Extensions of Complex Varieties across C^1 Manifolds

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Introduction

More than twenty years ago, Shiffman [Sh] proved that a closed subset E in an open set $\Omega \subset \mathbf{C}^n$ with zero $(2k-1)$-dimensional Hausdorff measure does not obstruct complex varieties, in the sense that if V is a k-dimensional complex variety in $\Omega \backslash E$ and if $\bar{V}$ is the closure of V in Ω, then $\bar{V} \cap \Omega$ is also a k-dimensional complex variety in Ω. Intuitively speaking, the set E is too small to obstruct the variety V, since the topological dimensional of E does not exceed $2k-2$ while the boundary of V has dimension less than or equal to $2k-1$. Later on, Alexander [A] and Becker [Be] considered the case when E is a real-analytic set with dimension as large as $2k-1$ but, in addition, V has certain symmetric properties. In this paper, we consider the following problem: If E does not obstruct the variety V topologically (for the precise definition see 1.5 below), is it necessary that $\bar{V} \cap \Omega$ also be a complex variety? We give an affirmative answer to this question when E is a $(2k-1)$-dimensional C^1 manifold. In a subsequent paper we will show that the same conclusion holds when E is either a rectifiable curve or a high-dimensional rectifiable set subject to certain requirements. For a general closed set E, the problem is far from being solved.

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1. Basic Definitions and Main Result

We start with some basic definitions and properties of analytic varieties.

Suppose that Ω is an open subset in $\mathbf{C}^n$. A subset V of Ω is a *complex analytic variety* if, for each point p in Ω, there exists a neighborhood U of p and a family of functions that are holomorphic in U such that $V \cap U$ is the set of common zeros for the functions in the family.

A complex variety V is called *irreducible* if it cannot be decomposed as $V = V_1 \cup V_2$, where V_i are two distinct proper subvarieties of V. A variety is irreducible if and only if the set of its regular points, $V_{\text{reg}} = V - V_{\text{sing}}$, is a connected set. The irreducible components of the variety V are the sets $\bar{W}_i$, where the sets $\{W_i\}$ are the connected components of V_{reg}.

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A complex variety V is said to be *irreducible at a point* $a \in V$ if there is a fundamental system of neighborhoods $\{U_i\}$ about a such that each $V \cap U_i$ is irreducible. Therefore the variety V is locally irreducible if and only if it is irreducible at every point.

Throughout this paper, we shall use Λ^m to denote the m-dimensional Hausdorff measure in $\mathbf{C}^n$.

DEFINITION 1.1. Let E be a closed subset in an open set $\Omega \subset \mathbf{C}^n$, and let V be a complex variety in $\Omega \backslash E$ such that $\bar{V} \cap \Omega = V \cup E$. For p in E we say that the pair (V, E) has *property*$(\mathfrak{F})$ *at* p if the following condition is satisfied: *For every neighborhood B of p in Ω, there exist a neighborhood $U \subseteq B$ and a positive integer $\lambda = \lambda_{(V,E)}(p)$ such that*

$$(\bar{V} \backslash E) \cap U = \bigcup_{i=1}^{\lambda} A_i,$$

where the sets $\{A_i\}$ are the connected components of $(\bar{V} \backslash E) \cap U$. If the condition is satisfied at every point of E, then we say that the pair (V, E) has property$(\mathfrak{F})$.

From the preceding definition, we can make the following observations.

(1) If $\Lambda^{2k-1}(E) = 0$, then $\Omega \backslash E$ is connected. Moreover, E is nowhere separating in Ω; that is, if B is an open connected subset of Ω, then $B \backslash E$ is also connected. But this is not true if Ω is replaced by analytic varieties, as the following example shows: Let $k = n = 1$, and let V be the union of two open unit discs with centers $(-1, 0)$ and $(1, 0)$, respectively; if we choose E to be the origin, then $\bar{V}$ is connected, while $\bar{V} \backslash E$ has two connected components at the origin.

(2) If E is a $(2n-1)$-dimensional C^1 submanifold in a domain Ω, then $\lambda_{(\Omega \backslash E, E)} = 2$. Therefore, the pair $(\Omega \backslash E, E)$ has property$(\mathfrak{F})$.

We now give some examples in which the pair (V, E) does not have property$(\mathfrak{F})$ either at finitely many points or at a set of points with nonzero linear measure.

EXAMPLE 1.2. Let $k = n = 1$, and let E be defined by

$$E = \left\{[0,1] \backslash \bigcup_{m=1}^{\infty} \left(\frac{3m+2}{3m(m+1)}, \frac{3m+4}{3m(m+1)} \right) \right\} \cup [-1, 0]$$

$$\bigcup_{m=1}^{\infty} \left\{ (x, y) \in \mathbf{R}^2 : \left(x - \frac{1}{m} \right)^2 + y^2 \le \left(\frac{1}{3m(m+1)} \right)^2 \right\}.$$

Then, by letting $V = \mathbf{C}^1 \backslash E$, (V, E) has property$(\mathfrak{F})$ at all $p \in E \backslash (0, 0)$, but $\lambda_{(V,E)}(0, 0) = \infty$.

