# Rigidity of Minimal Submanifolds in Spheres in Terms of Higher Fundamental Forms

G. TOTH

## 1. Preliminaries and Statement of Results

In this paper we will give a generalization of the DoCarmo–Wallach theory for rigidity and nonridigity of minimal submanifolds in spheres, introducing higher-order contact conditions for the submanifolds expressed in terms of their higher-order fundamental forms and osculating bundles.

Given an isometric immersion $F: M \to S^N$ of a Riemannian manifold $M$ of dimension $m$ into the Euclidean $N$-sphere $S^N$, we denote by $\beta_l(F)$, $l = 1, \dots, p_F$, the $l$th fundamental form of $F$, and by $\mathcal{O}_F^l$ the $l$th osculating bundle of $F$, all defined on a (maximal) nonempty open set $D_F$ of $M$ (cf. [1; 6]). For $x \in D_F$, $\beta_l(F)_x: S^l(T_xM) \to \mathcal{O}_{F;x}^l$ is a linear map of the $l$th symmetric power of $T_xM$ onto the fibre $\mathcal{O}_{F;x}^l$ of $\mathcal{O}_F^l$ at $x$ (also called the $l$th osculating space of $F$ at $x$). Recall that $\beta_1(F) = F_*$ is defined on $D_F^1 = M$ and, for $x \in D_F^1$, the first osculating space $\mathcal{O}_{F;x}^1$ is the image of $\beta_1(F)_x$. The higher fundamental forms and osculating bundles are then defined inductively by setting

$$\begin{gathered} \beta_{l+1}(F)_x(X^0, \dots, X^l) = (\nabla_{X^0}\beta_l(F))(X^1, \dots, X^l)^{\perp_l}, \\ X^0, \dots, X^l \in T_xM, \quad x \in D_F^l, \end{gathered} \tag{1}$$

where $\perp_l$ is the orthogonal projection with kernel $\mathcal{O}_{F;x}^0 \oplus \cdots \oplus \mathcal{O}_{F;x}^l$, $\mathcal{O}_{F;x}^0 = \mathbf{R} \cdot F(x)$, and $D_F^{l+1}$ is the set of points $x \in D_F^l$ at which the image $\mathcal{O}_{F;x}^{l+1}$ of $\beta_{l+1}(F)_x$ has maximal dimension. $\beta_{p_F}(F)$ is the highest nonvanishing fundamental form, and $p_F$ is said to be the *geometric degree* of $F$. We have $D_F = \bigcap_{l=0}^{p_F} D_F^l$. Finally, it is convenient to define the 0th fundamental form of $F$ to be $F$ itself. To be consistent with (1), we also put $\nabla_{X^0}\beta_0(F) = \beta_1(F)(X^0) = F_*(X^0)$. Note that if $M$ and $F$ are analytic then $D_F$ is dense in $M$.

Given two isometric immersions $F: M \to S^N$ and $f: M \to S^n$, we say that *$f$ is derived from $F$*, written as $f \leftharpoonup F$, if $f = A \cdot F$ for some linear map $A$: $\mathbf{R}^{N+1} \to \mathbf{R}^{n+1}$. If $F$ is full (i.e., the image of $F$ is not contained in a great sphere of $S^N$) then $A$ is uniquely determined. If $f$ is full then $A$ is onto; $f$ is (orthogonally) equivalent to $F$ if ($N = n$ and) $A$ is orthogonal.

Now let $f \leftharpoonup F$ via $f = A \cdot F$. We introduce

$$\langle f \rangle = A^{\mathrm{T}} \cdot A - I \in S^2(\mathbf{R}^{N+1}),$$

Received February 3, 1992.
Michigan Math. J. 40 (1993).

where $^{\mathrm{T}}$ stands for matrix transpose and $I$ is the identity. The condition that $f$ is an isometric immersion translates into $\langle f \rangle$ being perpendicular to

$$\mathcal{Z}_F = \operatorname{span}\{F(x) \odot F(x) \mid x \in M\} + \operatorname{span}\{F_*(X)^{\check{}} \odot F_*(Y)^{\check{}} \mid X, Y \in T_x M, x \in M\} \tag{2}$$

in $S^2(\mathbf{R}^{N+1})$ (w.r.t. the standard scalar product $\langle C, C' \rangle = \operatorname{trace}(C'^{\mathrm{T}} C)$, $C, C' \in S^2(\mathbf{R}^{N+1})$), where $\check{}: T\mathbf{R}^{N+1} \to \mathbf{R}^{N+1}$ is given by parallel translation. Indeed, $|f|^2 = 1$ translates into $\langle f \rangle$ being perpendicular to the first term on the right-hand side of (2), and the second term encodes the information that $f$ is isometric on the tangent spaces. As in [4], we obtain that the space of equivalence classes of full isometric immersions that are derived from $F$ can be parameterized by the convex body

$$\mathfrak{M}_F = \{C \in \mathfrak{F}_F \mid C + I \geq 0\}$$

in $\mathfrak{F}_F = (\mathcal{Z}_F)^{\perp}$. The parameterization is given by associating to (the equivalence class of) $f$ the symmetric matrix $\langle f \rangle$.

Assume now that $F$ is minimal. The Euler–Lagrange equation can be written as

$$\Delta^M F = m \cdot F,$$

to be satisfied by the induced vector-valued function $F: M \to \mathbf{R}^{N+1}$, where $\Delta^M$ is the Laplace–Beltrami operator on $M$. If $f \leftharpoondown F$, applying $A$ to both sides of this equation, we obtain that $f$ is automatically minimal. Hence $\mathfrak{M}_F$ parameterizes the equivalence classes of full isometric minimal immersions that are derived from $F$. Moreover, $\mathfrak{M}_F$ is compact if $M$ is compact.

(HISTORICAL) REMARK. The approach and the concepts developed here originate from the work of DoCarmo and Wallach on rigidity and nonrigidity of minimal isometric immersions into spheres (cf. [1; 6]). Note also that if $M$ is isotropy irreducible Riemannian homogeneous and $F = f_\lambda$ is the standard minimal immersion associated to an eigenvalue $\lambda$ of $\Delta^M$, then *any* full isometric minimal immersion $f: M \to S^n$ is derived from $f_\lambda$ provided that it induces the same metric on $M$ as $f_\lambda$.

