Affine Anosov Actions

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1. Introduction

In this paper we give a general procedure for constructing examples of affine Anosov actions without fixed points on nilmanifolds.

The simplest affine action is a *linear* or *standard* action, which is an affine action with a fixed point. The existence of a fixed point for an affine action on a flat torus is well known to be related to a group extension problem and the vanishing of corresponding cohomology groups. We also include in this paper the extension of these results to nilmanifolds, obtaining homological criteria for fixed points for affine actions on nilmanifolds.

There is a very active ongoing program to classify the Anosov actions of "large" groups on tori and nilmanifolds [8; 9; 10; 11; 12; 13; 14; 15; 18; 19]. The groups considered in this program are either free abelian groups of rank greater than 1, or they are a discrete subgroup of finite covolume of a connected semisimple algebraic $\mathbf{R}$-group G, where the $\mathbf{R}$-split rank of each factor of G is at least 2, G has finite center, and $G^0_{\mathbf{R}}$ has no compact factors. The group is considered to be "large" because nontrivial actions of the group on a manifold are expected to be rare owing to the additional structure imposed by the group properties. In fact, the basic program can be viewed as relating the "geometry at infinity" of the group with the geometry of an action, and as a consequence it can sometimes be shown that an Anosov action of such a group on a nilmanifold is always affine. One is thus left with the problem of classifying the affine Anosov actions of groups on tori and nilmanifolds. There is remarkably little known about this problem, and the constructions of examples in this paper are a first step toward its solution.

A point $x \in X$ is *periodic* for an action φ if the set

$$\Gamma(x) \stackrel{\text{def}}{=} \{\varphi(\gamma)(x) \mid \gamma \in \Gamma\}$$

is finite. Let $\Lambda(\varphi) \subset X$ denote the set of periodic points for φ.

THEOREM 1. *For each $n \geq 2$ and $p > 1$, there exists a lattice subgroup*

$$\Gamma(n,p) \subset SL(n,\mathbf{Z})$$

Received March 9, 1992. Revision received November 22, 1992.
The author was supported in part by NSF Grant DMS 91-03297.
Michigan Math. J. 40 (1993).

and an affine action φ with linear part given by the standard action of $\Gamma(n,p)$ on $\mathbf{T}^n$, such that:

(1) *$SL(n,\mathbf{Z})_{p^2} \subset \Gamma(n,p) \subset SL(n,\mathbf{Z})_p$, where $SL(n,\mathbf{Z})_p$ denotes the congruence p-subgroup;*

(2) *the restricted action of φ to $SL(n,\mathbf{Z})_{p^2}$ is the standard linear action; and*

(3) *the affine action φ of $\Gamma(n,p)$ has a dense set of periodic orbits but has no fixed points.*

A C^r-action $\varphi: \Gamma \times X$ of a group Γ on a compact manifold X is said to be *Anosov* if there exists at least one element, $\gamma_h \in \Gamma$, such that $\varphi(\gamma_h)$ is an Anosov diffeomorphism of X. We then say that γ_h is *φ-hyperbolic.* For an Anosov action, the set of periodic points $\Lambda(\varphi)$ is at most countable, as each power $p > 0$ of a φ-hyperbolic element $\varphi(\gamma_h^p)$ has a finite set of fixed points.

COROLLARY 1. *For each $n \geq 2$, there exists a subgroup $\Gamma \subset SL(n,\mathbf{Z})$ of finite index and an Anosov affine action $\varphi: \Gamma \times \mathbf{T}^n \to \mathbf{T}^n$ such that $\Lambda(\varphi)$ is dense, but there is no fixed point for the action of the full group Γ.*

The examples constructed in Theorem 1 make key use of the unipotent elements in the representation of the lattice. Therefore, the restrictions of these actions to an abelian semisimple subgroup of Γ will have a fixed point. However, for abelian subgroups there is a more direct technique for constructing actions without fixed points, described in Section 5, which yields the following theorem.

THEOREM 2. *Let Γ be a free abelian group of rank $r \geq 2$, and let*

$$\varphi_0: \Gamma \to SL(n,\mathbf{Z})$$

be a representation such that $\varphi_0(\gamma_h)$ is hyperbolic for some $\gamma_h \in \Gamma$. Then there exists a subgroup $\Gamma' \subset \Gamma$ of finite index and an affine Anosov action φ of Γ' on $\mathbf{T}^n$ with linear part $\varphi_0 | \Gamma'$ and no fixed points.

For example, given two commuting hyperbolic matrices $A, B \in SL(n,\mathbf{Z})$ which generate a free abelian group of rank 2, there is an affine action of $\mathbf{Z}^2$ on $\mathbf{T}^n$ with no fixed points, and the linear action is given by the group $\langle A^p, B^q \rangle$ for some integers $p, q \geq 1$. In another direction, let $\mathbf{Z}^2 = \langle \gamma_1, \gamma_2 \rangle$ act on $\mathbf{T}^2$, with $\varphi_0(\gamma_1) = \varphi_0(\gamma_2) = A \in SL(2,\mathbf{Z})$ for a hyperbolic matrix A which admits more than one fixed point. Then there exists an affine action $\varphi: \mathbf{Z}^2 \times \mathbf{T}^2 \to \mathbf{T}^2$ with linear part φ_0 and no fixed points.

A compact *nilmanifold* is a quotient $X = \Lambda \backslash \mathfrak{N}$ of a simply connected, nilpotent Lie group $\mathfrak{N}$ by a cocompact lattice $\Lambda \subset \mathfrak{N}$. A basis for the Lie algebra $\mathbf{n}$ of $\mathfrak{N}$ defines a set of *left-invariant* vector fields on $\mathfrak{N}$ which descend to a frame field for TX. A diffeomorphism $f: X \to X$ is said to be *affine* if the derivative map $Df: TX \to TX$ is implemented by a constant matrix map with respect to the framing of TX given by left-invariant vector fields. A differentiable action $\varphi: \Gamma \times X \to X$ is affine if the action of each $\gamma \in \Gamma$ is affine.

We introduce the descending series of commutators for the nilpotent Lie group $\mathfrak{N}$,

$$1=\mathfrak{N}_k \subset \mathfrak{N}_{k-1} \subset \cdots \subset \mathfrak{N}_1 \subset \mathfrak{N}_0 = \mathfrak{N}, \tag{1}$$

where $\mathfrak{N}_{i+1}=[\mathfrak{N}_i,\mathfrak{N}]$. The abelian quotient group $\mathfrak{A}_i=\mathfrak{N}_{i-1}/\mathfrak{N}_i$ is a real vector space. The automorphism group $\mathrm{Aut}(\mathfrak{N})$ preserves the descending series, and so induces actions on the quotient groups $\mathfrak{A}_i$ for all i. There is a corresponding descending series for Λ,

$$1=\Lambda_k \subset \Lambda_{k-1} \subset \cdots \subset \Lambda_1 \subset \Lambda_0 = \Lambda,$$

where $\Lambda_i=\Lambda \cap \mathfrak{N}_i$ is a cocompact lattice in $\mathfrak{N}_i$ (cf. [1, Chap. I]) with abelian intermediate quotient groups $\mathfrak{Q}_i=\Lambda_{i-1}/\Lambda_i$. The action φ preserves the algebraic structures associated to $\mathfrak{N}$, and so induces homomorphisms Φ_i: $\Gamma \to GL(\mathfrak{A}_i)$ and $\Phi_i\colon \Gamma \to \mathrm{Aut}(\mathfrak{Q}_i)$.