EXAMPLE 1.3. Let the sets E_1 and E_2 be given by

$$E_1 = \{(x, 0) : -1 \le x \le 1\} \quad \text{and}$$

$$E_2 = \{(x, x^3 \sin(1/x)) : -\tfrac{1}{2} \le x \le \tfrac{1}{2}\},$$

and let $V_i = D \backslash E_i$, where D is the unit disc in $\mathbf{C}^1$. Then $\lambda_{(V_1 \cap V_2, E_1 \cup E_2)}(0,0) = \infty$, while $\lambda_{(V_i, E_i)}(0,0) = 2$ for $i = 1, 2$.

EXAMPLE 1.4. In $\mathbf{C}^1$ we define, for $m = 1, 2, 3, \ldots$,

$$E_m = \bigcup_{l=1}^{2^m - 1} \left\{ \frac{2l-1}{2^m} \right\} \times \left[0, \frac{1}{2^{2m-1}} \right] \quad \text{and}$$

$$B_m = \bigcup_{l=1}^{2^m - 1} \left\{ (x, y) \in \mathbf{R}^2 : \left(x - \frac{2l-1}{2^m} \right)^2 + \left(y - \frac{3}{2^{2m}} \right)^2 = \frac{1}{2^{4m}} \right\},$$

and let

$$E = \bigcup_{m=1}^{\infty} \{E_m \cup B_m\} \cup \{[0,1] \times \{0\}\}.$$

The set E is thus a union of disjoint circles and lines in $[0,1] \times [0,1]$ such that all the points on $I = [0,1] \times \{0\}$ are limit points of the circles, and such that each circle is connected to I by a unique line in E. Note that $\Lambda^1(E) = 1 + \pi < \infty$. If we let $V = \mathbf{C}^1 \backslash E$, then (V, E) does not have property($\mathfrak{F}$) at any point $p \in [0,1] \times \{0\}$.

To consider continuation of analytic varieties, we noticed that property($\mathfrak{F}$) alone cannot give any positive result about the problem we are dealing with. A simple example to show this is to consider a 1-dimensional variety

$$V = \{(z_1, z_2) : z_1 = 0, |z_2| < 1/2\}$$

inside the unit ball $\mathbf{B}_2 \subset \mathbf{C}^2$. Then $E = \{(0, z_2) : |z_2| = 1/2\}$, and $\lambda_{(V,E)}(p) = 1$ for all $p \in E$. Certainly the set $\bar{V}$ is not a variety in $\mathbf{B}_2$. Notice that in this example the variety V has only one branch at every boundary point. To exclude this case, we introduce a definition.

DEFINITION 1.5. Suppose that (V, E) has property($\mathfrak{F}$) at a point $p \in E$, and that $\lambda_{(V,E)}(p) = 2$. Then we say (V, E) has property($\mathfrak{Q}$) at p. The pair (V, E) has property($\mathfrak{Q}$) if it has it at every point of E.

In other words, if the pair (V, E) has property($\mathfrak{Q}$), then there are two local components of V at every point on E. This excludes the example in the remark just before Definition 1.5. In fact, we can prove the following theorem as the main result of this paper.

THEOREM 1.6. *Let E be a closed $(2k-1)$-dimensional C^1 manifold in a domain $\Omega \subset \mathbf{C}^n$ and let V be a k-dimensional complex variety in $\Omega \backslash E$ with $(\bar{V} \backslash V) \cap \Omega \subset E$. If $(V, \bar{V} \cap E)$ has* property($\mathfrak{Q}$), *then $\bar{V} \cap \Omega$ is also a k-dimensional complex variety in Ω.*

Shiffman [Sh] proved that if E is a closed subset in an open set Ω with $\Lambda^{2k-1}(E) = 0$, and if V is a k-dimensional complex variety in $\Omega \backslash E$, then $\bar{V} \cap \Omega$ is a variety. Together with this, Theorem 1.6 leads to the following corollary.

COROLLARY 1.7. *Let E be a closed subset in $\Omega \subset \mathbf{C}^n$ such that:*

(1) *$E = E_1 \cup E_2$ with E_1 and E_2 closed in Ω; and*
(2) *$\Lambda^{2k-1}(E_1) = 0$, and E_2 is a $(2k-1)$-dimensional C^1 submanifold in $\Omega \backslash E_1$.*

If V is a k-dimensional complex variety in $\Omega \backslash E$ with $(\bar{V} \backslash V) \cap \Omega \subset E$, and if the pair $(V, \bar{V} \cap E)$ has property($\mathfrak{Q}$), *then $\bar{V} \cap \Omega$ is a complex variety in Ω.*

2. Some Prefatory Results

The idea of the proof given in the next section is to use a result of Harvey and Lawson that gives a characterization of odd-dimensional submanifolds in $\mathbf{C}^n$ to bound complex subvarieties. To make this paper complete, we quote this result and its related definitions as follows.

Suppose M is an oriented C^1 submanifold in $\mathbf{C}^n$ of dimension $2k-1$ with $k > 1$. We denote by J the standard complex structure in $\mathbf{C}^n$. If the complex tangent space

$$H_z(M) = T_z(M) \cap JT_z(M) = \{v \in T_z(M) : Jv \in T_z(M)\}$$

has complex dimension $k-1$ at each point $z \in M$, then M is said to be *maximally complex.* In case $k = 1$, M is maximally complex if the following *moment condition* is satisfied: *For every holomorphic* 1*-form ω in $\mathbf{C}^n$, $\int_M \omega = 0$.*

A compact set M in $\mathbf{C}^n$ is said to be a *scarred $2k-1$ cycle* (of class C^r) if it has the following properties:

(1) there is a closed subset S (called the *scar set*) in M such that S has zero $(2k-1)$-dimensional Hausdorff measure, and such that $M \backslash S$ is a C^r oriented $(2k-1)$-dimensional submanifold in $\mathbf{C}^n \backslash S$ for a positive integer r;
(2) the set $M \backslash S$ has finite volume; and
(3) the current $[M]$ defined by integrating over $M \backslash S$ has no boundary—that is, $\int_{M \backslash S} d\omega = 0$ for every C^∞ $(2k-2)$-form ω in $\mathbf{C}^n$ with compact support.