Let $F: M \to S^N$ and $f: M \to S^n$ be full isometric immersions, and let $1 \leq k \leq p_F$. We say that *$f$ has $k$th-order contact with $F$* if $D_f \cap D_F \neq \emptyset$ and, for $x \in D_f \cap D_F$, we have

$$\begin{aligned} &\langle (\nabla_{X^0} \beta_{l'}(f))(X^1, \dots, X^{l'}), \beta_{l''+1}(f)(Y^0, \dots, Y^{l''}) \rangle \\ &\quad = \langle (\nabla_{X^0} \beta_{l'}(F))(X^1, \dots, X^{l'}), \beta_{l''+1}(F)(Y^0, \dots, Y^{l''}) \rangle, \\ &\quad X^0, \dots, X^{l'}; Y^0, \dots, Y^{l''} \in T_x M, \ 0 \leq l'' \leq l' < k. \end{aligned} \tag{3}$$

REMARKS. 1. For $k = 1$, the contact condition is automatically satisfied.

2. For $l' = l'' = l - 1 < k$, (3) is equivalent to

$$\begin{aligned} &\langle \beta_l(f)(X^1, \dots, X^l), \beta_l(f)(Y^1, \dots, Y^l) \rangle \\ &\quad = \langle \beta_l(F)(X^1, \dots, X^l), \beta_l(F)(Y^1, \dots, Y^l) \rangle. \end{aligned} \tag{4}$$

3. If $M$, $F$, and $f$ are analytic then $D_f \cap D_F$ is dense in $M$. The special case we will be interested in is when $M$ is Riemannian homogeneous, $F$ is minimal (hence analytic), and $f \leftharpoonup F$ so that $f$ is also analytic.

THEOREM 1. *Let $F: M \to S^N$ and $f: M \to S^n$ be full isometric immersions with $f \leftharpoonup F$ and $D_F \subset \overline{D_f}$, and let $1 \le k \le p_F$. Then $f$ has $k$th-order contact with $F$ if and only if $\langle f \rangle \in \mathfrak{F}_F^k = (\mathcal{Z}_F^k)^\perp$, where*

$$\mathcal{Z}_F^k = \operatorname{span}\{\beta_{l'}(F)(X^1, \ldots, X^{l'})^\vee \odot \beta_{l''}(F)(Y^1, \ldots, Y^{l''})^\vee \mid X^1, \ldots, X^{l'}; Y^1, \ldots, Y^{l''} \in T_x M, x \in D_F, 0 \le l', l'' \le k\} \subset S^2(\mathbf{R}^{N+1}).$$

It follows that, for $M$ and $F$ analytic, the space of equivalence classes of isometric immersions $f: M \to S^n$ that are derived from $F$ and have $k$th-order contact with $F$ can be parameterized by the convex body

$$\mathfrak{M}_F^k = \mathfrak{M}_F \cap \mathfrak{F}_F^k \subset \mathfrak{F}_F^k. \tag{5}$$

We have the following rigidity theorem.

THEOREM 2. *Let $M$ be analytic and let $F: M \to S^N$ be a full analytic isometric immersion. Assume that $f: M \to S^n$ is a full isometric immersion derived from $F$ that has $k$th-order contact with $F$. If $p_F \le 2k+1$ then $f$ is equivalent to $F$.*

REMARK. For $k=1$ this is the rigidity theorem of DoCarmo and Wallach, which states that a full analytic isometric immersion of degree $\le 3$ is linearly rigid.

In view of Theorem 1, Theorem 2 can be rephrased by saying that $p_F \le 2k+1$ implies $\mathfrak{F}_F^k = \{0\}$ (for $M$ and $F$ analytic). For higher degree we have nonrigidity, as the following result shows.

THEOREM 3. *Let $f_{\lambda_p}: S^m_{m/\lambda_p} \to S^{n(\lambda_p)}$ be the standard minimal immersion associated to the $p$th eigenvalue $\lambda_p = p(p+m-1)$ of $\Delta^{S^m}$, $m \ge 3$, where the subscript $m/\lambda_p$ indicates the curvature of the metric on $S^m$ induced by $f_{\lambda_p}$. Assume $2k+1 < p\ (= p_{f_{\lambda_p}})$. Then we have*

$$\dim \mathfrak{M}_{f_{\lambda_p}}^k = \dim \mathfrak{F}_{f_{\lambda_p}}^k \ge \sum_{(a,b) \in \Delta_k^p;\, a, b \text{ even}} \dim_{\mathbf{C}} V_{m+1}^{(a,b,0,\ldots,0)}, \tag{6}$$

*where $\Delta_k^p \subset \mathbf{R}^2$ is the closed convex triangle with vertices $(2k+2, 2k+2)$, $(p,p)$, and $(2(p-k), 2k+2)$, and where $V_{m+1}^{(\sigma_1,\ldots,\sigma_r)}$, $r = [|(m+1)/2|]$, is the irreducible complex $SO(m+1)$-module with highest weight $(\sigma_1, \ldots, \sigma_r)$ relative to the standard maximal torus in $SO(m+1)$.*

Just like Theorem 2, Theorem 3 (for $k=1$) specializes to the nonrigidity theorem of DoCarmo and Wallach. Note that the lower bound of the dimension in (6) can be computed by the Weyl dimension formula. In particular, we have

$$\dim_{\mathbf{C}} V_{m+1}^{(2k+2,2k+2,0,\dots,0)} \geq \dim_{\mathbf{C}}(V_4^{(2k+2,2k+2)} \oplus V_4^{(2k+2,-2k-2)}) = 8k+10.$$

The proofs are arranged as follows. In Section 2 we introduce the two technical tools; a method to compare, for $f \leftharpoondown F$, the higher fundamental forms of $F$ and $f$, and the space $\mathfrak{F}_F^k$ that encodes the information of the contact condition. Theorem 1 is proved there and is used in Section 3 to prove rigidity. In Section 4 we first specialize the treatment to $M$ isotropy irreducible Riemannian homogeneous, and use equivariance of the standard minimal immersion to realize $\mathfrak{F}_{f_{\lambda_p}}^k$ as a representation space of the acting group of isometries on $M$. We then further specialize to spherical domains, and use the decomposition of the symmetric square of the space of spherical harmonics into irreducible components given by DoCarmo–Wallach to derive Theorem 3. Finally, for $k=2$, we work out the contact condition explicitly and point out some connections of $\mathfrak{M}_{f_{\lambda_p}}^k$ with isotropic minimal immersions.

## 2. Higher Fundamental Forms of the Derived Map

We begin with the following observation.