For a representation $\rho\colon\Gamma\to\mathrm{Aut}(\mathbf{V})$ on an abelian group $\mathbf{V}$, let $H^*(\Gamma;\mathbf{V}_\rho)$ denote the cohomology groups of Γ with coefficients in the Γ-module $\mathbf{V}$. We abuse notation for the cases of $\mathbf{V}=\mathfrak{A}_i$ and $\mathfrak{Q}_i$ by denoting the cohomology groups associated to the actions Φ_i induced from φ simply by $H^1(\Gamma;\mathfrak{A}_i)$ and $H^1(\Gamma;\mathfrak{Q}_i)$. The following result is almost folklore, but we include the proof here as the author has not found it in the literature, and the techniques of proof are useful in the general study of affine actions on nilmanifolds.

THEOREM 3. *Let $\varphi\colon\Gamma\times X\to X$ be an affine action of a finitely generated group Γ on a compact nilmanifold X. Suppose that $H^1(\Gamma;\mathfrak{A}_i)=0$ for each $1\le i\le k$; then the action has a periodic orbit and the periodic points $\Lambda(\varphi)$ are dense in X. If, in addition, $H^2(\Gamma;\mathfrak{Q}_i)$ is torsion-free for each $1\le i\le k$, then φ has a fixed point.*

A finitely generated group Γ is a said to be a *higher-rank lattice* if Γ is a discrete subgroup of a connected semisimple algebraic $\mathbf{R}$-group G, where the $\mathbf{R}$-split rank of each factor of G is at least 2, G has finite center, $G^0_{\mathbf{R}}$ has no compact factors, and G/Γ has finite volume. A remarkable result of Margulis [16, Thm. 3] implies that $H^1(\Gamma;\mathbf{R}^m_\rho)=0$ for Γ a higher-rank lattice, where ρ is any representation.

COROLLARY 2. *Let $\varphi\colon\Gamma\times X\to X$ be an affine action of a higher-rank lattice Γ on a compact nilmanifold X. Then the periodic orbits of φ are dense in X.*

The existence of a fixed point for an affine action of a higher-rank lattice is more delicate to show, for this requires knowledge of the torsion in the groups $H^2(\Gamma;\mathfrak{Q}_i)$ for $1\le i\le k$. Very little seems to be known about torsion in the second cohomology group of lattices with coefficients in a $\mathbf{Z}$-module (cf. [4; 5]).

We conclude our discussion of results with the following problem.

PROBLEM 1. Let $\varphi\colon\Gamma\times X\to X$ be an Anosov action of a higher-rank lattice Γ on a compact manifold X, and suppose that there exists a probability measure μ on X invariant under the action. Show that the set of periodic points $\Lambda(\varphi)$ is dense in X.

R. Trapp has observed that there is an affine action of the semidirect product $\Gamma = SL(n, \mathbf{Z}) \times \mathbf{Z}^n$ on $\mathbf{T}^n$ without fixed points, and this action preserves the Haar measure. The group Γ is a uniform lattice in $SL(n, \mathbf{R}) \times \mathbf{R}^n$, which is not semisimple. Thus, the hypothesis in the problem that Γ is a lattice in a semisimple Lie group is necessary.

The remainder of the paper is organized as follows. The special case of an affine action on a torus is discussed in Section 2. Generalities about affine actions on nilmanifolds are discussed in Section 3, where we give the general form of the criteria for the existence of a fixed point for the action. Section 4 gives examples and constructions that prove Theorem 1, and Section 5 gives the constructions for abelian actions proving Theorem 2.

The author is grateful to J. Lewis for helpful conversations, especially for his careful explanations of the material of Section 2 on toral actions, and for raising the question of whether it was possible to have Anosov actions without fixed points.

2. Affine Actions on Tori

The study of affine actions on the torus $X = \mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n$ is the most straightforward to describe, as it can be approached from a purely group-theoretic perspective using the abelian group structure of $\mathbf{T}^n$. In this section, we recall this basic background material (cf. the discussion preceding [14, Lemma 2.6]).

Fix an action of Γ on $\mathbf{T}^n$ by automorphisms,

$$\varphi_0 : \Gamma \to \mathrm{Aut}(\mathbf{T}^n) \cong GL(n, \mathbf{Z}).$$

A map $\tau : \Gamma \to \mathbf{T}^n$ is a 1-cocycle over the action φ_0 if for each $\gamma_1, \gamma_2 \in \Gamma$,

$$\tau(\gamma_1\gamma_2) = \tau(\gamma_1) \cdot \varphi_0(\gamma_1)(\tau(\gamma_2)) \tag{2}$$

A 1-cocycle is *trivial* if there exists a point $x_0 \in \mathbf{T}^n$ such that

$$\tau(\gamma) = \varphi_0(\gamma)(x_0)^{-1} \cdot x_0 \tag{3}$$

for all $\gamma \in \Gamma$. The cohomology group $H^1(\Gamma; \mathbf{T}^n_{\varphi_0})$ is the quotient space of the 1-cocycles modulo the trivial 1-cocycles.

The basic structure of affine actions is expressed in terms of 1-cocycles and 1-cohomology in the following manner.

PROPOSITION 1. *Let Γ be a finitely generated group, and fix a representation $\varphi_0 : \Gamma \to GL(n, \mathbf{Z})$.*

(1) *There is a one-to-one correspondence between the affine actions φ of Γ on $\mathbf{T}^n$ with linear part φ_0 and the 1-cocycles τ over the action φ_0.*

Let τ_φ denote the 1-cocycle over φ_0 associated to an action φ. The translational class *associated to φ is the cohomology class $[\tau_\varphi] \in H^1(\Gamma; \mathbf{T}^n_{\varphi_0})$.*

(2) *An affine action φ with linear part φ_0 has a fixed point if and only if $[\tau_\varphi]$ is trivial.*
(3) *An affine action φ with linear part φ_0 has a dense set of periodic points if and only if $[\tau_\varphi]$ is torsion.*

Proof. (1) Given an affine transformation $\varphi(\gamma)$, define the translational part $\tau_\varphi(\gamma) \in \mathbf{T}^n$ by its action on the identity element $\tau(\gamma)(0) = \varphi(\gamma)(0)$. The group law for the action φ translates into the cocycle law (2) over the linear action φ_0. Conversely, given a 1-cocycle $\tau: \Gamma \to \mathbf{T}^n$ over φ_0, define an affine action by the rule $\varphi(\gamma)(x) = \tau(\gamma) \cdot \varphi_0(\gamma)(x)$ for $x \in \mathbf{T}^n$ and $\gamma \in \Gamma$.

