf{D}$ and thus for $z \in \mathbf{D}$,

$$\left|\frac{\varphi(z)-\varphi(\zeta)}{1-\overline{\varphi(\zeta)}\varphi(z)}\right| \leq |s|\frac{|a|+|s|}{1+|as|}, \tag{2.3}$$

$$\left|\frac{\varphi(z)-\varphi(\zeta)}{1-\overline{\varphi(\zeta)}\varphi(z)}-as\right| \leq \frac{1-|a|^2}{1-|as|}|s|^2. \tag{2.4}$$

The analytic function ψ satisfies $\psi(\mathbf{D}) \subset \mathbf{D}$, $\psi(0)=0$, and $\psi'(0)=a$ and is therefore (by [Ne, p. 171]) univalent in $\{|s|<\rho\}$ where $\rho=|a|/(1+\sqrt{1-|a|^2})$. Hence (2.1) shows that

$$\varphi \text{ is univalent in } \{|(z-\zeta)/(1-\bar{\zeta}z)|<\rho\}. \tag{2.5}$$

We also need the elementary fact that, for $0 \leq \beta_n \leq 1$,

$$\sum_{n=1}^{\infty} \beta_1 \cdots \beta_{n-1}(1-\beta_n)=1-\lim_{n\to\infty} \beta_1 \cdots \beta_n \leq 1. \tag{2.6}$$

Proof. Since $f_n=\varphi_n \circ f_{n-1}$, it follows from (1.3) that

$$g_n(z)=\frac{\varphi_n(f_{n-1}(z))-\varphi_n(z_{n-1})}{1-\overline{\varphi_n(z_{n-1})}\varphi_n(f_{n-1}(z))} \quad (z \in \mathbf{D}). \tag{2.7}$$

We apply (2.3) and (2.4) with $\varphi=\varphi_n$ and

$$\zeta=z_{n-1}, \quad s=\frac{f_{n-1}(z)-z_{n-1}}{1-\bar{z}_{n-1}f_{n-1}(z)}=g_{n-1}(z), \quad a=\frac{1-|z_{n-1}|^2}{1-|z_n|^2}\varphi_n'(z_{n-1}).$$

Then $(\zeta+s)/(1+\bar{\zeta}s)=f_{n-1}(z)$ and we obtain, by (2.7) and (1.5), that

$$|g_n(z)| \leq |g_{n-1}(z)|\frac{\alpha_n+|g_{n-1}(z)|}{1+\alpha_n|g_{n-1}(z)|} \leq |g_{n-1}(z)| \tag{2.8}$$

and

$$\left|g_n(z)-\frac{1-|z_{n-1}|^2}{1-|z_n|^2}\varphi_n'(z_{n-1})g_{n-1}(z)\right| \leq \frac{1-\alpha_n^2}{1-|z|}|g_{n-1}(z)|^2, \tag{2.9}$$

because $|g_{n-1}(z)| \leq |z|$ by repeated application of (2.8). Since

$$(u+v)/(1+uv) \leq u+2(1-u)v \quad \text{for } 0 \leq u<1 \text{ and } 0 \leq v,$$

it follows from (2.8) and (1.5) that

$$|g_n(z)| \leq \alpha_n|g_{n-1}(z)|(1+2(1-\alpha_n)\alpha^{-1}|g_{n-1}(z)|). \tag{2.10}$$

Let $0<r<1$ and let $M_1, M_2, \ldots$ be constants that depend only on α and r. Since $(1-x)/\log(1/x)$ is increasing in $0<x<1$, we obtain from (1.5) that

$$\frac{\alpha_n+r}{1+\alpha_n r}\le \exp\left[-\frac{1-r}{2}(1-\alpha_n)\right]\le \exp\left[-c\log\frac{1}{\alpha_n}\right]=\alpha_n^c,$$

where $c=(1-r)(1-\alpha)/(2\log(1/\alpha))>0$. Now let $|z|\le r$. Then $|g_{n-1}(z)|\le r$, and it follows by repeated application of (2.8) that

$$|g_n(z)|\le \prod_{k=1}^{n}\frac{\alpha_k+r}{1+\alpha_k r}\le \prod_{k=1}^{n}\alpha_k^c.$$