A scarred $(2k-1)$-cycle M is said to be maximally complex if the submanifold $M \backslash S$ is maximally complex.

THEOREM (Harvey and Lawson [HL]). *Suppose M is a maximally complex scarred $2k-1$ cycle of class C^1 in $\mathbf{C}^n$. (It suffices to assume that $M \backslash S$ is an oriented immersed submanifold of $\mathbf{C}^n \backslash S$ instead of an embedded submanifold.) Then there exists a unique holomorphic k-chain T in $\mathbf{C}^n \backslash M$, with* supp($T$) $\subset\subset \mathbf{C}^n$ *and with finite mass, such that $dT = [M]$ in $\mathbf{C}^n$. Furthermore, there is a compact subset A of M with zero $(2k-1)$-dimensional Hausdorff measure such that, for each point of $M \backslash A$ near which M is of class C^l, $1 \le l \le \infty$, there is a neighborhood in which* supp(T) $\cup M$ *is a regular C^l submanifold with boundary. In particular, if M is connected, then there exists a unique precompact irreducible complex k-dimensional subvariety V in $\mathbf{C}^n \backslash M$ such that $d[V] = \pm [M]$ with boundary regularity as above.*

In order to verify that the current $[M]$ is closed, we need to invoke the following result of Chirka, which allows us to apply a generalized Stokes's formula [St, p. 10] to complex varieties with C^1 boundaries.

THEOREM (Chirka [C1]). *Let D be a domain in $\mathbf{C}^n$, M a submanifold of D of dimension $2k-1$ of class C^l with $l \geq 1$, and V a purely k-dimensional complex variety of $D \backslash M$ with $M \subset \bar{V}$. Then there exists a (possibly empty) closed subset $E \subset M$ with zero $(2k-1)$-dimensional Hausdorff measure such that the pair (V, M) is a C^l manifold with boundary in a neighborhood of every point in $M \backslash E$.*

To obtain volume finiteness required in the definition of scarred cycle, we need a technical lemma. We shall use the notation that for $p \in \mathbf{C}^n$, $G_{n,k}(p)$ is the set of all complex k-planes through the point p, and π_Σ is the orthogonal projection from $\mathbf{C}^n$ to Σ for every $\Sigma \in G_{n,k}(p)$.

LEMMA 2.1. *Let E be a closed subset in a domain $\Omega \subset \mathbf{C}^n$, $p \in E$, and let V be a k-dimensional complex variety in $\Omega \backslash E$ such that $\bar{V} \cap \Omega = V \cup E$. Furthermore, suppose that near the point p,*

(1) *$E = E_1 \cup E_2$ with E_1, E_2 closed subsets, $\Lambda^{2k-1}(E_1) = 0$, $\Lambda^{2k}(E_2) = 0$, and*
(2) *there is a subset $\mathcal{G} \subset G_{n,k}(p)$ of positive measure such that for every $\Sigma \in \mathcal{G}$, the pair $(\pi_\Sigma(V \cup E_1), \pi_\Sigma(E_2))$ has* property($\mathfrak{F}$) *at the point $\pi_\Sigma(p)$.*

Then $\bar{V}$ has locally finite volume at p; that is, there exists a neighborhood $U(p) \subset \Omega$ of p such that

$$\Lambda^{2k}(\bar{V} \cap U(p)) < \infty.$$

Proof. By using Shiffman's result, we can assume without loss of generality that E_1 is empty, and hence $E = E_2$. We need to show that there exists a neighborhood $U(p)$ about p in Ω such that

$$\Lambda^{2k}(U(p) \cap \bar{V}) < \infty.$$

We can assume that $p \in E \cap \bar{V}$ is the origin of $\mathbf{C}^n$. First we prove a claim about projection mappings, which also can be found in Bishop [Bi], Shiffman [Sh], and Stolzenberg [St]:

Let U be a domain in $\mathbf{C}^n$ with $0 \in U$, and let S be a closed subset in U such that $\Lambda^{2k+1}(S) = 0$. Then, for almost all coordinate systems $z_1, \ldots, z_n$ of $\mathbf{C}^n$ and for every permutation σ of $(1, \ldots, n)$, there exist a neighborhood Δ_σ^k of $0 \in \mathbf{C}^k$ and a neighborhood Δ_σ^{n-k} of $0 \in \mathbf{C}^{n-k}$ such that, for $\Delta_\sigma = \Delta_\sigma^k \times \Delta_\sigma^{n-k}$, the projection map $\pi_\sigma : S \cap \Delta_\sigma \to \Delta_\sigma^k$ defined by

$$\pi_\sigma(z_1, \ldots, z_n) = (z_{\sigma(1)}, \ldots, z_{\sigma(k)})$$

is a proper mapping onto its image.