(2) Let x_0 be a fixed point for the action. For each $\gamma \in \Gamma$, $\tau(\gamma) \cdot \varphi_0(\gamma)(x_0) = x_0$ or

$$\tau(\gamma) = x_0 \cdot \varphi_0(\gamma)(x_0)^{-1} = \varphi_0(\gamma)(x_0)^{-1} \cdot x_0,$$

so that τ is a trivial 1-cocycle. Conversely, let x_0 satisfy equation (3); then calculate

$$\varphi_\tau(\gamma)(x_0) = \tau(\gamma) \cdot \varphi_0(\gamma)(x_0) = \tau(\gamma) \cdot \tau(\gamma)^{-1} \cdot x_0 = x_0.$$

(3) Suppose that φ is an affine action with a periodic orbit $x_0 \in \mathbf{T}^n$. The stabilizer of the full orbit of x_0 is a normal subgroup $\Gamma' \subset \Gamma$ with finite index. Let $R: H^1(\Gamma; \mathbf{T}^n_{\varphi_0}) \to H^1(\Gamma'; \mathbf{T}^n_{\varphi_0})$ be the restriction map, and let $T: H^1(\Gamma'; \mathbf{T}^n_{\varphi_0}) \to H^1(\Gamma; \mathbf{T}^n_{\varphi_0})$ be the transfer. The restriction class $R[\tau_\varphi] \in H^1(\Gamma'; \mathbf{T}^n_{\varphi_0})$ is zero by (2). The composition $T \circ R = [\Gamma : \Gamma'] \cdot \mathrm{Id}$ (cf. [6, Prop. 10.1]), so that $0 = T \circ R[\tau_\varphi] = [\Gamma : \Gamma'] \cdot [\tau_\varphi]$, which implies that $[\tau_\varphi]$ is annihilated by the order $[\Gamma : \Gamma']$ and hence is a torsion class.

Conversely, suppose that there exists $p > 1$ such that

$$p \cdot \tau(\gamma) = \varphi_0(\gamma)(x_0)^{-1} \cdot x_0$$

for all $\gamma \in \Gamma$. Then choose $y_0 \in \mathbf{T}^n$ so that $p \cdot y_0 = x_0$, and calculate that

$$p \cdot (\tau(\gamma) \cdot \varphi_0(\gamma)(y_0) \cdot y_0^{-1}) = e \in \mathbf{T}^n.$$

Hence τ is cohomologous to a cocycle τ' which takes values in the subgroup $((1/p) \cdot \mathbf{Z})/\mathbf{Z} \subset \mathbf{T}^n$. The proof of the proposition is now completed by the following lemma.

LEMMA 1. *Let φ be an affine action defined by a translational cocycle τ'_φ with values in a finite subgroup $\mathcal{P} \subset \mathbf{T}^n$. Then the action φ has a dense set of periodic points.*

Proof. The isotropy subgroup $\Gamma_{\mathcal{P}} \subset \Gamma$ of the set $\mathcal{P}$ for the linear action φ_0 is a normal subgroup of finite index. The restriction of τ'_φ to $\Gamma_{\mathcal{P}}$ is a homomorphism $\tau'_\varphi: \Gamma_{\mathcal{P}} \to \mathcal{P}$ with kernel $\Gamma' \subset \Gamma_{\mathcal{P}}$ of finite index also. The restriction of the action φ to Γ' is just the standard action. It follows that every rational point of $\mathbf{T}^n$ is a periodic point for the action φ. □

Next we consider the long exact sequence of homology groups associated with the exact sequence of Γ-modules $1 \to \mathbf{Z}^n \to \mathbf{R}^n \to \mathbf{T}^n \to 1$:

$$\cdots \to H^1(\Gamma; \mathbf{R}^n_\varphi) \xrightarrow{q} H^1(\Gamma; \mathbf{T}^n_{\varphi_0}) \xrightarrow{\delta} H^2(\Gamma; \mathbf{Z}^n_{\varphi_0}) \xrightarrow{\iota} H^2(\Gamma; \mathbf{R}^n_{\varphi_0}) \to \cdots. \quad (4)$$

As pointed out to the author by J. Lewis, one can identify the kernel of $H^2(\Gamma;\mathbf{Z}^n_{\varphi_0}) \xrightarrow{\iota} H^2(\Gamma;\mathbf{R}^n_{\varphi_0})$ with the torsion subgroup of $H^2(\Gamma;\mathbf{Z}^n_{\varphi_0})$. We therefore have this corollary.

COROLLARY 3. *Let Γ be a finitely generated group, and fix a representation $\varphi_0\colon \Gamma \to GL(n,\mathbf{Z})$. Suppose that*

$H^1(\Gamma;\mathbf{R}^n_{\varphi_0})=0$ and
$H^2(\Gamma;\mathbf{Z}^n_{\varphi_0})$ is torsion free.

Then every affine action φ with linear part φ_0 has a fixed point.

3. Group Structure of Affine Actions

In this section we consider the general case of an affine action $\varphi\colon \Gamma\times X\to X$ on a nilmanifold $X=\Lambda\backslash\mathfrak{N}$ of a simply connected, nilpotent Lie group $\mathfrak{N}$ by a cocompact lattice $\Lambda\subset\mathfrak{N}$. Examples of single Anosov diffeomorphisms on nilmanifolds have been constructed in varying degrees of generality in the literature [2; 3; 7; 17]. These constructions can be extended to yield linear Anosov actions of arithmetic lattices on nilmanifolds, and the methods of the next section yield affine actions on the same nilmanifolds, without fixed points. In this section, we establish some basic properties of affine actions on compact nilmanifolds. Propositions 2 and 3 extend the conclusions of Proposition 1 to the case of nilmanifolds. We use the definitions and terminology formulated in Section 1.

Let $\mathcal{G}$ denote the real algebraic group of (left) affine transformations of $\mathfrak{N}$. Thus, $\mathcal{G}$ is an extension of the linear algebraic group $\mathrm{Aut}(\mathfrak{N})$ of Lie group automorphisms by the normal subgroup $\mathfrak{N}$ of left affine translations. We write a typical element $g\in\mathcal{G}$ in the form (t_g, A_g), with $A_g\in\mathrm{Aut}(\mathfrak{N})$ and $t_g\in\mathfrak{N}$; the product is given by

$$(t_{g_1},A_{g_1})\cdot(t_{g_2},A_{g_2})=(t_{g_1}\cdot A_{g_1}(t_{g_2})),A_{g_1}A_{g_2}), \tag{5}$$

corresponding to the left action on $\mathfrak{N}$, first by $\mathrm{Aut}(\mathfrak{N})$, then by translation.