Since $1-\alpha_k\le M_1(1-\alpha_k^c)$, we conclude from (2.6) that

$$\sum_{n=1}^{\infty}(1-\alpha_n)|g_{n-1}(z)|\le M_1\sum_{n=1}^{\infty}(1-\alpha_n^c)\alpha_1^c\cdots\alpha_{n-1}^c\le M_1.$$

Hence we obtain from (2.10) that

$$|g_n(z)|\le \prod_{k=1}^{n}(\alpha_k\exp[M_2(1-\alpha_k)|g_{k-1}(z)|])\le M_3\alpha_1\cdots\alpha_n.$$

Dividing (2.9) by $|g_n'(0)|$ we therefore obtain from (1.4) and (1.5) that, for $|z|\le r$,

$$\left|\frac{g_n(z)}{g_n'(0)}-\frac{g_{n-1}(z)}{g_{n-1}'(0)}\right|\le\frac{1-\alpha_n^2}{1-r}\frac{|g_{n-1}(z)|^2}{\alpha_1\cdots\alpha_n}$$
$$\le\frac{M_3^2(1-\alpha_n^2)}{(1-r)\alpha_n}\alpha_1\cdots\alpha_{n-1}\le M_4(1-\alpha_n)\alpha_1\cdots\alpha_{n-1},\qquad(2.11)$$

and it follows from (2.6) that the limit (1.6) exists locally uniformly in $\mathbf{D}$.

We see from (1.5) and (2.5) that φ_n is univalent in $\{|z-z_{n-1}|/|1-\bar z_{n-1}z|<\rho\}$. Since

$$|(f_{n-1}(z)-z_{n-1})/(1-\bar z_{n-1}f_{n-1}(z))|=|g_{n-1}(z)|<\rho\quad\text{for } |z|<\rho,$$

and since $f_n=\varphi_n\circ f_{n-1}$, it follows by induction that f_n is univalent in $\{|z|<\rho\}$ for all n and thus g is also, by (1.6).

Finally, it follows from (2.11) that, for $|z|\le r$ and $m<n$,

$$\left|\frac{g_m(z)}{g_m'(0)}-\frac{g_n(z)}{g_n'(0)}\right|\le M_4\sum_{k=m+1}^{n}(1-\alpha_k)\alpha_1\cdots\alpha_{k-1}$$
$$=M_4(\alpha_1\cdots\alpha_m-\alpha_1\cdots\alpha_n),$$

which implies (1.8), by (1.4) and (1.5). □

3. Composition with Fixed Point 0

We now consider the case where $\varphi_n(0)=0$ and thus $z_n=0$ for all n.

COROLLARY. *Let $\varphi_n:\mathbf{D}\to\mathbf{D}$ be analytic,*

$$\varphi_n(z)=a_nz+b_nz^2+\cdots,\quad |a_n|\ge\alpha>0,\qquad(3.1)$$

and $f_n=\varphi_n\circ\cdots\circ\varphi_1$ *for* $n=1,2,\ldots$. *If* $f_n'(0)=a_1\cdots a_n\to 0$ *as* $n\to\infty$ *then*

$$g(z)=\lim_{n\to\infty}\frac{f_n(z)}{f_n'(0)}=z+\left(\sum_{\nu=1}^{\infty}\frac{b_\nu}{a_\nu}a_1\cdots a_{\nu-1}\right)z^2+\cdots \tag{3.2}$$

exists locally uniformly in $\mathbf{D}$ *and is univalent in* $\{|z|<\alpha/(1+\sqrt{1-\alpha^2})\}$.

This is an immediate consequence of our theorem; see (1.10). The coefficient in (3.2) is obtained as follows: We see from (3.1) that

$$f_{n+1}''(0)=\varphi_{n+1}''(0)f_n'(0)^2+\varphi_{n+1}'(0)f_n''(0)=2b_{n+1}(a_1\cdots a_n)^2+a_{n+1}f_n''(0)$$

and thus by induction that

$$f_n''(0)=2a_1\cdots a_n\sum_{\nu=1}^{n}\frac{b_\nu}{a_\nu}a_1\cdots a_{\nu-1}\quad (n=1,2,\ldots).$$

Our first example shows that the assumption $|a_n|\geq\alpha>0$ in (3.1) cannot be replaced by $a_n\neq 0$; compare also (1.5).