LEMMA 1. *Let $F: M \to S^N$ be an isometric immersion. For $1 \leq l' < l-1$, we have*

$$\langle (\nabla_{X^0}\beta_l(F))(X^1,\dots,X^l), \beta_{l'}(F)(Y^1,\dots,Y^{l'})\rangle = 0 \tag{7}$$

*for $X^0,\dots,X^l; Y^0,\dots,Y^{l'} \in T_xM$, $x \in D_F$.*

*Proof.* We use vector fields that are defined locally near $x \in D_F$. We have

$$\begin{aligned}
&\langle (\nabla_{X^0}\beta_l(F))(X^1,\dots,X^l), \beta_{l'}(F)(Y^1,\dots,Y^{l'})\rangle \\
&\quad = \langle \nabla_{X^0}(\beta_l(F)(X^1,\dots,X^l)), \beta_{l'}(F)(Y^1,\dots,Y^{l'})\rangle \\
&\quad = X^0\langle \beta_l(F)(X^1,\dots,X^l), \beta_{l'}(F)(Y^1,\dots,Y^{l'})\rangle \\
&\qquad - \langle \beta_l(F)(X^1,\dots,X^l), \nabla_{X^0}(\beta_{l'}(F)(Y^1,\dots,Y^{l'}))\rangle.
\end{aligned}$$

The first term on the right-hand side is zero by orthogonality of the fundamental forms. By definition, $\nabla_{X^0}(\beta_{l'}(F)(Y^1,\dots,Y^{l'}))$ is a local section of $\mathcal{O}_F^0 \oplus \cdots \oplus \mathcal{O}_F^{l'+1}$. Since $l'+1 < l$, the second term also vanishes and the proof is complete. □

From now on through the rest of this section, $F: M \to S^N$ and $f: M \to S^n$ are full isometric immersions and $f \leftharpoondown F$ via $f = A \cdot F$. Given a vector field $v$ along $F$, we define $v^A$ to be the vector field along $f$ by

$$(v^A)^\vee = A \cdot \check{v} - \langle A \cdot \check{v}, f\rangle f.$$

LEMMA 2. *For any vector field $X$ on $M$, we have*

$$(\nabla_X v)^A = \nabla_X(v^A) + \langle A \cdot \check{v}, f\rangle f_*(X). \tag{8}$$

*Proof.* Let $w$ be a vector field along $f$. Using $\langle A \cdot F, \check{w}\rangle = \langle f, \check{w}\rangle = 0$, we have

$$\begin{aligned}\langle(\nabla_X v)^A, w\rangle &= \langle A\cdot(\nabla_X v)^\vee, \check{w}\rangle\\ &= \langle A\cdot X(\check{v}), \check{w}\rangle - \langle A\langle X(\check{v}), F\rangle F, \check{w}\rangle = \langle A\cdot X(\check{v}), \check{w}\rangle.\end{aligned}$$

As for the first term on the right-hand side of (8), we compute

$$\begin{aligned}\langle\nabla_X(v^A), w\rangle &= \langle X(v^A)^\vee, \check{w}\rangle\\ &= \langle X(A\cdot\check{v}), \check{w}\rangle - \langle X(\langle A\cdot\check{v}, f\rangle f), \check{w}\rangle\\ &= \langle X(A\cdot\check{v}), \check{w}\rangle - \langle A\cdot\check{v}, f\rangle\langle Xf, \check{w}\rangle.\end{aligned}$$

Now, $f_*(X)^\vee = Xf$ and we are done. □

Using (8), $f_* = \beta_1(f)$, and orthogonality of the fundamental forms, for $l' \geq 2$ we have

$$\begin{aligned}&X^0\langle\beta_l(F)(X^1,\dots,X^l)^A, \beta_{l'}(f)(Y^1,\dots,Y^{l'})\rangle\\ &\quad = \langle[\nabla_{X^0}\beta_l(F)(X^1,\dots,X^l)]^A, \beta_{l'}(f)(Y^1,\dots,Y^{l'})\rangle\\ &\qquad + \langle\beta_l(F)(X^1,\dots,X^l)^A, \nabla_{X^0}\beta_{l'}(f)(Y^1,\dots,Y^{l'})\rangle, \qquad (9)\end{aligned}$$

where the vector fields are given on an open set in $D_f \cap D_F$. We will refer to (9) as the *differentiation formula*. Next, we show that $\beta_l(F)(X^1,\dots,X^l)^A$ is a local section of $\mathcal{O}_f^1 \oplus \cdots \oplus \mathcal{O}_f^l$, with the $\mathcal{O}_f^l$-component being equal to $\beta_l(f)(X^1,\dots,X^l)$.

THEOREM 4. *For $l \geq 1$, we have on $D_f \cap D_F$:*

$$\begin{aligned}&\langle\beta_l(F)(X^1,\dots,X^l)^A, \beta_{l'}(f)(Y^1,\dots,Y^{l'})\rangle\\ &\quad = \begin{cases}\langle\beta_l(f)(X^1,\dots,X^l), \beta_{l'}(f)(Y^1,\dots,Y^l)\rangle, & l' = l;\\ 0, & l' > l.\end{cases} \qquad (10)\end{aligned}$$

*Proof.* We use induction with respect to $l$. For $l = 1$, we have

$$A\beta_1(F) = AF_* = f_* = \beta_1(f)$$

and the statement is clear. For the general induction step $1,\dots,l \Rightarrow l+1$, we use the differentiation formula (9) to differentiate the left-hand side of (10). Let $l' \geq l+1$. We obtain

$$\begin{aligned}&\langle[\nabla_{X^0}\beta_l(F)(X^1,\dots,X^l)]^A, \beta_{l'}(f)(Y^1,\dots,Y^{l'})\rangle\\ &\quad + \langle\beta_l(F)(X^1,\dots,X^l)^A, \nabla_{X^0}\beta_{l'}(f)(Y^1,\dots,Y^{l'})\rangle = 0. \qquad (11)\end{aligned}$$

According to Lemma 1, $\nabla_{X^0}\beta_l(F)(X^1,\dots,X^l)$ is a local section of $\mathcal{O}_F^{l+1} \oplus \mathcal{O}_F^l \oplus \mathcal{O}_F^{l-1}$, the $\mathcal{O}_F^{l+1}$-component being equal to $\beta_{l+1}(F)(X^0,\dots,X^l)$. Using the induction hypothesis, the first term on the left-hand side of (11) may be rewritten as

$$\langle\beta_{l+1}(F)(X^1,\dots,X^l)^A, \beta_{l'}(f)(Y^1,\dots,Y^{l'})\rangle. \qquad (12)$$