LEMMA 2. *For each $\gamma\in\Gamma$, the diffeomorphism $\varphi(\gamma)$ of X lifts to a diffeomorphism $\widetilde{\varphi(\gamma)}\in\mathcal{G}$ of $\mathfrak{N}$.*

Proof. Choose a lift $\widetilde{\varphi(\gamma)}$ of $\varphi(\gamma)$. Set $\tau(\widetilde{\varphi(\gamma)})=\widetilde{\varphi(\gamma)}(e)\in\mathfrak{N}$, and compose with translation on the left by $\tau(\widetilde{\varphi(\gamma)})^{-1}$ to obtain a diffeomorphism γ_A which fixes the identity element $e\in\mathfrak{N}$. It suffices to show that $\gamma_A\in\mathrm{Aut}(\mathfrak{N})$. By the assumption that $\varphi(\gamma)$ is affine, $D\gamma_A$ induces a linear map on the Lie algebra $\mathbf{n}$ of left-invariant vector fields on $\mathfrak{N}$. The induced action is a Lie algebra automorphism, as a diffeomorphism preserves the Lie bracket operation on vector fields. Hence γ_A is a Lie group automorphism of $\mathfrak{N}$. □

Let $\Gamma_A\subset\mathcal{G}$ be the transformation group of $\mathfrak{N}$ generated by the lifts $\widetilde{\varphi(\gamma)}$ of the diffeomorphisms $\varphi(\gamma)$ to the universal cover $\mathfrak{N}$ of X, together with the left translations by elements of Λ.

Corollary 4. *The action φ determines an exact sequence of groups*

$$1 \to \Lambda \to \Gamma_A \xrightarrow{\pi} \Gamma_X \to 1, \tag{6}$$

where Γ_X is the image of Γ in $\mathrm{Diff}(X)$ *and π is the restriction of the quotient map* $\mathcal{G} \to \mathrm{Diff}(X)$.

The basic plan for extending the results of section 2 is to use the fact that every compact nilmanifold X admits a fibration with base a torus and fiber a nilmanifold of lower rank. This suggests an obvious inductive approach, which can be implemented using two additional observations: This fibration is defined algebraically, and so is preserved by the actions of elements of $\mathrm{Aut}(\mathfrak{N})$ which induce actions on X; the translational actions of elements of $\mathfrak{N}$ on X induce translational actions on the toral quotient. In the following, we carefully establish the technical aspects of this inductive procedure, but the reader is advised to view the process as "finding a fixed point or periodic orbit one torus factor at a time".

First, consider the translational action. For each $g \in \mathfrak{N}$ and $h \in \mathfrak{N}_i$, the conjugate $ghg^{-1}h^{-1} \in \mathfrak{N}_{i+1}$. Note that this implies that for any right coset $\mathfrak{N}_{i+1} \cdot h$, $g \cdot \mathfrak{N}_{i+1} \cdot h = \mathfrak{N}_{i+1} \cdot gh$, so that left translation by any $g \in \mathfrak{N}$ preserves the right cosets. We conclude that for each $1 \le i \le k$ there is an induced translational action of $g \in \mathfrak{N}_{i-1}$ on the quotient manifold

$$\mathbf{T}^{n_i} = (\Lambda_{i-1} \cdot \mathfrak{N}_i) \backslash \mathfrak{N}_{i-1}.$$

If we restrict to $g \in \Lambda$, then conjugation by g maps Λ_i into Λ_{i+1} and hence $g \cdot \Lambda_i h = \Lambda_i gh$.

Next define the homomorphisms $\Phi_i : \Gamma \to \mathrm{Aut}(\mathcal{Q}_i)$ of Section 1. Fix $1 \le i \le k$; then, for each $\gamma \in \Gamma$, choose $\varphi(\gamma) \in \Gamma_A$ which maps to $\widetilde{\varphi(\gamma)}$ and let $\gamma_A \in \mathrm{Aut}(\mathfrak{N})$ be the projection defined in Lemma 2. The action of $\mathrm{Aut}(\mathfrak{N})$ on $\mathfrak{N}$ preserves the filtration (1), so for $y \in \mathfrak{N}_{i-1}$ set

$$\Phi_i(\gamma)(y \bmod \mathfrak{N}_i) = \gamma_A(y) \bmod \mathfrak{N}_i;$$

we obtain a well-defined action on $\mathcal{A}_i = \mathfrak{N}_i \backslash \mathfrak{N}_{i-1}$. We need the following extension of this remark to the action on lattices as well.

Lemma 3. *The induced action of Γ on $\mathfrak{N}_i$ preserves the subgroup $\Lambda_i \cdot \mathfrak{N}_{i+1}$.*

Proof. For each $\gamma \in \Gamma$, the action of $\widetilde{\varphi(\gamma)}$ on $\mathfrak{N}$ descends to an action on each quotient space $X_i = \Lambda_i \backslash \mathfrak{N}_i$ (as these quotients are themselves coverings of X), so that $\widetilde{\varphi(\gamma)}(\Lambda_i \cdot x) = \Lambda_i \cdot \widetilde{\varphi(\gamma)}(x)$ for all $x \in \mathfrak{N}_i$. That is,

$$\tau(\widetilde{\varphi(\gamma)}) \cdot \gamma_A(\Lambda_i \cdot x) = \Lambda_i \cdot \tau(\widetilde{\varphi(\gamma)}) \cdot \gamma_A(x);$$

hence $\tau(\widetilde{\varphi(\gamma)}) \cdot \gamma_A(\Lambda_i) = \Lambda_i \cdot \tau(\widetilde{\varphi(\gamma)})$. The commutator of $\tau(\widetilde{\varphi(\gamma)})$ with Λ_i lies in $\mathfrak{N}_{i+1}$, so this yields $\gamma_A(\Lambda_i \cdot \mathfrak{N}_{i+1}) = \Lambda_i \cdot \mathfrak{N}_{i+1}$. □

Let ϕ_i denote the induced action on the toral quotient

$$\mathbf{T}^{n_i} = (\Lambda_{i-1} \cdot \mathfrak{N}_i) \backslash \mathfrak{N}_{i-1} = \mathcal{Q}_i \backslash \mathcal{A}_i.$$

PROPOSITION 2. *Let φ be an affine action of Γ on a compact nilmanifold X. Suppose that $H^1(\Gamma;\mathbf{T}^{n_i}_{\phi_i})=0$ for $1\le i\le k$. Then the action of φ has a fixed point.*

