Proper Holomorphic Self-Mappings of Hartogs Domains in C^2

Xiaojun Huang & Yifei Pan

It has been an open problem whether a proper holomorphic self-mapping of a smooth bounded pseudoconvex domain in C^n is biholomorphic [B]. In [A], Alexander showed that each proper holomorphic self-mapping of the unit ball is an automorphism. This result has been generalized to several cases; for example, in the case of strictly pseudoconvex domains, it is due to Pincuk [Pi]. Bedford and Bell [BB] verified the result in the case of smooth real-analytic pseudoconvex domains. Recently, we have been able to prove that Alexander's theorem remains true in the case of smooth pseudoconvex Reinhardt domains whose Levi determinant vanishes to finite order and in the case of Reinhardt domains with real-analytic boundary (not necessarily pseudoconvex) [P1; P2]. We note that all previous results involve a kind of finite type condition for weakly pseudoconvex boundary points. In this note, we shall provide a class of domains in which many points of infinite type may occur and Alexander's theorem remains valid.

To state our result, we need some basic notation. Let Ω be a smooth bounded domain in C^2 and let r be a smooth defining function of Ω defined in C^2 such that $\Omega=\{r<0\}$ and $dr\neq 0$ on $b\Omega$. Define a function $\Lambda_r\colon C^2\to R$ by

$$\Lambda_r=-\det\begin{pmatrix} 0 & r_{\bar z} & r_{\bar w} \\ r_z & r_{z\bar z} & r_{\bar z w} \\ r_w & r_{z\bar w} & r_{w\bar w}\end{pmatrix},$$

which we call the *Levi determinant*. Note that if Ω is pseudoconvex then $\Lambda_r(z)\geq 0$ for $z\in b\Omega$ and that the set $W(b\Omega)=\{z\in b\Omega;\ \Lambda(z)=0\}$ is precisely that of all weakly pseudoconvex boundary points of Ω. We will study pseudoconvex Hartogs domains in C^2 which admit a defining function of the form

$$r(z,w)=|w|^2+\phi(z),$$

where $\phi(z)$ is a real-valued smooth function on C, so that

$$\Omega=\{(z,w)\in C^2;\ |w|^2+\phi(z)<0\}.$$

Let $E=\{(z,0)\}\cap\Omega$. E is called the *base domain* of the Hartogs domain Ω. We see that $E=\{(z,0)\in C^2;\ \phi(z)<0\}$.

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Since $r=|w|^2+\phi(z)$, it follows that the Levi determinant of Ω becomes

$$\Lambda_r=|\phi_z|^2-\phi\phi_{z\bar{z}},$$

and is defined on E. Therefore the structure of $W(b\Omega)$ is the Cartesian product of E_0 with a circle, where $E_0=\{(z,0)\in E;\ \Lambda_r(z)=0\}$. In this paper we prove the following theorem.

THEOREM. *Let $\Omega=\{|w|^2+\phi(z)<0\}$ be a smooth bounded pseudoconvex Hartogs domain in C^2 with $\phi(z)$ a smooth function. Suppose that E_0 has no interior limit points in E and that for each point (z_0,w_0) in $W(b\Omega)$, either the Levi determinant of Ω vanishes to finite order or $\partial^k\phi(z_0)/\partial z^k=0$ for all integers $k>0$. Then each proper holomorphic self-mapping of Ω is biholomorphic.*

REMARK. That $\partial^k\phi(z_0)/\partial z^k=0$ for all k implies that the Levi determinant at (z_0,w_0) vanishes to infinite order and therefore the point (z_0,w_0) is of infinite type. On the other hand, we note that the theorem contains a result in [P1] as a special case.

EXAMPLE. Let D be the convex domain

$$\left\{(z,w):|w|^2+2\exp\left[-\frac{1}{|z|^2}\right]<1\right\}.$$

Then each proper holomorphic self-mapping of D is biholomorphic by the preceding theorem.