Again by Lemma 1, $\nabla_{X^0}\beta_{l'}(f)(Y^1,\dots,Y^{l'})$ is a local section of $\mathcal{O}_f^{l'+1} \oplus \mathcal{O}_f^l \oplus \mathcal{O}_f^{l'-1}$. Since $l' \geq l+1$, by the induction hypothesis, the $\mathcal{O}_f^{l'+1}$- and $\mathcal{O}_f^{l'}$-components of the second term on the left-hand side of (11) cancel. If $l' > l+1$

then the $\mathcal{O}_f^{l'-1}$-component also cancels since (12) vanishes. Thus, in this case we are done with the induction and the second statement of (10) follows. It remains to study the case $l'=l+1$. As noted above, only the $\mathcal{O}_f^l$-component of $\nabla_{X^0}\beta_{l+1}(f)(Y^1,\dots,Y^{l+1})$ can have a nonzero contribution in

$$\langle \beta_l(F)(X^1,\dots,X^l)^A, \nabla_{X^0}\beta_{l+1}(f)(Y^1,\dots,Y^{l+1})\rangle. \tag{13}$$

Again by the induction hypothesis, the $\mathcal{O}_f^l$-component of $\beta_l(F)(X^1,\dots,X^l)^A$ is $\beta_l(f)(X^1,\dots,X^l)$. Hence (13) may be rewritten as

$$\begin{aligned}&\langle \beta_l(f)(X^1,\dots,X^l), \nabla_{X^0}\beta_{l+1}(f)(Y^1,\dots,Y^l)\rangle\\ &\quad= -\langle \nabla_{X^0}\beta_l(f)(X^1,\dots,X^l), \beta_{l+1}(f)(Y^0,\dots,Y^l)\rangle\\ &\quad= -\langle \beta_{l+1}(f)(X^0,\dots,X^l), \beta_{l+1}(f)(Y^0,\dots,Y^l)\rangle.\end{aligned}$$

Putting this together with (11) and (12), the induction step is completed and the first statement of (10) follows. □

COROLLARY 1. *Let $f \leftharpoondown F$ with $D_f \cap D_F \neq \emptyset$. Then $p_f \le p_F$.*

*Proof.* Let $0 \neq w \in \mathcal{O}_{f;x}^{p_f}$ for some $x \in D_f \cap D_F$. By Theorem 4, there exists $v \in \mathcal{O}_{F;x}^{p_f}$ such that the $\mathcal{O}_{f;x}^{p_f}$-component of $v^A$ at $x$ is $w$. In particular, $v \neq 0$ and the claim follows. □

We are now ready to prove Theorem 1. Assume that $f \leftharpoondown F$ via $f = A\cdot F$ and that $f$ has $k$th-order contact with $F$, $1 \le k \le p_F$. We first show that, on $D_f \cap D_F$, we have

$$A\cdot\beta_l(F)(X^1,\dots,X^l)^{\check{}} = \beta_l(f)(X^1,\dots,X^l)^{\check{}},\quad 0\le l\le k. \tag{14}$$

The proof is by induction with respect to $l$; the cases $l=0,1$ are clear. To perform the general step $0,\dots,l \Rightarrow l+1$ $(\le k)$, we first observe that by Theorem 4 it is enough to show that

$$\langle \beta_{l+1}(F)(X^0,\dots,X^l)^A, \beta_{l'}(f)(Y^1,\dots,Y^{l'})\rangle = 0,\quad 1\le l' < l+1. \tag{15}$$

Let $x \in D_f \cap D_F$, and choose a normal coordinate neighborhood $U$ in $D_f \cap D_F$ centered at $x$. Given $X^0,\dots,X^l; Y^1,\dots,Y^l \in T_xM$, we extend each of these tangent vectors to a vector field on $U$ by parallel transport along geodesics emanating from $x$ and denote the extension by the same symbol. Thus the covariant derivative of all vector fields considered vanishes at $x$. In what follows all computations are at $x$ and, to simplify the notation, the presence of $x$ is suppressed. We have

$$\begin{aligned}&\langle \beta_{l+1}(F)(X^0,\dots,X^l)^A, \beta_{l'}(f)(Y^1,\dots,Y^{l'})\rangle\\ &\quad= \langle [\nabla_{X^0}\beta_l(F)(X^1,\dots,X^l)]^A, \beta_{l'}(f)(Y^1,\dots,Y^{l'})\rangle\\ &\qquad -\langle [(\nabla_{X^0}\beta_l(F)(X^1,\dots,X^l))^{\tau_l}]^A, \beta_{l'}(f)(Y^1,\dots,Y^{l'})\rangle,\end{aligned} \tag{16}$$

where $\tau_l$ is orthogonal projection to $\mathcal{O}_F^1 \oplus \cdots \oplus \mathcal{O}_F^l$. The induction hypothesis implies that $\langle A\cdot\beta_l(F)(X^1,\dots,X^l)^{\check{}}, f\rangle = \langle \beta_l(f)(X^1,\dots,X^l)^{\check{}}, f\rangle = 0$. Using this in the second term of the right-hand side of (8) together with the induction hypothesis (14), the first term on the right-hand side of (16) may be rewritten as

$$\begin{aligned}&\langle\nabla_{X^0}[\beta_l(F)(X^1,\dots,X^l)]^A,\beta_{l'}(f)(Y^1,\dots,Y^{l'})\rangle\\&\quad=\langle\nabla_{X^0}\beta_l(f)(X^1,\dots,X^l),\beta_{l'}(f)(Y^1,\dots,Y^{l'})\rangle\\&\quad=\langle\nabla_{X^0}\beta_l(F)(X^1,\dots,X^l),\beta_{l'}(F)(Y^1,\dots,Y^{l'})\rangle,\end{aligned}$$

where, in the last equality, we used to the contact condition (4). We now consider the second term of the right-hand side of (16). For $1\le l''\le l$, denote by $\beta_{l''}(F)(Z^1,\dots,Z^{l''})$ the $\mathcal{O}_F^{l''}$-component of $\nabla_{X^0}\beta_l(F)(X^1,\dots,X^l)$. By the induction hypothesis and the contact condition (3), we obtain

$$\begin{aligned}&\langle[\beta_{l''}(F)(Z^1,\dots,Z^{l''})]^A,\beta_{l'}(f)(Y^1,\dots,Y^{l'})\rangle\\&\quad=\langle\beta_{l''}(f)(Z^1,\dots,Z^{l''}),\beta_{l'}(f)(Y^1,\dots,Y^{l'})\rangle\\&\quad=\langle\beta_{l''}(F)(Z^1,\dots,Z^{l''}),\beta_{l'}(F)(Y^1,\dots,Y^{l'})\rangle,\end{aligned}$$

so that the second term on the right-hand side of (16) takes the form

$$-\langle(\nabla_{X^0}\beta_l(F)(X^1,\dots,X^l))^{\intercal_l},\beta_{l'}(F)(Y^1,\dots,Y^{l'})\rangle.$$