Proof. As noted above, for a lift $\widetilde{\varphi(\gamma)}$ of $\varphi(\gamma)$ there are induced actions of γ_A and $\tau(\widetilde{\varphi(\gamma)})$ on the toral quotient $\mathbf{T}^{n_1}=(\Lambda\cdot\mathfrak{N}_1)\backslash\mathfrak{N}$. Moreover, the induced action of $\tau(\widetilde{\varphi(\gamma)})$ is independent of the choice of lift $\widetilde{\varphi(\gamma)}$, since the action of Λ on $(\Lambda\cdot\mathfrak{N}_1)\backslash\mathfrak{N}$ is trivial. Thus there is an induced affine action φ^1 of Γ on $\mathbf{T}^{n_1}$. By our cohomology assumption and Proposition 1, there is a fixed point $x_1\in\mathbf{T}^{n_1}$ for φ^1. Choose a lift $\tilde{x}_1\in\mathfrak{N}$ of x_1; then define a new action of Γ on X by translating *on the right* by the element $\tilde{x}_1$. That is,

$$\varphi_1(\gamma)=R(\tilde{x}_1)^{-1}\circ\varphi(\gamma)\circ R(\tilde{x}_1).$$

Then the lift $\tilde{\varphi}_1(\gamma)$ of $\varphi_1(\gamma)$ to $\mathcal{G}$ has translational part $\tau(\widetilde{\varphi_1(\gamma)})\in\mathfrak{N}_1$.

This procedure sets up an induction: Given the action φ_l of Γ on X whose elements lift to $\mathcal{G}$ with translational components $\tau(\widetilde{\varphi_l(\gamma)})\in\mathfrak{N}_l$, we use $H^1(\Gamma;\mathbf{T}^{n_{l+1}})=0$ to obtain $\tilde{x}_{l+1}\in\mathfrak{N}$, so that

$$\varphi_{l+1}(\gamma)=R(\tilde{x}_{l+1})^{-1}\circ\varphi_l(\gamma)\circ R(\tilde{x}_{l+1})$$

has translational component in $\mathfrak{N}_{l+1}$.

Finally, we obtain that φ^k fixes a point in $\mathbf{T}^{n_k}$. The corresponding φ_k fixes the identity coset in X, and hence φ fixes the point $x_*=x_1\cdots x_k$. □

We deduce that the fixed point conclusion of Theorem 3 follows from Proposition 2, the cohomology exact sequences

$$\cdots\to H^1(\Gamma;\mathfrak{A}_i)\to H^1(\Gamma;\mathbf{T}^{n_i}_{\phi_i})\to H^2(\Gamma;\mathfrak{Q}_i)\to H^2(\Gamma;\mathfrak{A}_i)\to\cdots,\tag{7}$$

and the remark that the kernel of the map $H^2(\Gamma;\mathfrak{Q}_i)\to H^2(\Gamma;\mathbf{R}_i)$ is the torsion subgroup of $H^2(\Gamma;\mathfrak{Q}_i)$.

The preceding proof can be modified to yield periodic orbits for the action on X under the weaker hypothesis that $H^1(\Gamma;\mathfrak{A}_i)=0$ for $1\le i\le k$. We prove the following, which implies that the action has a periodic point.

PROPOSITION 3. *Let φ be an affine action of Γ on a compact nilmanifold X. Suppose that $H^1(\Gamma;\mathfrak{A}^{n_i}_{\phi_i})=0$ for $1\le i\le k$. Then there is a finite-index subgroup $\Gamma'\subset\Gamma$ such that the restricted action $\varphi(\Gamma')$ has a fixed point.*

Proof. Let φ^1 denote the induced affine action of Γ on $\mathbf{T}^{n_1}$, as defined in the proof of Proposition 3. The hypotheses $H^1(\Gamma;\mathfrak{A}^{n_1}_{\phi_1})=0$ and Proposition 1(3) imply that there is a periodic orbit for the action of Γ on $\mathbf{T}^{n_1}$. Therefore, there is a subgroup of finite index $\Gamma_1\subset\Gamma$ which fixes a point $x_1\in\mathbf{T}^{n_1}$.

We then proceed as before: Choose a lift $\tilde{x}_1\in\mathfrak{N}$ of x_1, and then define a new action of Γ_1 on X by translating *on the right* by the element $\tilde{x}_1$. That is,

$$\varphi_1(\gamma)=R(\tilde{x}_1)^{-1}\circ\varphi(\gamma)\circ R(\tilde{x}_1)$$

Then the lift $\tilde{\varphi}_1(\gamma)$ of $\varphi_1(\gamma)$ to $\mathcal{G}$ has translational part $\tau(\widetilde{\varphi_1(\gamma)}) \in \mathfrak{N}_1$ for all $\gamma \in \Gamma_1$. Repeat this construction inductively to obtain an action φ^k of a finite index normal subgroup $\Gamma_k \subset \Gamma_{k-1} \subset \cdots \subset \Gamma$ which fixes a point in $\mathbf{T}^{n_k}$. The corresponding action $\varphi_k(\Gamma_k)$ fixes the identity coset in X, and hence the restricted action $\varphi(\Gamma_k)$ fixes the point $x_* = x_1 \cdots x_k$. Then take $\Gamma' = \Gamma_k$. □

The proof of Proposition 3 shows somewhat more than asserted above. The inductive procedure first finds a periodic point for the quotient action on a torus. This yields a compact fiber which is invariant for a subgroup of finite index. The induced action on this fiber has a quotient action on a torus, which also has a periodic point, and so forth. At each stage, there is a "free choice" of the periodic orbit in the quotient torus. By Proposition 1(3), each quotient toral action has a dense set of periodic orbits to choose from. Therefore, we can conclude the following also.

COROLLARY 5. *Let φ be an affine action of Γ on a compact nilmanifold X. Suppose that $H^1(\Gamma; \mathfrak{A}^{n_i}_{\phi_i}) = 0$ for $1 \leq i \leq k$. Then the set of periodic points $\Lambda(\varphi)$ is dense.*

The existence of a dense set of periodic orbits can also be deduced from arithmetic considerations, as in the proof of the Theorem in [3, §3].

4. Affine Actions without Fixed Points

In this section, we construct affine actions that have dense sets of periodic orbits yet no fixed points. The method is first developed for the congruence subgroups of $SL(n, \mathbf{Z})$, using the cohomology interpretation of affine actions on $\mathbf{T}^n$. The method produces groups $\Gamma(n,p) \subset SL(n,\mathbf{Z})$ with nontrivial torsion classes in $H^1(\Gamma(n,p); \mathbf{T}^n_{\varphi_0})$, where φ_0 is the standard linear action of $\Gamma(n,p)$ on $\mathbf{T}^n$. By Proposition 1, this yields affine actions on $\mathbf{T}^n$ with dense periodic points but no fixed points. We then give a dynamical reformulation of the method which is applicable to more general affine actions.