For if $\Lambda^{2k+1}(S)=0$, then for almost all $\Sigma\in G_{n,n-k}(0)$, $\Lambda^1(\Sigma\cap S)=0$. Thus for almost all coordinate systems $(z_1,\dots,z_n)$ of $\mathbf{C}^n$, $\Lambda^1(S\cap\pi_\sigma^{-1}(0))=0$. Hence we can find an $(n-k)$-ball $B_\sigma^2\subset\pi_\sigma^{-1}(0)$ such that

$$bB_\sigma^2\cap S=\emptyset.$$

Since the set S is closed, we can also find a k-ball $B_\sigma^1\subset(\pi_\sigma^{-1}(0))^\perp$ such that

$$bB_\sigma^2\cap S\cap\pi_\sigma^{-1}(B_\sigma^1)=\emptyset.$$

Thus by taking $\Delta_\sigma^k=B_\sigma^1$ and $\Delta_\sigma^{n-k}=B_\sigma^2$, the projection mapping π_σ is proper from $S\cap\Delta_\sigma$ onto its image. This establishes the claim. □

Applying this claim to the closed set $S=E\cap\bar{V}$ and using the hypothesis in (2), we can assume that the following conditions are satisfied for some coordinate systems $(z_1,\dots,z_n)$.

(a) For each fixed permutation σ, a small neighborhood $\Delta_\sigma=\Delta_\sigma^k\times\Delta_\sigma^{n-k}$ exists such that the projection $\pi_\sigma\colon\Delta_\sigma\cap(V\cup E)\to\Delta^k$ is a proper mapping onto its image. For the sake of simplicity, we drop the lower index σ for the neighborhood Δ_σ in the following argument.

(b) The map $\pi_\sigma\colon\Delta\cap E\to\Delta^k$ is proper to its image, since the set E is closed.

(c) $\pi_\sigma\colon\Delta\cap V\setminus\pi_\sigma^{-1}\pi_\sigma(E\cap\Delta)\to\Delta^k$ is also a proper mapping onto its image.

(d) $\pi_\sigma\colon\Delta\cap V\to\Delta^k$ is a locally biholomorphic mapping outside a proper subvariety $W_\sigma^1\subset V$. Note that $\Lambda^{2k-1}(W_\sigma^1)=0$.

(e) The set $\pi_\sigma(\bar{V}\cap\Delta)\setminus\pi_\sigma(E\cap\Delta)\subset\Delta^k\setminus\pi_\sigma(E\cap\Delta)$ has only finitely many connected components.

We denote by W_σ^2 the preimage set $\pi_\sigma^{-1}(\pi_\sigma(E\cap\Delta))$, and let $W_\sigma^3=W_\sigma^1\cup W_\sigma^2$. Then the map

$$\pi_\sigma\colon(V\setminus W_\sigma^3)\cap\Delta\to\pi_\sigma(\Delta\cap(V\setminus W_\sigma^1))\setminus\pi_\sigma(E\cap\Delta)$$

is a finite-sheeted covering map, since the map $\pi_\sigma|_{(V\setminus W_\sigma^3)\cap\Delta}$ is again a proper map to its image. As the map is nonsingular, it must be a finite-sheet covering map. Let

$$\pi_\sigma(\Delta\cap(V\setminus W_\sigma^1))\setminus\pi_\sigma(E\cap\Delta)=\bigcup_{j=1}^{n_\sigma}E_\sigma^j,$$

where E_σ^j is connected and n_σ is a positive integer. If we let m_σ^j be the number of sheets of π_σ over each component E_σ^j, then $m_\sigma^j<\infty$. Therefore we can take $U=\bigcap_\sigma\Delta_\sigma$ and obtain

$$\begin{aligned}\Lambda^{2k}(U\cap\bar{V})&\le\sum_\sigma\{\Lambda^{2k}(\Delta_\sigma\cap(V\setminus W_\sigma^3))+\Lambda^{2k}(\Delta_\sigma\cap V_{\text{reg}}\cap W_\sigma^3)\}\\&=\sum_\sigma\sum_{j=1}^{n_\sigma}m_\sigma^j\Lambda^{2k}(E_\sigma^j)+\sum_\sigma\Lambda^{2k}(\Delta_\sigma\cap V_{\text{reg}}\cap W_\sigma^3).\end{aligned}$$

The first part on the right-hand side of above inequality is finite, since it is at most $(\sum_\sigma\sum_{j=1}^{n_\sigma}m_\sigma^j)\Lambda^{2k}(\Delta_\sigma^k)$. The second part is also finite, as we demonstrate below.

For each fixed σ we denote Δ_σ by Δ, and apply Eilenberg's inequality from geometric measure theory [BZ, p. 101; F, p. 188] to obtain

$$\int^*_{\pi_\sigma(V\cap\Delta)} \Lambda^0(E\cap\Delta\cap\pi_\sigma^{-1}(y))\,d\Lambda^{2k}(y)\le C_{k,k}\Lambda^{2k}(E\cap\Delta)=0.$$