Putting this together with the result for the first term above, we obtain

$$\begin{aligned}&\langle\beta_{l+1}(F)(X^0,\dots,X^l)^A,\beta_{l'}(f)(Y^1,\dots,Y^{l'})\rangle\\&\quad=\langle\beta_{l+1}(F)(X^0,\dots,X^l),\beta_{l'}(F)(Y^1,\dots,Y^{l'})\rangle=0\end{aligned}$$

since $l'\le l$. Hence (14) follows. Now (4), (14), and orthogonality of the fundamental forms imply that $\langle f\rangle$ is perpendicular to all $\beta_{l'}(F)(X^1,\dots,X^{l'})^\vee\odot\beta_{l''}(F)(Y^1,\dots,Y^{l''})^\vee$ for $X^1,\dots,X^{l'};Y^1,\dots,Y^{l''}\in T_xM$, $0\le l',l''\le k$, and $x\in D_f\cap D_F$. Now $D_F\subset\overline{D_f}$ so that the same is true for $x\in D_F$, and $\langle f\rangle\in\mathfrak{F}_F^k$ follows.

For the converse, we assume that $\langle f\rangle\in\mathfrak{F}_F^k$. As an intermediate step, we again claim that (14) holds. Using induction, we need only perform the general step $0,\dots,l\Rightarrow l+1\ (\le k)$. By the induction hypothesis, for $1\le l'<l+1$ we have

$$\begin{aligned}&\langle\beta_{l+1}(F)(X^0,\dots,X^l)^A,\beta_{l'}(f)(Y^1,\dots,Y^{l'})\rangle\\&\quad=\langle A\cdot\beta_{l+1}(F)(X^0,\dots,X^l)^\vee,\beta_{l'}(f)(Y^1,\dots,Y^{l'})^\vee\rangle\\&\quad=\langle A\cdot\beta_{l+1}(F)(X^0,\dots,X^l)^\vee,A\cdot\beta_{l'}(F)(Y^1,\dots,Y^{l'})^\vee\rangle\\&\quad=\langle\beta_{l+1}(F)(X^0,\dots,X^l),\beta_{l'}(F)(Y^1,\dots,Y^{l'})\rangle=0,\end{aligned}$$

where in the last equality we used $\langle f\rangle\in\mathfrak{F}_F^k$. Theorem 4 completes the induction.

We now turn to the proof of the contact condition (3). Let $0\le l''\le l'<k$, and consider all vector fields defined on a normal coordinate neighborhood in $D_f\cap D_F$ centered at $x$, as in the first part of the proof. Using (14) and the assumption $\langle f\rangle\in\mathfrak{F}_F^k$, we compute at $x$:

$$\begin{aligned}&\langle(\nabla_{X^0}\beta_{l'}(f))(X^1,\dots,X^{l'}),\beta_{l''+1}(f)(Y^0,\dots,Y^{l''})\rangle\\&\quad=\langle\nabla_{X^0}\beta_{l'}(f)(X^1,\dots,X^{l'}),\beta_{l''+1}(f)(Y^0,\dots,Y^{l''})\rangle\end{aligned}$$

$$=\langle X^0(\beta_{l'}(f)(X^1,\dots,X^{l'})^\vee),\beta_{l''+1}(f)(Y^0,\dots,Y^{l''})^\vee\rangle$$
$$=\langle A\cdot X^0(\beta_{l'}(F)(X^1,\dots,X^{l'})^\vee),\beta_{l''+1}(f)(Y^0,\dots,Y^{l''})^\vee\rangle$$
$$=\langle A\cdot(\nabla_{X^0}\beta_{l'}(F)(X^1,\dots,X^{l'}))^\vee,\beta_{l''+1}(f)(Y^0,\dots,Y^{l''})^\vee\rangle$$
$$=\langle A\cdot(\nabla_{X^0}\beta_{l'}(F)(X^1,\dots,X^{l'}))^\vee,A\cdot\beta_{l''+1}(F)(Y^0,\dots,Y^{l''})^\vee\rangle$$
$$=\langle\nabla_{X^0}\beta_{l'}(F)(X^1,\dots,X^{l'}),\beta_{l''+1}(F)(Y^0,\dots,Y^{l''})\rangle,$$

and (3) follows. □

REMARK. Equation (14) implies that if $f: M\to S^n$ is derived from $F$: $M\to S^N$ (via $f=A\cdot F$) and $f$ has $k$th-order contact with $F$ then, for $0\le l\le k$, $\mathcal{O}_F^l$ and $\mathcal{O}_f^l$ are isomorphic over $D_f\cap D_F$ with fibrewise isometry (given by $A: \check{\mathcal{O}}_{F;x}^l\to\check{\mathcal{O}}_{f;x}^l$, $x\in D_f\cap D_F$). In particular, for $k\le p_f$, we have

$$\sum_{l=0}^{k}\dim\mathcal{O}_{F;x}^l=\sum_{l=0}^{k}\dim\mathcal{O}_{f;x}^l\le n+1,\quad x\in D_f\cap D_F.$$

## 3. Rigidity

To prove Theorem 2, we assume that $M$ is analytic and that $F: M\to S^N$ is a full analytic isometric immersion. In accordance with our earlier notation, we put $\mathcal{Z}_F^0=\operatorname{span}\{F(x)\odot F(x)\mid x\in M\}$ (cf. Theorem 1). Given $x\in D_f\cap D_F$, let $\gamma: (-\epsilon,\epsilon)\to M$ be a geodesic with $\gamma(0)=x$ whose image lies entirely in $D_f\cap D_F$. Setting $\sigma=F\circ\gamma$, we first note that $\sigma^{(s)}(0)\in\check{\mathcal{O}}_{F;x}^0\oplus\cdots\oplus\check{\mathcal{O}}_{F;x}^s$ with $\check{\mathcal{O}}_{F;x}^s$-components, for various $\gamma$, spanning $\check{\mathcal{O}}_{F;x}^s$. We first claim that

$$\sigma^{(s)}(0)\odot\sigma(0)\in\mathcal{Z}_F^r,\quad 0\le s\le 2r+1. \tag{17}$$