EXAMPLE 1. *If* $0<b<1$, $0<c<1$, *and*

$$\varphi_n(z)=z\frac{\alpha_n+z}{1+\alpha_n z}\quad \textit{with}\quad \alpha_n=bc^{2^{n-1}}\quad (n=1,2,\ldots), \tag{3.3}$$

then $f_n(x)/f_n'(0)\to\infty$ $(n\to\infty)$ *for some* $x\in\mathbf{D}$.

Proof. Let $0<x<1$. We write $p_n=f_n(x)/\alpha_{n+1}$ and see from (3.3) that

$$p_{n+1}=\frac{f_n(x)}{bc^{2^{n+1}}}\frac{bc^{2^n}+f_n(x)}{1+bc^{2^n}f_n(x)}=bp_n\frac{1+p_n}{1+\alpha_{n+1}^2p_n}.$$

The positive fixed point ξ_n of $bx(1+x)/(1+\alpha_{n+1}^2x)$ satisfies

$$\xi_n=\frac{1-b}{b-b^2c^{2^{n+1}}}>\xi_{n+1}>\frac{1-b}{b}. \tag{3.4}$$

We choose x with $(1-b)c/(1-bc^2)<x<1$. Then $p_0=x/\alpha_1>\xi_0$, and if $p_n>\xi_n$ then $p_{n+1}>p_n>\xi_n>\xi_{n+1}$ by (3.4). Hence $p_n>\xi_n$ holds for all n and thus, by (3.4),

$$\frac{f_n(x)}{f_n'(0)}=\frac{bc^{2^n}p_n}{b^nc^{2^n-1}}>\frac{(1-b)c}{b^n}\to\infty. \qquad \square$$

EXAMPLE 2. *For* $k=1,2,\ldots$, *let* $0<a_{kn}<1$ *and*

$$\varphi_{kn}(z)=z\frac{a_{kn}+z}{1+a_{kn}z}=a_{kn}z+(1-a_{kn}^2)z^2+\cdots. \tag{3.5}$$

If $a_{k1}\cdots a_{kn}\to 0$ $(n\to\infty)$ *for each* k *and* $\inf_n a_{kn}\to 1$ $(k\to\infty)$, *then* $f_{kn}=\varphi_{kn}\circ\cdots\circ\varphi_{1n}$ *satisfies*

$$\lim_{n\to\infty}\frac{f_{kn}(z)}{f_{kn}'(0)}\to\frac{z}{(1-z)^2}\quad \textit{as } k\to\infty,\quad a\in\mathbf{D}. \tag{3.6}$$

Since $\varphi_{kn}(z) \to z$ as $k \to \infty$ uniformly in n, one might have expected that the limit function in (3.6) is z. This shows that the convergence in (3.2) is not uniform in the function data if $|\varphi_n'(0)|$ is near to 1.

Proof. Let $k = 1, 2, \ldots$. Since $f_{kn}'(0) = a_{k1} \cdots a_{kn} \to 0$ as $n \to \infty$, we conclude that

$$h_k(z) \equiv \lim_{n\to\infty} \frac{f_{kn}(z)}{f_{kn}'(0)} = z + c_k z^2 + \cdots \quad \text{and}$$

$$c_k \equiv \sum_{\nu=1}^{\infty} \frac{1-a_{k\nu}^2}{a_{kn}} a_{k1} \cdots a_{k,\nu-1} > 2 \sum_{\nu=1}^{\infty} (1-a_{k\nu}) a_{k1} \cdots a_{k,\nu-1} = 2 \tag{3.7}$$

by (2.6). Since $\inf_n a_{kn} \to 1$ as $k \to \infty$, the corollary shows that every limit function $h(z) = z + cz^2 + \cdots$ of (h_k) is univalent in $\mathbf{D}$. We see from (3.7) that $c \geq 2$ so that $h(z) = z/(1-z)^2$, the Koebe function [Du, p. 30]. □

Our final example from iteration theory emphasizes this non-uniformity. We denote the iterates by f^n and the unit circle $\partial\mathbf{D}$ by $\mathbf{T}$. See [Yo, II.2.6] for the case where λ_0 is not a root of unity; see [HR] for further results.