(Note that the projection π_σ is a Lipschitz map.) Thus there is a subset $W_\sigma^4\subset \pi_\sigma(V\cap\Delta)$ with $\Lambda^{2k}(W_\sigma^4)=0$ such that $\Lambda^0(E\cap\Delta\cap\pi_\sigma^{-1}(y))=0$ for all $y\in \pi_\sigma(V\cap\Delta)\backslash W_\sigma^4$. That is to say that for those y, $E\cap\Delta$ is disjoint from $\pi_\sigma^{-1}(y)$. Therefore $E\cap\Delta\cap\pi_\sigma^{-1}(\pi_\sigma((V\cap\Delta)\backslash\pi_\sigma^{-1}(\Sigma_\sigma^4)))=E\cap\Delta\cap\pi_\sigma^{-1}(\pi_\sigma(V\cap\Delta)\backslash\Sigma_\sigma^4)$ is the empty set; that is,

$$\pi_\sigma(E\cap\Delta)\cap\pi_\sigma((V\cap\Delta)\backslash\pi_\sigma^{-1}(\Sigma_\sigma^4))=\emptyset.$$

Consequently,

$$\pi_\sigma^{-1}(\pi_\sigma(E\cap\Delta))\cap((V\cap\Delta)\backslash\pi_\sigma^{-1}(\Sigma_\sigma^4))=\emptyset.$$

Thus, to estimate the volume of the set $V\cap\Delta\cap W_\sigma^3$, we need only consider its subset $V\cap\Delta\cap\pi_\sigma^{-1}(\pi_\sigma(E\cap\Delta))\cap\pi_\sigma^{-1}(W_\sigma^4)$, which can be decomposed as $A_\sigma^1\cup A_\sigma^2$, where

$$A_\sigma^1=\pi_\sigma^{-1}(\pi_\sigma(E\cap\Delta))\cap\pi_\sigma^{-1}(W_\sigma^4)\cap W_\sigma^1\cap\Delta$$

and

$$A_\sigma^2=\pi_\sigma^{-1}(\pi_\sigma(E\cap\Delta))\cap\pi_\sigma^{-1}(W_\sigma^4)\cap(V\backslash W_\sigma^1)\cap\Delta.$$

Since $\Lambda^{2k}(W_\sigma^1)=0$, $\Lambda^{2k}(W_\sigma^4)=0$; since π_σ is a local biholomorphism on $V\cap\Delta\backslash W_\sigma^1$, $\Lambda^{2k}(A_\sigma^1)=\Lambda^{2k}(A_\sigma^2)=0$ and the result follows. This completes our proof of the lemma. □

3. Proof of Theorem 1.6

We can assume that $E\cup V=\bar V\cap\Omega$; otherwise we simply replace E by $E\cap\bar V$. Our problem is purely local, so we suppose that $\Omega=\mathbf{B}_N$. Without loss of generality, suppose that $0\in E$. We need to show that $\bar V\cap\Omega$ is an analytic variety at the origin. To do this we first introduce some notation.

Let E be a closed subset in $\mathbf{C}^n$, and let V be a k-dimensional complex variety in $\mathbf{C}^n\backslash E$. For $t>0$ we define

$$S^t=\{z\in\mathbf{C}^n:|z|=t\},$$

$$B^t=\{z\in\mathbf{C}^n:|z|<t\},$$

$$V^t=V\cap S^t,\quad\text{and}$$

$$(V_{\text{reg}})^t=V_{\text{reg}}\cap S^t,\qquad (V_{\text{sing}})^t=V_{\text{sing}}\cap S^t.$$

Note that V^t is a real-analytic subvariety in $S^t\backslash E$ for all $t>0$. Therefore we let $(V^t)_{\text{reg}}$ be the set of regular points in V^t, and let $(V^t)_{\text{sing}}$ be the set of singular points in V^t.

(1) *Local finite volume of $\bar V$ near E*: Since E is a $(2k-1)$-dimensional C^1 manifold, $\Lambda^{2k}(E)=0$. For almost all $\Sigma\in G_{n,k}(0)$, $\pi_\Sigma(E)$ is also a $(2k-1)$-dimensional submanifold near $0\in\Sigma$. Thus, by Lemma 2.1, $\bar V$ has locally finite volume near the origin.

(2) *Generic position of* V^t: Let

$$T_1 = \{0 < t < 1: V^t \text{ is a } (2k-1)\text{-dimensional real analytic variety}\}.$$

Then $\Lambda^1((0,1)\backslash T_1) = 0$. Let $T_2 = \{t \in T_1: \Lambda^{2k-1}(\overline{V^t}) < \infty\}$. Then T_2 differs from T_1 by a set of zero linear measure. For if α is a number close to 1 then $\Lambda^{2k}(\bar{V} \cap \rho^{-1}[0,\alpha]) < \infty$, where $\rho(z) = |z|$. The map ρ satisfies a Lipschitz condition, and we can apply Eilenberg's inequality to obtain

$$\int_{[0,\alpha]}^{*} \Lambda^{2k-1}(\bar{V} \cap \rho^{-1}(t))\, d\Lambda^1(t) \leq C_{2k,1}\Lambda^{2k}(A) < \infty.$$

Thus $\Lambda^{2k-1}(\overline{V^t}) < \infty$ for almost all $0 < t \leq \alpha$, and the conclusion follows. Applying a similar argument to the closed set E satisfying $\Lambda^{2k}(E) = 0$ gives us a set $T_3 = \{t \in T_2: \Lambda^{2k-1}(E^t) = 0\}$ with the property that $\Lambda^1(T_2 \backslash T_3) = 0$. Hence T_3 differs from $(0,1)$ by a null set.