By definition, for $|t|<\epsilon$ we have

$$\sigma^{(l)}(t)\odot\sigma^{(l)}(t)\in\mathcal{Z}_F^l,\quad l\ge 0.$$

Differentiating, we obtain

$$\sigma^{(l+1)}(t)\odot\sigma^{(l)}(t)\in\mathcal{Z}_F^l.$$

Differentiating again, we get

$$\sigma^{(l+2)}(t)\odot\sigma^{(l)}(t)+\sigma^{(l+1)}(t)\odot\sigma^{(l+1)}(t)\in\mathcal{Z}_F^l,$$

so that

$$\sigma^{(l+2)}(t)\odot\sigma^{(l)}(t)\in\mathcal{Z}_F^{l+1}.$$

Differentiating this, we obtain

$$\sigma^{(l+3)}(t)\odot\sigma^{(l)}(t)+\sigma^{(l+2)}(t)\odot\sigma^{(l+1)}(t)\in\mathcal{Z}_F^{l+1}$$

and, by the above, this reduces to

$$\sigma^{(l+3)}(t)\odot\sigma^{(l)}(t)\in\mathcal{Z}_F^{l+1}.$$

Repeating this procedure, we obtain in general that

$$\sigma^{(l+2r)}(t)\odot\sigma^{(l)}(t),\ \sigma^{(l+2r+1)}(t)\odot\sigma^{(l)}(t)\in\mathcal{Z}_F^{l+r}.$$

Taking $l=0$, (17) follows.

We now assume that $f: M \to S^n$ is a full isometric immersion that is derived from $F$ and has $k$th-order contact with $F$. By Theorem 1, $\langle f \rangle \in \mathfrak{F}_F^k$; that is, we have $\langle f \rangle \perp \mathfrak{Z}_F^k$. Using (17), we have

$$\langle\langle f \rangle, \sigma^{(l)}(0) \odot \sigma(0) \rangle = \langle \langle f \rangle \sigma(0), \sigma^{(l)}(0) \rangle = 0, \quad 0 \leq l \leq 2k+1. \tag{18}$$

On the other hand, analyticity of $F$ implies that, for $x \in D_F$, we have

$$\check{\mathcal{O}}_{F;x}^0 \oplus \cdots \oplus \check{\mathcal{O}}_{F;x}^{p_F} = \mathbf{R}^{N+1}$$

(cf. [1]). Since the $\check{\mathcal{O}}_{F;x}^l$-components of $\sigma^{(l)}(0)$ span $\check{\mathcal{O}}_{F;x}^l$, combining (18) and $p_F \leq 2k+1$ yields

$$\langle f \rangle \sigma(0) = 0.$$

Since $\sigma(0) \in D_f \cap D_F$ was arbitrary and $D_f \cap D_F$ spans $\mathbf{R}^{N+1}$, we obtain that $\langle f \rangle = 0$. Hence $f$ is equivalent to $F$, and the theorem follows. □

Corollary 2. *Let $M$ be analytic and $F: M \to S^N$ a full analytic isometric immersion. Let $1 \leq k \leq p_F$, and assume that*

$$N \leq k + \sum_{l=0}^{k} \dim \mathcal{O}_{F;x}^l \tag{19}$$

*holds for some $x \in D_F$. Then any full isometric immersion $f: M \to S^n$ derived from $F$ that has $k$th-order contact with $F$ is equivalent to $F$.*

*Proof.* Using analyticity of $F$, we have

$$N+1 = \sum_{l=0}^{p_F} \dim \mathcal{O}_{F;x}^l \geq \sum_{l=0}^{k} \dim \mathcal{O}_{F;x}^l + p_F - k.$$

Combining this with (19), we obtain $2k+1 \geq p_F$ and Theorem 2 applies. □

## 4. Nonrigidity

We now assume that $F: M \to S^N$ is a full isometric immersion that is equivariant with respect to a homomorphism $\rho_F: G \to O(N+1)$, where $G$ is a closed subgroup of the group of isometries of $M$. This means that, for $a \in G$, we have

$$F \circ a = \rho_F(a) \cdot F.$$

The homomorphism $\rho_F$, which is uniquely determined by fullness of $F$, defines an orthogonal $G$-module structure on $\mathbf{R}^{N+1}$ and thereby on $S^2(\mathbf{R}^{N+1})$ as well. Since $G$ acts on $M$ by isometries, the higher fundamental forms and osculating bundles are also $G$-invariant. It follows that $\mathfrak{Z}_F^k$ and hence $\mathfrak{F}_F^k$ are $G$-invariant subspaces of $S^2(\mathbf{R}^{N+1})$. Moreover, for $f \leftharpoonup F$ and $a \in G$, we have

$$a \cdot \langle f \rangle = \langle f \circ a^{-1} \rangle;$$

in particular, $\mathfrak{M}_F^k$ is a $G$-invariant subset of $\mathfrak{F}_F^k$.

We now assume that $M = G/K$ is an isotropy irreducible Riemannian homogeneous space, and that $F$ is the standard minimal immersion $f_\lambda$: $M \to S^{n(\lambda)}$ defined by having its components comprise an orthonormal basis

$\{f_\lambda^i\}_{i=0}^{n(\lambda)}$ in the eigenspace $V_\lambda$ of $\Delta^M$ corresponding to $\lambda \in \operatorname{Spec}(M)$. Here, $V_\lambda$ is endowed with the normalized $L^2$-scalar product

$$\langle \phi, \phi' \rangle = \frac{n(\lambda)+1}{\operatorname{vol}(M)} \int_M \phi\phi' \cdot v_M, \quad \phi, \phi' \in V_\lambda,$$

where $v_M$ stands for the Riemannian volume element (cf. [4] for details). If $g$ is the original Riemannian metric on $M$ then $f_\lambda$ is isometric with respect to the Riemannian metric $\lambda/m \cdot g$ on $M$. The standard minimal immersion $f_\lambda$: $M \to S^{n(\lambda)}$ is equivariant with respect to the homomorphism

$$\rho_\lambda : G \to O(n(\lambda)+1)$$

that defines the orthogonal $G$-module structure on $V_\lambda$ ($\cong \mathbf{R}^{n(\lambda)+1}$) so that the general setting above applies. Note also that $D_{f_{\lambda_p}} = M$ and hence the higher fundamental forms and osculating bundles are defined everywhere. Now, any full minimal isometric immersion $f: M \to S^n$ of the Riemannian metric $\lambda/m \cdot g$ on $M$ is automatically derived from $f_\lambda$. This follows since, by minimality, the components of $f$ are in $V_\lambda$. Since $M$ and $f_\lambda$ are analytic, we obtain that $\mathfrak{M}_{f_\lambda}^k$ parameterizes the equivalence classes of full isometric minimal immersions (of the metric $\lambda/m \cdot g$ on $M$) that have $k$th-order contact with $f_\lambda$.