Let $\mathbf{Z}/p\mathbf{Z}$ denote the cyclic group of order p. For each $n > 1$, there is a natural "mod p" quotient map $\Pi_p: SL(n,\mathbf{Z}) \to SL(n, \mathbf{Z}/p\mathbf{Z})$ whose kernel is the congruence p-subgroup denoted by $SL(n,\mathbf{Z})_p$. Given a subgroup $\Gamma \subset SL(n,\mathbf{Z})$, define $\Gamma_p = \Gamma \cap SL(n,\mathbf{Z})_p$. Observe then that Γ_p is normal with finite index in Γ.

Recall that $\varphi_0: \Gamma \times \mathbf{T}^n \to \mathbf{T}^n$ is the standard action. For a subgroup $\Gamma \subset SL(n,\mathbf{Z})$, define the subgroup of Γ-invariants in $\mathbf{T}^n$,

$$\mathcal{I}(\Gamma) = (\mathbf{T}^n)^\Gamma = \{x \in \mathbf{T}^n \mid x = \varphi_0(\gamma)(x) \text{ for all } \gamma \in \Gamma\}.$$

LEMMA 4. *There is a natural map* $\operatorname{Hom}(\Gamma, \mathcal{I}(\Gamma)) \xrightarrow{i_*} H^1(\Gamma; \mathbf{T}^n_{\varphi_0})$.

Proof. φ_0 restricts to the trivial action on $\mathcal{I}(\Gamma)$, so the inclusion $i: \mathcal{I}(\Gamma) \subset \mathbf{T}^n$ induces a well-defined map $\operatorname{Hom}(\Gamma, \mathcal{I}(\Gamma)) \cong H^1(\Gamma; \mathcal{I}(\Gamma)_\varphi) \xrightarrow{i_*} H^1(\Gamma; \mathbf{T}^n_{\varphi_0})$. □

We will show that this map is nonzero for appropriate choices of finite-index subgroups $\Gamma \subset SL(n,\mathbf{Z})$. Let $(1/p)\mathbf{Z}$ denote the additive group of fractions generated by $1/p$, and let $\mathbf{T}^n_p = ((1/p)\mathbf{Z})^n \bmod \mathbf{Z}^n$ be the "$(1/p)$-points" for the n-torus.

LEMMA 5. *For each* $p \geq 1$, $\mathcal{I}(SL(n,\mathbf{Z}_p)) = \mathbf{T}^n_p$.

Proof. Each element $\gamma \in SL(n,\mathbf{Z}_p)$ is an invertible matrix with integer entries, with $\gamma - I = pA$ for some integer matrix A. Given a vector $x = (x_1, \ldots, x_n) \in ((1/p)\mathbf{Z})^n$, calculate

$$\gamma \cdot x = x + pA \cdot x = x \bmod \mathbf{Z}^n.$$

Conversely, if $\gamma \cdot x = x \bmod \mathbf{Z}^n$ for all $x \in ((1/p)\mathbf{Z})^n$, then $\gamma - I$ maps $((1/p)\mathbf{Z})^n$ to the integral lattice $\mathbf{Z}^n$, which implies that $\gamma - I = pA$ for some integer matrix A. □

We now give the proof of Theorem 1 by exhibiting a lattice subgroup $\Gamma(n,p)$ such that $SL(n,\mathbf{Z})_{p^2} \subset \Gamma(n,p) \subset SL(n,\mathbf{Z})_p$ and

$$i_*: \mathrm{Hom}(\Gamma(n,p), \mathbf{T}^n_p) \to H^1(\Gamma(n,p); \mathbf{T}^n_\varphi)$$

is not trivial. Let U_p denote the $n \times n$ matrix with all entries zero, except for the top right entry which equals p, and let Id be the $n \times n$ identity matrix. Then $A = \mathrm{Id} + U_p \in SL(n,\mathbf{Z})_p$ is an upper triangular matrix. Note that $A^p = \mathrm{Id} + U_{p^2}$, so that A generates a cyclic subgroup of order p in the quotient group $SL(n,\mathbf{Z})_p / SL(n,\mathbf{Z})_{p^2}$. Define $\Gamma(n,p)$ to be the group generated by A and the subgroup $SL(n,\mathbf{Z})_{p^2}$. Let $\Pi: \Gamma(n,p) \to \mathbf{Z}/p\mathbf{Z}$ be the quotient map onto the cyclic group of order p, where the kernel of Π is $SL(n,\mathbf{Z})_{p^2}$ and $\Pi(A) = 1 \in \mathbf{Z}/p\mathbf{Z}$.

For $1 \leq i \leq n$, define homomorphisms $\tau_i: \Gamma(n,p) \to \mathbf{T}^n_p$ by

$$\tau_i(\gamma) = \frac{\Pi(\gamma)}{p} \cdot \vec{e}_i \bmod \mathbf{Z}^n,$$

where $\vec{e}_i = (0, \ldots, 1, \ldots, 0)$ is the ith basis vector of $\mathbf{Z}^n$.

LEMMA 6. *The elements* $\{i_*[\tau_2], \ldots, i_*[\tau_n]\} \in H^1(\Gamma(n,p); \mathbf{T}^n_{\varphi_0})$ *are linearly independent over* $\mathbf{Z}/p\mathbf{Z}$.

Proof. Let $a_2, \ldots, a_n$ be integers such that $i_*[a_2 \cdot \tau_2 + \cdots + a_n \cdot \tau_n] = 0$ in $H^1(\Gamma(n,p); \mathbf{T}^n_{\varphi_0})$. By Proposition 1, there exists $z_0 \in \mathbf{T}^n$ such that

$$\varphi(\gamma)(x_0) = \tau_2(\gamma)^{a_2} \cdots \tau_n(\gamma)^{a_n} \cdot x_0 \tag{8}$$

for all $\gamma \in \Gamma(n,p)$. In particular, for $\gamma \in SL(n,\mathbf{Z})_{p^2}$, this implies that

$$\varphi(\gamma)(x_0) = x_0;$$

hence $x_0 \in \mathbf{T}^n_{p^2}$ by Lemma 5. Next, evaluate the identity (8) for $\gamma^l = A = I + U^l_p$ and multiply by x_0^{-1} to obtain

$$\varphi(U_p^l)(x_0)=\frac{l\cdot a_2}{p}\vec{e}_2\cdots\frac{l\cdot a_n}{p}\vec{e}_n \bmod \mathbf{Z}^n. \tag{9}$$

This identity says that x_0 is a point whose translation under the "derivation" $\varphi(U_p^l)(x_0)=\varphi(\gamma)(x_0)\cdot x_0^{-1}$ is in the span of the translates by the basic vectors $\{(1/p)\vec{e}_2,\dots,(1/p)\vec{e}_n\}$. However, $\varphi(U_p)(x_0)$ is calculated by left multiplication by the matrix U_p on the coset x_0, so the elements $\varphi(U_p^l)(x_0)$ and $\varphi(U_p^k)(x_0)$ differ by an element of the subgroup of $\mathbf{T}^n$ generated by the coset of $(1/p)\vec{e}_1$. The subgroup of $\mathbf{T}^n$ generated by the set $\{(1/p)\vec{e}_2,\dots,(1/p)\vec{e}_n\}$ intersects the subgroup generated by $(1/p)\vec{e}_1$ in the identity element, so we must have that $a_i=0 \bmod p$ for $2\le i\le n$, as was to be shown. □

COROLLARY 6. *There is an inclusion* $(\mathbf{Z}/p\mathbf{Z})^{n-1}\subset H^1(\Gamma(n,p);\mathbf{T}^n_{\varphi_0})$.