(3) *Maximally complex scarred cycle in* $V \cap S^t$ *for* $t \in T_3$: To apply the theorem of Harvey and Lawson, we fix a $t_0 \in T_3$. By Chirka's result in Section 2, there exists a closed subset $E^\# \subset E$ with $\Lambda^{2k-1}(E^\#) = 0$ such that (V,E) is a manifold pair at each point $p \in E \backslash E^\#$. Let $M = \bar{V} \cap S^{t_0}$. Then M is a compact subset in $\mathbf{B}^n$. Define the scar set S of M by $(V^{t_0})_{\text{sing}} \cup (V_{\text{sing}})^{t_0} \cup E^{t_0}$. Then $M \backslash S \subset (V^{t_0})_{\text{reg}} \cap (V_{\text{reg}})^{t_0}$ is an oriented $(2k-1)$-dimensional submanifold with orientation inherited from V_{reg}. By our choice of t_0, $\Lambda^{2k-1}(M) < \infty$ and $\Lambda^{2k-1}(S) = 0$.

LEMMA 3.1. *The current* $[M]$ *is closed; that is,* $d[M] = 0$.

Proof. Let φ be a $(2k-2)$-form in $\mathbf{C}^n$ with compact support. We want to show that

$$(d[M])(\varphi) = [M](d\varphi) = \int_{M \backslash S} i^* d\varphi = 0,$$

where $i: M \backslash S \to \mathbf{C}^n$ is the natural inclusion map. To apply the generalized Stokes's formula to the manifold $V_{\text{reg}} \cap B^{t_0}$, we write

$$\overline{V_{\text{reg}} \cap B^{t_0}} \backslash (V_{\text{reg}} \cap B^{t_0}) = (M \backslash S) \cup \delta \cup \{(E \backslash E^\#) \cap \overline{B^{t_0}}\},$$

where $\delta = (V_{\text{sing}} \cup E^\# \cup S) \cap \overline{B^{t_0}}$ is a compact subset of $\overline{B^{t_0}}$ with $\Lambda^{2k-1}(\delta) = 0$, and $(V_{\text{reg}} \cap B^{t_0}, (M \backslash S) \cup (E \backslash E^\#) \cap B^{t_0})$ is a manifold pair. Then, for every $(2k-1)$-form $d\varphi$,

$$0 = \int_{V_{\text{reg}} \cap \overline{B^{t_0}}} d(d\varphi) = \int_{(M \backslash S) \cup ((E \backslash E^\#) \cap \overline{B^{t_0}})} i^* d\varphi.$$

Since $\pi_\Sigma(E)$ is a $(2k-1)$-dimensional manifold near $0 \in \Sigma \cong \mathbf{C}^k$, there exists a neighborhood $\Delta \subset \mathbf{C}^k$ of $0 \in \mathbf{C}^k$ such that $\pi_\Sigma(E)$ divides Δ into two parts, say Δ_+ and Δ_-. By the property($\mathfrak{Q}$), we can choose a neighborhood $U \subset \mathbf{C}^n$ of 0 so small that $\pi_\Sigma(U) \subset \Delta$, and so that $V \cap U = A_1 \cup A_2$, where A_1 and A_2 are two disjoint connected sets. Thus $\pi_\Sigma(A_1)$ and $\pi_\Sigma(A_2)$ are two subdomains of Δ. By shrinking Δ if necessary, we can assume that $b\Delta_+$ and $b\Delta_-$

are of class C^1 and that both contain $\pi_\Sigma(E\cap U)$. Moreover, we assume that $\pi_\Sigma(A_1)=\Delta_+$. We claim that $\pi_\Sigma(A_2)\subset\Delta_-$. In fact, we can show that

$$\pi_\Sigma(A_1)\cap\pi_\Sigma(A_2)=\emptyset.$$

To prove this, let us first assume that π_Σ is proper on A_1 and on A_2. Since $\pi_\Sigma(A_i)$ is an open set, $b\pi_\Sigma(A_i)=\pi_\Sigma(bA_i)\subset b\Delta\cup\pi_\Sigma(E\cap U)$ and $A_i\cap\pi_\Sigma^{-1}\pi_\Sigma(E\cap U)=\emptyset$ for $i=1,2$. Therefore $b\pi_\Sigma(A_1)\cap b\pi_\Sigma(A_2)=\pi_\Sigma(E\cap U)$. Let $\mathfrak{N}=\pi_\Sigma(A_1)\cap\pi_\Sigma(A_2)$. If $\mathfrak{N}\neq\emptyset$, then it is an open set contained in Δ_+. If $\mathfrak{N}\neq\Delta_+$, then we can find a point p that lies in $b\mathfrak{N}\cap\Delta_+$. Thus

$$p\in\Delta_+\cap(\pi_\Sigma(E\cap U)\cup(b(\Delta_-)\backslash\pi_\Sigma(E\cap U))),$$

since $b\mathfrak{N}\subset b\Delta\cup\pi_\Sigma(E\cap U)$. The set Δ_+ meets the $\overline{\Delta_-}$ only along the set $\pi_\Sigma(E\cap U)$. Therefore $\Delta_+\cap(b\Delta_-\backslash\pi_\Sigma(E\cap U))=\emptyset$. Thus $p\in\Delta_+\cap\pi_\Sigma(E\cap U)$. But the latter set is an empty set, since $A_1\cap\pi_\Sigma^{-1}\pi_\Sigma(E\cap U)=\emptyset$. This implies that either $\mathfrak{N}=\emptyset$ or $\mathfrak{N}=\Delta_+=\pi_\Sigma(A_1)$. We show next that the latter case will never happen.