The standard minimal immersion $f_\lambda : M \to S^{n(\lambda)}$ is equivariant with respect to the homomorphism $\rho_\lambda : G \to O(n(\lambda)+1)$ that defines the orthogonal $G$-module structure on $V_\lambda$ ($\cong \mathbf{R}^{n(\lambda)+1}$) so that the general setting above applies. Note also that $D_{f_{\lambda_p}} = M$, so that the higher fundamental forms and osculating bundles are defined everywhere.

Even more specifically, we will be interested in the case when $M = S^m$, $m \geq 2$, with $\lambda = \lambda_p = p(p+m-1) \in \operatorname{Spec}(S^m)$, $p \geq 2$. Then $V_{\lambda_p} = \mathcal{H}_m^p$ is the linear space of spherical harmonics of order $p$ on $S^m$, and $f_{\lambda_p} : S^m \to S^{n(\lambda_p)}$ is the standard minimal immersion with geometric degree $p_{f_{\lambda_p}} = p$. (For $m = 2$, $f_{\lambda_p} : S^2 \to S^{2p}$ is nothing but the classical Veronese map.) Setting the origin $o$ at $(1, 0, \ldots, 0)$ with corresponding isotropy subgroup $SO(m) = [1] \oplus SO(m) \subset SO(m+1)$, we have

$$\mathcal{H}_m^p \,|\, _{SO(m)} = \mathcal{H}_{m-1}^0 \oplus \cdots \oplus \mathcal{H}_{m-1}^p,$$

so that it follows, as in [1], that

$$\beta_l(f_{\lambda_p})_o : S^l(T_o S^m) \to \mathcal{H}_{m-1}^l, \quad 1 \leq l \leq p,$$

is a surjective $SO(m)$-module homomorphism. In particular, $\mathcal{O}_{f_{\lambda_p};o}^l \cong \mathcal{H}_{m-1}^l$, $1 \leq l \leq p$, as $SO(m)$-modules. Using the standard coordinates $x_1, \ldots, x_m$ on $T_o S^m$, the $l$th power $S^l(T_o S^m)$ becomes the space of homogeneous polynomials in $x_1, \ldots, x_m$ of degree $l$; it also follows that this homomorphism is (up to a constant multiple) the harmonic projection operator given explicitly in [5]. Before the proof of Theorem 3, we digress here to make some specific observations. First, according to the remark at the end of Section 2, if a full minimal isometric immersion $f: S_{m/\lambda_p}^m \to S^n$ has $k$th-order contact with $f_{\lambda_p}$, then $k \leq p_f$ and

$$n(\lambda_k)+1=\dim \mathcal{H}_m^k \leq n+1.$$

This follows from the discussion above. Second, we wish to give an explicit form of the contact condition (3) for $k=2$. This amounts to the determination of the second fundamental form of $\beta_2(f_{\lambda_p})$ which, for the sake of completeness, we include here in detail. First of all, we have

$$S^2(T_o S^m)=S^2(\mathcal{H}_{m-1}^1)=\mathcal{H}_{m-1}^0 \oplus \mathcal{H}_{m-1}^2$$

as $SO(m)$-modules. In terms of matrices, the first term on the right-hand side contains the constant multiples of the identity, while the second consists of all traceless matrices. Hence

$$\beta_2(f_{\lambda_p})\colon S^2(\mathcal{H}_{m-1}^1) \to \mathcal{H}_{m-1}^2$$

is a constant multiple $c$, say, of the map

$$B \mapsto B-\frac{1}{m}\operatorname{trace} B \cdot I.$$

Using this, simple computation shows that, for $X, Y, U, V \in T_x S^m$, $x \in S^m$, we have

$$\begin{aligned}\langle \beta_2(f_{\lambda_p})(X,Y), \beta_2(f_{\lambda_p})(U,V)\rangle = {} & \frac{c^2}{2}(\langle X,U\rangle\langle Y,V\rangle+\langle X,V\rangle\langle Y,U\rangle) \\ & -\frac{c^2}{m}\langle X,Y\rangle\langle U,V\rangle. \end{aligned} \tag{20}$$

The value of the constant $c$ can be easily determined from the Gauss equation, since $f_{\lambda_p}$ induces the metric on $S^m$ with constant curvature $m/\lambda_p$. We obtain

$$c^2=\frac{1-m/\lambda_p}{1/2-1/m}. \tag{21}$$

On the other hand, for $k=2$ the contact condition is equivalent to

$$\langle \beta_2(f)(X,Y), \beta_2(f)(U,V)\rangle = \langle \beta_2(f_{\lambda_p})(X,Y), \beta_2(f_{\lambda_p})(U,V)\rangle,$$
$$X, Y, U, V \in T_x S^m, \quad x \in S^m.$$

Combining this with (20), we obtain that a full isometric minimal immersion $f\colon S^m_{m/\lambda_p} \to S^n$ has second-order contact with $f_{\lambda_p}$ if and only if

$$\begin{aligned}\langle \beta_2(f)(X,Y), \beta_2(f)(U,V)\rangle = {} & \frac{c^2}{2}(\langle X,U\rangle\langle Y,V\rangle+\langle X,V\rangle\langle Y,U\rangle) \\ & -\frac{c^2}{m}\langle X,Y\rangle\langle U,V\rangle. \end{aligned} \tag{22}$$

Theorem 2 immediately yields the following.

COROLLARY 3. *Let $f\colon S^m_{m/\lambda_p} \to S^n$ be a full isometric minimal immersion, and assume that* (22) *holds. If $p \leq 5$ then $f$ is equivalent to $f_{\lambda_p}$.*

REMARK. An isometric immersion $f: M \to S^n$ is said to be *constant isotropic* if $|\beta_2(f)(X,X)|^2$ is constant on the unit sphere bundle of $M$ (cf. [3]). If $f: S^m_{m/\lambda_p} \to S^n$ is an isometric minimal immersion that has second-order contact with $f_{\lambda_p}$ then, again by (22), $f$ is constant isotropic. In particular, for $p \geq 6$, Theorem 3 provides an abundance of constant isotropic minimal immersions between spheres.