The points in the rational torus $\mathbf{Q}^n/\mathbf{Z}^n$ are all periodic for the standard action, and also for the translation action by a rational number. It follows that $\mathbf{Q}^n/\mathbf{Z}^n$ is contained in the set of periodic orbits for each affine action φ_τ associated to a class $[\tau]=a_2\cdot[\tau_2]+\cdots+a_n\cdot[\tau_n]\in H^1(\Gamma(n,p);\mathbf{T}^n_{\varphi_0})$, hence $\Lambda(\varphi_\tau)$ is dense. When $[\tau]\ne 0$, there are no fixed points for the affine action φ_τ. Theorem 1 now follows. □

We next reformulate the arithmetic constructions above with their dynamical counterparts, thus broadening the scope of our class of examples without fixed points. Assume that $\varphi:\Gamma\times\mathbf{T}^n\to\mathbf{T}^n$ is an affine Anosov action, with $\Lambda(\varphi)$ nonempty. A set $\Sigma\subset\mathbf{T}^n$ is said to be *φ-saturated* if $x\in\Sigma$ implies that $\varphi(\gamma)(x)\in\Sigma$ for all $\gamma\in\Gamma$.

DEFINITION 1. A *filtration* of $\Lambda(\varphi)$ is an ascending sequence of φ-saturated finite sets $\Lambda_1\subset\Lambda_2\subset\cdots\subset\Lambda_p\subset\cdots$ whose union is all of $\Lambda(\varphi)$.

LEMMA 7. *An affine Anosov action φ admits a natural length filtration on $\Lambda=\Lambda(\varphi)$.*

Proof. For each positive integer p, let $\Lambda_p\subset\Lambda$ be the subset of points whose φ-orbit $\Gamma(x)$ contains at most p points. Clearly, $\Lambda_p\subset\Lambda_{p+1}$ with the union over all p yielding Λ. We must check that each Λ_p is a finite set. Let $\gamma_h\in\Gamma$ be φ-hyperbolic. Observe that each $x\in\Lambda_p$ is a periodic point for $\varphi(\gamma_h)$, and hence is a fixed point for some power $\varphi(\gamma_h)^q$ with $0<q\le p$. Thus, Λ_p is contained in the set of fixed points for the Anosov diffeomorphism $\varphi(\gamma_h)^{p!}$, and hence is finite. □

For each $p\ge 1$ define Γ_p to be the stabilizer subgroup of Λ_p. That is,

$$\gamma\in\Gamma_p \Leftrightarrow \varphi(\gamma)(x)=x \quad \text{for all } x\in\Lambda_p.$$

Clearly, $\Gamma_{p+1}\subset\Gamma_p$, and Λ dense in $\mathbf{T}^n$ implies that the intersection over all Γ_p is the set of $\gamma\in\Gamma$ which act as the identity on $\mathbf{T}^n$.

LEMMA 8. *Γ_p is a normal subgroup of Γ with finite index,*

$$[\Gamma_p : \Gamma] \leq \{\mathrm{Card}(\Lambda_p)\}!.$$

Proof. The action of Γ on Λ_p defines a representation $\Gamma \to \mathrm{Perm}(\Lambda_p)$ into the permutation group on the set Λ_p with kernel Γ_p, and the finite group of permutations has order $\{\mathrm{Card}(\Lambda_p)\}!$. □

We also introduce $X_p = \{x \in \mathbf{T}^n \mid \varphi_0(\gamma)(x) = x \text{ for all } \gamma \in \Gamma_p\}$, which is the Γ_p-"saturation" of the set Λ_p. The inclusion $\Gamma_q \subset \Gamma_p$ for $p < q$ implies that $X_p \subset X_q$.

LEMMA 9. *If $\varphi_0 : \Gamma \times \mathbf{T}^n \to \mathbf{T}^n$ is a linear action, then X_p is a finite subgroup of $\mathbf{T}^n$.*

Proof. Each $A \in GL(n, \mathbf{Z})$ acts as a group automorphism of $\mathbf{T}^n$, so that $A(xy) = A(x)A(y) = xy$ if x and y are fixed by A. This holds for each A in the image of φ_0, hence X_p is a subgroup. For each $x \in X_p$, the orbit $\Gamma(x)$ contains at most $\{\mathrm{Card}(\Lambda_p)\}!$ points, so that $X_p \subset \Lambda_{p!}$. Thus X_p is a finite set. □

The p-adic filtration used previously can now be replaced with the length filtration, and the remainder of the arguments work as before. Fix a pair of integers $q > p > 1$. (In the previous case, we let $q = p^2$.) We introduce the following notation:

for $\gamma \in \Gamma_p$, let $\Gamma(q, \gamma) \subset \Gamma_p$ be the subgroup generated by Γ_q and γ;
given an auxiliary subgroup $\Gamma' \subset \Gamma_p$, $\mathrm{Der}_q(\Gamma') \subset X_p$ will denote the subgroup generated by the subset

$$\{\varphi_0(\delta)(x) \cdot x^{-1} \mid \delta \in \Gamma' \text{ and } x \in X_q\} \cap X_p \subset X_p.$$

PROPOSITION 4. *Let $\Gamma \subset GL(n, \mathbf{Z})$ contain a hyperbolic element. Then for each $q > p > 1$ and $\gamma \in \Gamma_p$ there is a homomorphism*

$$\mathrm{Hom}(\Gamma(q, \gamma)/\Gamma_q, X_p) \cong H^1(\Gamma(q, \gamma)/\Gamma_q; X_p) \xrightarrow{j} H^1(\Gamma(q, \gamma); \mathbf{T}^n_{\varphi_0}) \qquad (10)$$

with kernel contained in the subgroup $\mathrm{Hom}(\Gamma(q, \gamma)/\Gamma_q, \mathrm{Der}_q(\Gamma(q, \gamma)))$.