EXAMPLE 3. *For $\lambda \in \mathbf{D}$, $\lambda \neq 0$, let*

$$f_\lambda(z) = z\frac{\lambda + z}{1+\bar{\lambda}z} = \lambda z + (1-|\lambda|^2) \sum_{k=2}^{\infty} \bar{\lambda}^{k-2}(-z)^k \tag{3.8}$$

and let $g_\lambda = \lim_{n\to\infty} \lambda^{-n} f_\lambda^n$. If $\lambda_0 \in \mathbf{T}$ is not a root of unity then, as $\lambda \to \lambda_0$,

$$g_\lambda(z) \to z \ \textit{ locally uniformly in } z \in \mathbf{D}. \tag{3.9}$$

If λ_0 is a primitive mth *root of unity then, as $\rho \to 1$,*

$$g_{\rho\lambda_0}(z) \to z(1+(-z)^m)^{-2/m} \ \textit{ locally uniformly in } \mathbf{D} \tag{3.10}$$

whereas (3.9) *holds if $\lambda \to \lambda_0$ tangentially to* $\mathbf{R}$.

Proof. It is easy to see [Va, p. 116] that

$$\lambda g_\lambda(z) = g_\lambda(f_\lambda(z)) \quad \text{for } z \in \mathbf{D},\ 0 < |\lambda| < 1, \tag{3.11}$$

the Koenigs functional equation. We write

$$g_\lambda(z) = \sum_{k=1}^{\infty} b_k(\lambda) z^k \quad (z \in \mathbf{D}), \qquad b_1(\lambda) \equiv 1 \tag{3.12}$$

and obtain from (3.8) and (3.11) that, as $|\lambda| \to 1$,

$$\sum_{k=1}^{\infty} \lambda b_k(\lambda) z^k = \sum_{k=1}^{\infty} \lambda^k b_k(\lambda) z^k + O(1-|\lambda|)$$

locally uniformly in $z \in \mathbf{D}$. Hence, for each fixed k,

$$(1-\lambda^{k-1}) b_k(\lambda) = O(1-|\lambda|) \quad \text{as } |\lambda| \to 1. \tag{3.13}$$

If λ_0 is not a root of unity it follows that

$$b_k(\lambda) \to 0 \quad (\lambda \to \lambda_0) \quad \text{for } k \geq 2. \tag{3.14}$$

This also holds if λ_0 is a root of unity and $\lambda \to \lambda_0$ tangentially to **T**, because then $(1-|\lambda|)/|\lambda_0-\lambda| \to 0$.

Now let λ_0 be a primitive mth root of unity and let $\lambda = \rho\lambda_0$, $0<\rho<1$. It follows from (3.13) that

$$b_k(\lambda) = O(1-\rho)(\rho \to 1) \quad \text{for } k \neq \nu m+1 \ (\nu = 0, 1, \ldots) \tag{3.15}$$

so that, by (3.8) and (3.11),

$$\sum_{\nu=0}^{\infty} b_{m\nu+1}(\lambda) f_\lambda(z)^{m\nu+1} + \sum_{k \neq \nu m+1}^{\infty} \lambda^k b_k(\lambda) z^k = \sum_{k=1}^{\infty} \lambda b_k(\lambda) z^k + O((1-\rho)^2).$$

Comparing the coefficients of z^{m+1} we thus obtain from (3.8) and $b_1(\lambda) = 1$ that

$$(-1)^{m+1}(1-\rho^2)\bar{\lambda}^{m-1} + \lambda^{m+1} b_{m+1}(\lambda) = \lambda b_{m+1}(\lambda) + O((1-\rho)^2).$$

Since $\lambda_0^m = 1$ it follows that, as $\rho \to 1$,

$$b_{m+1}(\lambda) = (-1)^{m+1}\frac{1-\rho^2}{\rho(1-\rho^m)} + O(1-\rho) \to (-1)^{m+1}\frac{2}{m}. \tag{3.16}$$

The corollary shows that every limit function $h(z) = z + c_2 z^2 + \cdots$ of (g_λ) as $|\lambda| \to 1$ is univalent in **D**. If λ_0 is not a root of unity or if the approach is tangential then (3.14) shows that $h(z) = z$. Now let λ_0 be a root of unity. Then, for radial approach,

$$h(z) = z + \sum_{\nu=1}^{\infty} c_{m\nu+1} z^{m\nu+1}, \qquad c_{m+1} = (-1)^{m+1}\frac{2}{m},$$

by (3.15) and (3.16). The function

$$[h(z^{1/m})]^m = z + 2(-1)^{m+1} z^2 + \cdots$$

is again analytic and univalent and is therefore $= z/(1+(-1)^m z)^2$ [Du, p. 30], which implies (3.10). □

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