Finally, we turn to the proof of Theorem 3. We give a lower estimate for the $SO(m+1)$-module $\mathfrak{F}^k_{\lambda_p}$. (In what follows, to simplify the notation, we omit $f$ from the lower indices.) By equivariance of $f_{\lambda_p}$, we have $\mathcal{Z}^k_{\lambda_p} = SO(m+1) \cdot \mathcal{Z}^k_o$, where

$$\mathcal{Z}^k_o = \operatorname{span}\{\mathcal{O}^{l'}_{\lambda_p;o} \odot \mathcal{O}^{l''}_{\lambda_p;o} \mid 0 \leq l', l'' \leq k\}.$$

Let $\bar{\mathfrak{F}}^k_{\lambda_p}$ be the sum of those $SO(m+1)$-submodules of $S^2(\mathcal{H}^p_m)$ that, when restricted to $SO(m)$, do not contain any irreducible component of $\mathcal{Z}^k_o$. Frobenius reciprocity applies to this situation, yielding

$$\bar{\mathfrak{F}}^k_{\lambda_p} \subset \mathfrak{F}^k_{\lambda_p}$$

(cf. [1] or Corollary 3.2 on p. 72 in [4]). On the other hand, we have

$$\begin{aligned}\mathcal{Z}^k_o &= \operatorname{span}\{\mathcal{H}^{l'}_{m-1} \odot \mathcal{H}^{l''}_{m-1} \mid 0 \leq l', l'' \leq k\}\\ &= \operatorname{span}\{v' \odot v'' \mid v', v'' \in \mathcal{H}^0_{m-1} \oplus \cdots \oplus \mathcal{H}^k_{m-1}\}\\ &= S^2(\mathcal{H}^0_{m-1} \oplus \cdots \oplus \mathcal{H}^k_{m-1})\\ &= \sum_{l=0}^{k} S^2(\mathcal{H}^l_{m-1}) \oplus \sum_{0 \leq l'' < l' \leq k} \mathcal{H}^{l'}_{m-1} \otimes \mathcal{H}^{l''}_{m-1}\end{aligned}$$

as $SO(m)$-modules.

The proof of Theorem 3 will be finished if we show the following.

THEOREM 5. *Let $m \geq 3$ and $2k+1 < p$. Then we have*

$$\bar{\mathfrak{F}}^k_{\lambda_p} = \sum_{(a,b) \in \Delta^p_k;\ a,b \text{ even}} V^{(a,b,0,\ldots,0)}_{m+1},$$

*where the notation is as in Theorem 3.*

*Proof.* According to the decomposition theorem of DoCarmo and Wallach [1], for $l'' < l'$ we have

$$\mathcal{H}^{l'}_{m-1} \otimes \mathcal{H}^{l''}_{m-1} = \sum_{s=0}^{l''} V^{(l'+l''-s,s,0,\ldots,0)}_m \oplus (\mathcal{H}^{l'-1}_{m-1} \otimes \mathcal{H}^{l''-1}_{m-1}) \tag{24}$$

and

$$S^2(\mathcal{H}^l_{m-1}) = \sum_{(c,d) \in \Delta^l_0;\ c,d \text{ even}} V^{(c,d,0,\ldots,0)}_m, \tag{25}$$

where we used the notation introduced in Section 1. (Note also that, for $m=4$, $V^{(c,d,0,\ldots,0)}_m$ means $V^{(c,d)}_4 \oplus V^{(c,-d)}_4$.) A quick comparison of (24) and (25) with (23) shows that $\mathcal{Z}^k_o$ has the following two (overlapping) sets of $SO(m)$-components (with multiplicity):

$$V^{(c,d,0,\ldots,0)}_m, \quad (c,d) \in \tilde{\Delta}_k, \tag{26}$$

where $\tilde{\Delta}_k \subset \mathbf{R}^2$ is the closed convex triangle with vertices $(1,0)$, $(k,k-1)$, and $(2k-1,0)$, and

$$V_m^{(c,d,0,\dots,0)}, \quad (c,d) \in \Delta_0^k; \ c,d \text{ even}.$$

Applying (25) to $S^2(\mathcal{H}_{m+1}^p)$, we obtain

$$S^2(\mathcal{H}_{m+1}^p) = \sum_{(a,b)\in\Delta_0^p;\ a,b \text{ even}} V_{m+1}^{(a,b,0,\dots,0)} \tag{28}$$

so that we must determine those $V_{m+1}^{(a,b,0,\dots,0)}$ in (28) that, when restricted to $SO(m)$, do not contain any $V_m^{(c,d,0,\dots,0)}$ in (26) and (27). This is done by the branching rule [1]:

$$V_{m+1}^{(\sigma_1,\dots,\sigma_r)}|_{SO(m)} = \sum_\tau V_m^\tau,$$

where the summation runs over all $\tau \in \mathbf{Z}^{[|m/2|]}$ for which

$$\sigma_1 \geq \tau_1 \geq \cdots \geq \sigma_r \geq |\tau_r| \quad \text{if } m+1 = 2r+1,$$

and

$$\sigma_1 \geq \tau_1 \geq \cdots \geq \tau_{r-1} \geq |\sigma_r| \quad \text{if } m+1 = 2r.$$

Assume first that $m \geq 5$. The branching rule for the components in (26) and (27) boils down to

$$a \geq c \geq b \geq d \geq 0.$$

Comparing the possible ranges of $(a,b)$ and $(c,d)$, this condition is equivalent to

$$b \geq 2k+1,$$

which restricts the range of $(a,b)$ to $\Delta_k^p$, and the theorem follows. The proof for $m = 3, 4$ is similar. □

## References


[1] M. DoCarmo and N. Wallach, *Minimal immersions of spheres into spheres,* Ann. of Math. (2) 93 (1971), 43–62.

[2] J. Eells and L. Lemaire, *Selected topics in harmonic maps,* Reg. Conf. Ser. in Math., 50, Amer. Math. Soc., Providence, RI, 1982.

[3] H. Nakagawa, *A characterization of the 3rd standard immersions of spheres into a sphere,* J. Differential Geom. 16 (1981), 511–527.

[4] G. Toth, *Harmonic maps and minimal immersions through representation theory,* Academic Press, Boston, 1990.

[5] N. I. Vilenkin, *Special functions and the theory of group representations,* Translations of Mathematical Monographs, 22, Amer. Math. Soc., Providence, RI, 1968.

[6] N. Wallach, *Minimal immersions of symmetric spaces into spheres,* Symmetric spaces, Pure and Appl. Math., 8, pp. 1–40, Dekker, New York, 1972.



Department of Mathematics
Rutgers University
Camden, NJ 08102