Proof. For $\alpha \in \mathrm{Hom}(\Gamma(q, \gamma)/\Gamma_q, X_p)$, define $\tau_\alpha : \Gamma(q, \gamma) \to \mathbf{T}^n$ to be the composition

$$\Gamma(q, \gamma) \to \Gamma(q, \gamma)/\Gamma_q \xrightarrow{\alpha} X_p \subset \mathbf{T}^n.$$

Define $j[\alpha] = [\tau_\alpha]$. Suppose that $[\tau_\alpha] = [\tau_\beta]$; then there exists $x_0 \in \mathbf{T}^n$ such that $\tau_\alpha(\delta)\tau_\beta(\delta)^{-1} = \varphi_0(\delta)(x_0) \cdot x_0^{-1}$ for all $\delta \in \Gamma(q, \gamma)$. In particular, $\varphi(\Gamma_q)(x_0) = x_0$, so that $x_0 \in X_q$. As $\tau_\alpha(\delta)\tau_\beta(\delta)^{-1} \in X_p$ by construction, we have that $\tau_\alpha(\delta)\tau_\beta(\delta)^{-1} \in \mathrm{Der}_q(\Gamma(q, \gamma))$. □

COROLLARY 7. *Let $\Gamma \subset GL(n, \mathbf{Z})$ contain a hyperbolic element. Suppose there exist $q > p > 1$ and $\gamma \in \Gamma_p$ such that*

$$\mathrm{Hom}(\Gamma(q,\gamma)/\Gamma_q, \mathrm{Der}_q(\Gamma(q,\gamma))) \neq \mathrm{Hom}(\Gamma(q,\gamma)/\Gamma_q, X_p).$$

Then there exists an affine action φ of $\Gamma(q,\gamma)$ on $\mathbf{T}^n$ without fixed points, such that the restriction $\varphi|\Gamma_q$ is the standard action φ_0 with fixed point set X_q.

5. Affine Actions of Abelian Groups

Assume that $\varphi: \Gamma \times \mathbf{T}^n \to \mathbf{T}^n$ is an affine Anosov action of a finitely generated abelian group Γ, with φ-hyperbolic element $\gamma_h \in \Gamma$. The linear action associated to φ defines a representation $\varphi_0: \Gamma \to GL(n, \mathbf{Z})$. The fixed point set $\mathrm{Fix}(\varphi(\gamma_h)) \subset \mathbf{T}^n$ for $\varphi(\gamma_h)$ is finite and (as Γ is abelian) is invariant under the full action of $\varphi(\Gamma)$. Thus, the existence of a fixed point for the action of φ is equivalent to the existence of a fixed point for the restricted action of Γ on $\mathrm{Fix}(\varphi(\gamma_h))$. We use this remark to reformulate the cohomological obstructions to the existence of a fixed point in Proposition 1.

We introduce the relative cohomology group $H^1(\Gamma, \gamma_h; \mathrm{Fix}(\varphi_0(\gamma_h))_{\varphi_0})$ spanned by the 1-cocycles $\tau: \Gamma \to \mathrm{Fix}(\varphi_0(\gamma_h))$ over φ_0 which vanish on γ_h. Note that this defines a subcomplex as $\tau(\gamma_h^m) = \tau(\gamma_h)^m$, using that $\varphi_0(\gamma_h)$ acts trivially on $\mathrm{Fix}(\varphi_0(\gamma_h))$.

PROPOSITION 5. *Let $\varphi_0: \Gamma \times \mathbf{T}^n \to \mathbf{T}^n$ be a linear Anosov action of a finitely generated abelian group Γ with $\varphi_0(\gamma_h)$-hyperbolic element $\gamma_h \in \Gamma$. Then the set of affine actions of Γ with linear part φ_0 is indexed by the cohomology group $H^1(\Gamma, \gamma_h; \mathrm{Fix}(\varphi_0(\gamma_h))_{\varphi_0})$. In particular, each nonzero class in $H^1(\Gamma, \gamma_h; \mathrm{Fix}(\varphi_0(\gamma_h))_{\varphi_0})$ gives rise to an affine Anosov action without fixed points, but with dense periodic orbits.*

Proof. There is a natural map

$$H^1(\Gamma, \gamma_h; \mathrm{Fix}(\varphi_0(\gamma_h))_{\varphi_0}) \to H^1(\Gamma; \mathbf{T}^n_{\varphi_0}). \tag{11}$$

By Proposition 1, it suffices to show that this map is injective, and that each affine action φ as in the proposition yields a 1-cohomology class in the image of this map.

First, assume that $\tau: \Gamma \to \mathrm{Fix}(\varphi_0(\gamma_h))$ is a 1-cocycle over φ_0 which vanishes on γ_h and is a coboundary as a map into $\mathbf{T}^n$. Then the corresponding affine action φ_τ admits a fixed point, and so is conjugate to the linear action φ_0 via translation by some $x_0 \in \mathbf{T}^n$. Translation by x_0 maps the fixed point set of φ to that of φ_0. The hypothesis $\tau(\gamma_h) = 0$ implies that $\mathrm{Fix}(\varphi(\gamma_h))$ contains 0, so

$$x_0 \in \mathrm{Fix}(\varphi_0(\gamma_h))_{\varphi_0}$$

and thus τ also defines the zero class in $H^1(\Gamma; \mathrm{Fix}(\varphi_0(\gamma_h))_{\varphi_0})$. Note the corollary of this argument: the fixed point sets $\mathrm{Fix}(\varphi(\gamma_h))$ and $\mathrm{Fix}(\varphi_0(\gamma_h))$ agree whenever $0 \in \mathrm{Fix}(\varphi(\gamma_h))$.

Given an affine action φ as in the proposition, conjugate the action by a translation so that $0 \in \mathrm{Fix}(\varphi(\gamma_h))$. Let $\tau_\varphi: \Gamma \to \mathbf{T}^n$ denote the corresponding

1-cocycle. $\text{Fix}(\varphi(\gamma_h))$ is invariant under the action $\varphi(\Gamma)$, so $\tau_\varphi(\gamma) = \varphi(\gamma_i)(0) \in \text{Fix}(\varphi(\gamma_h)) = \text{Fix}(\varphi_0(\gamma_h))$. That is, we have shown that the class of $[\tau_\varphi]$ is in the image of (11). □

Proposition 5 almost immediately yields a proof of Theorem 2. Fix the hyperbolic element $\gamma_h \in \Gamma$, and let $\Gamma' \subset \Gamma$ be the subgroup of finite index consisting of elements which act trivially when restricted to $\text{Fix}(\varphi(\gamma_h))$. For any abelian group $\mathfrak{A}$, let $\text{Hom}(\Gamma', \gamma_h; \mathfrak{A})$ denote the group homomorphisms that map γ_h to the trivial element. Theorem 2 now follows from Proposition 1 and the following observation.

LEMMA 10. *There is an inclusion*

$$\text{Hom}(\Gamma', \gamma_h; \text{Fix}(\varphi_0(\gamma_h))) \subset H^1(\Gamma', \gamma_h; \text{Fix}(\varphi_0(\gamma_h))_{\varphi_0}).$$

The group $\text{Hom}(\Gamma', \gamma_h; \text{Fix}(\varphi_0(\gamma_h)))$ can easily be evaluated using the structure theory for finitely generated abelian groups. In particular, with the hypotheses of Theorem 2 there always exists a nontrivial homomorphism in this group.

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