For if it occurs, then $\pi_\Sigma(A_2)\subset\pi_\Sigma(A_1)$. For $q\in\pi_\Sigma(A_2)$ we let $D_\pi(q)$ be the discriminant polynomial for the analytic covering map π_Σ [C2, p. 44]. Then D_π is a nonzero holomorphic function defined on $\pi_\Sigma(A_2)$. But on the other hand, since $\overline{A_1}\cap\overline{A_2}\subset E$, $D_\pi(q)\to 0$ when $q\to\pi_\Sigma(E\cap U)\subset b\pi_\Sigma(A_2)$. Therefore the uniqueness theorem for holomorphic functions gives that $D_\pi\equiv 0$, a contradiction that shows our claim.

For the general case, the mapping π_Σ is proper on $V\cap U\backslash\pi_\Sigma^{-1}\pi_\Sigma(E\cap U)$. Therefore

$$\pi_\Sigma(A_1\backslash\pi_\Sigma^{-1}\pi_\Sigma(E\cap U))\cap\pi_\Sigma(A_2\backslash\pi_\Sigma^{-1}\pi_\Sigma(E\cap U))=\emptyset.$$

This implies that $\pi_\Sigma(A_1)\cap\pi_\Sigma(A_2)-\pi_\Sigma(E\cap U)=\emptyset$, that is,

$$\pi_\Sigma(A_1)\cap\pi_\Sigma(A_2)\subset\pi_\Sigma(E\cap U).$$

But the set $\pi_\Sigma(E\cap U)$ is a closed set with empty interior, while the set $\pi_\Sigma(A_1)\cap\pi_\Sigma(A_2)$ is open. The contradiction from the above inclusion proves our claim for the general case.

By what we proved above, if on $(E\backslash E^\#)\cap B^{t_0}$ we use the induced orientation from V_{reg}, then at every point of $(E\backslash E^\#)\cap B^{t_0}$ we will have exactly two opposite orientations, one induced from A_1 and the other from A_2. This leads to

$$\int_{(E\backslash E^\#)\cap B^{t_0})} i^*d\varphi=0,$$

and hence establishes the lemma. □

Next we show that the manifold $M\backslash S$ satisfies the maximally complex condition or the moment condition. If $k>1$, we need to show that

$$\dim_{\mathbb{C}}(T_p(M\backslash S)\cap JT_p(M\backslash S))=k-1.$$

Let $W_1 = V_{\text{reg}} \cap B^{t_0}$. Then W_1 is a smooth complex submanifold of (complex) dimension k. Since $M \setminus S \subset bW_1$, the above equality holds naturally. In fact, $T_p(M \setminus S) = T_p(bW_1)$ by considering their real dimensions. Similarly, if $k=1$ and if ω is an arbitrary holomorphic 1-form in $\mathbf{B}_n$, then from the generalized Stokes's formula and from the proof of Lemma 3.1 we have

$$\int_M \omega = \int_{W_1} d\omega = \int_{W_1} \partial\omega = 0.$$

This shows that M satisfies the moment condition.

(4) *Completion of the proof:* By what we have obtained so far and by the result of Harvey and Lawson, there exists a unique irreducible k-dimensional complex variety W in $\mathbf{B}_n^{t_0} \setminus M$ such that $d[W] = \pm[M]$. The sign before $[M]$ indicates the consistency of two orientations, one from W and the other from $[M]$. Let us assume that they coincide, that is, $d[W] = [M]$. So for every $(2k-1)$-form ψ in $\mathbf{B}^n$ with compact support,

$$\int_W d\psi = \int_{M \setminus S} \psi.$$

On the other hand, we already know that $\int_{M \setminus S} \psi = \int_{\bar{V} \cap B^{t_0}} d\psi$. Therefore

$$d[W] = [M] = d[\bar{V} \cap B^{t_0}].$$

(Note that $\bar{V} \cap B^{t_0}$ defines a current by the choice of the number t_0.) Uniqueness implies that $W|_{B^{t_0} \setminus E} = V|_{B^{t_0}}$, and thus $\bar{V} \cap B^{t_0}$ is an analytic variety in B^{t_0}. The variety $V \cap B^{t_0}$ thus can be extended across $E \cap B^{t_0}$ for every $t_0 \in T_3$. In particular, $\bar{V}$ is an analytic variety at the origin. This completes our proof of the theorem. □

4. Final Comments

Before ending this paper, we would like to make some comments about our proof and raise some questions that are likely to be solved by the method adopted here.

1. In the proof of Lemma 3.1 we showed that if A_1 and A_2 are two local irreducible components of V near a point $p \in E$, then for almost all projections π_Σ to k-dimensional complex planes Σ, $\pi_\Sigma(A_1)$ and $\pi_\Sigma(A_2)$ are disjoint. This shows that the number of local irreducible components of V at p cannot exceed 2. Thus our property($\mathfrak{Q}$) can be generalized to have the following weak form:

The pair (V, E) has property($\mathfrak{Q}'$) at point p if, for every neighborhood B of p, there exists a neighborhood $U \subset B$ and an integer $\lambda(p) \geq 2$ such that

$$V \cap U = \bigcup_{i=1}^{\lambda(p)} A_i,$$

where $A_1, \ldots, A_{\lambda(p)}$ are disjoint k-dimensional proper subvarieties of V.

The pair (V, E) has property($\mathfrak{Q}'$) if and only if property($\mathfrak{Q}'$) holds at almost all (with respect to $(2k-1)$-dimensional Hausdorff measure) points of E.

From the proof of Lemma 3.1, we know that if (V, E) has property($\mathfrak{Q}'$) then V can only have two irreducible components near almost all points $p \in E$. Therefore the conclusion in Theorem 1.6 still holds if we replace property($\mathfrak{Q}$) by property($\mathfrak{Q}'$). Thus, as a corollary of our Theorem 1.6, we have the following result, which (when $m=2$ and E is connected) is proved by Chirka [C2, p. 258] using a different method.

COROLLARY 4.1. *Let E be a closed $(2k-1)$-dimensional C^1 submanifold of a domain $\Omega \subset \mathbf{C}^n$, and let $V_1, \dots, V_m$ be distinct k-dimensional irreducible complex varieties in $\Omega \backslash E$ $(m \geq 2)$ such that*

$$E \subset \bigcup_{i \neq j} (\bar{V}_i \cap \bar{V}_j \cap \Omega).$$

Then $(\bigcup_{i=1}^m V_i \cup E) \cap \Omega$ is a k-dimensional variety in Ω.

2. The proof of our main result could be carried out if, instead of using the result from Harvey and Lawson, we used the following structure theorem [K]:

Let S be a locally rectifiable current in $\mathbf{C}^n$ of dimension $2k$. Suppose that $bS = 0$ and that S is a positive current. Then $S \in Z_k(\mathbf{C}^n)$; that is, $S \in \Sigma n_i [X_i]$, where $\{X_i\}$ are k-dimensional subvarieties in $\mathbf{C}^n$ and $\{n_i\}$ are nonzero integers.

The advantage of using the result of Harvey and Lawson is that the method also works for varieties in $\mathbf{P}^n - \mathbf{P}^{n-q}$, where $\mathbf{P}^n$ is the complex projective space. (Note that $\mathbf{C}^n = \mathbf{P}^n - \mathbf{P}^{n-1}$.) This may provide a way to solve the first problem stated in comment 5.

3. Generally, suppose that E is a $(2k-1)$-dimensional orientable rectifiable set and that $\bar{V} = V \cup E$. Furthermore, suppose that $d[\bar{V}] = [E]$. Then $\bar{V} \cap \Omega$ is a complex variety, provided that (V, E) has property($\mathfrak{Q}$).

4. We cannot in general claim that the variety $\bar{V} \cap \Omega$ is regular at all boundary points in E, though it is regular on $E \backslash E_0$ for some closed subset $E_0 \subset E$ with $\Lambda^{2k-1}(E_0) = 0$. But a result from Harvey and Lawson [HL] and from Rossi [R] leads to the conclusion that $\bar{V} \cap \Omega$ is regular at E (and therefore has only finitely many isolated singularities) if E is class C^2 and Levi nonflat at every point.

5. Two problems arise naturally from our paper. (a) Let E be a closed C^1 manifold in an open subset $\Omega \subset \mathbf{C}^n$ of dimension $2n-1$, and let T be a closed positive current in $\Omega \backslash E$. Find a similar topological property for the pair (T, E) so that the trivial extension $\tilde{T}$ of T is a closed positive current in Ω. (b) Formulate some topological properties (similar to property($\mathfrak{Q}$)) for a pair (V, E) in $\mathbf{P}^n - \mathbf{P}^{n-q}$ $(q > 1)$ to ensure the extendibility of the variety $V \subset (\mathbf{P}^n - \mathbf{P}^{n-q}) \backslash E$.

References

[A] H. Alexander, *Continuing 1-dimensional analytic sets,* Math. Ann. 191 (1971), 143–144.

[Be] J. Becker, *Continuing analytic sets across* $\mathbf{R}^n$, Math. Ann. 195 (1972), 103–106.

[Bi] E. Bishop, *Conditions for the analyticity of certain sets,* Michigan Math. J. 11 (1964), 289–304.

[BZ] Yu. Burago and V. Zalgaller, *Geometric inequalities,* Springer, Berlin, 1988.

[C1] E. Chirka, *Regularity of the boundaries of analytic sets,* Math. USSR-Sb. 45 (1983), 291–335.

[C2] ———, *Complex analytic sets,* Kluwer, Dordrecht, 1989.

[F] H. Federer, *Geometric measure theory,* Springer, Berlin, 1969.

[GR] R. Gunning and H. Rossi, *Analytic functions of several complex variables,* Prentice-Hall, Englewood Cliffs, NJ, 1965.

[HL] R. Harvey and B. Lawson, *On boundaries of complex analytic varieties I,* Ann. of Math. (2) 102 (1975), 223–290.

[K] J. King, *The currents defined by analytic varieties,* Acta Math. 127 (1971), 185–220.

[R] H. Rossi, *Attaching analytic spaces to an analytic space along a pseudoconcave boundary,* Proc. Conf. Complex Analysis (Minneapolis), Springer, Berlin, 1965.

[Sh] B. Shiffman, *On the continuation of analytic sets,* Math. Ann. 185 (1970), 1–12.

[St] G. Stolzenberg, *Volumes, limits, and extensions of analytic varieties,* Lecture Notes in Math., 19, Springer, Berlin, 1966.

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