This paper is based on a recent result, due to Boas and Straube [BS], which concludes that each smooth complete Hartogs domain in C^2 satisfies the condition R in the sense of Bell. As a consequence, proper holomorphic self-mappings of such domains are smooth up to the boundary. In order to prove that a self-map f of a smooth bounded domain is biholomorphic, it suffices (by a result of Pincuk [Pi]) to show that f is unbranched, that is, that $\det[f']\neq 0$ in Ω. We introduce the branch locus of a proper map $f:\Omega\to\Omega$ as $V_f=\{z\in\Omega:\det[f']=0\}$, a complex analytic variety of codimension 1. By the proper mapping theorem, $f(V_f)$ is an analytic variety in Ω and f induces a proper mapping $f:V_f\to f(V_f)$. We need the following fact from [P1].

LEMMA 1. *The branch locus V_f of a proper holomorphic mapping is a C^∞ manifold with boundary near most points of $\bar{V}_f\cap b\Omega$.*

LEMMA 2. *The domain $\Omega=\{|w|^2+\phi(z)<0\}$ is strictly pseudoconvex at the boundary points where $w=0$.*

Proof. Let $r=|w|^2+\phi(z)$. The smoothness gives that $dr=(\partial r/\partial z,\partial r/\partial w)=(\phi_z(z),\bar{w})\neq 0$ on $b\Omega$. In particular, $\phi_z(z)\neq 0$ when $w=0$ and $z\in b\Omega$. On the other hand, the Levi determinant is $\Lambda=|\phi_z(z)|^2+|w|^2\phi_{z\bar{z}}(z)$, which proves Lemma 2. □

LEMMA 3. *Let $f: \Omega \to \Omega$ be a proper mapping, where Ω is as in the theorem. If f is branched then there exist finitely many points $z_1, \dots, z_n \in E_0$ such that, for all iterations f^k of f,*

$$V_{f^k} \subset \bigcup_{i=1}^{n} \{(z_i, w)\} \cap \Omega.$$

Proof. Let V be an irreducible component of V_f. We first show that there exists a point $z_0 \in E_0$ such that $V = \{(z_0, w)\} \cap \Omega$. Assuming otherwise, it follows from Lemma 1 that $\bar{V} \cap \Omega$ must be a smooth curve near a boundary point so that the projection of the curve to the base domain E of Ω has 1-dimensional positive measure. This is a contradiction, since the projection of that curve must lie in E_0 which is assumed to not have any interior limit points. Here we have used the fact that boundary points where a proper map is branched must be weakly pseudoconvex; this follows from the identity

$$\Lambda_{r \circ f}(z) = |\det[f']|^2 \Lambda_r(f(z)),$$

where $r \circ f$ is still a defining function for Ω by an argument of Fornaess [F]. Thus V_f is of the form $\Omega \cap (E_1 \times C)$, where E_1 is a subset of E_0. But since E_0 is assumed to be discrete in E and Lemma 2 shows that E_0 is relatively compact in E, E_0 itself must be finite. The same argument handles V_{f^k}, so the proof of Lemma 3 is complete. □

Now we are in a position to prove the theorem.

Proof of Theorem. Let $F = (f, g)$ be a proper holomorphic self-mapping of Ω. If we assume the assertion of the theorem is false so the branch locus V_F is not empty, then it follows from Lemma 3 that there exist finitely many points $z_1, \dots, z_n$ in E_0 such that

$$V_{F^k} \subset \sum_{i=1}^{n} \{(z_i, w)\} \cap \Omega$$

for all k. Iterating F, if necessary, we may assume that F maps V to itself, where $V = \{(z_0, w)\} \cap \Omega \subset V_f$ for some $z_0 \in E_0$. So we may conclude that $f(z_0, w) = z_0$ and that $g(z_0, w)$ is a proper self-map of V. We may also assume $\phi(z_0) = -1$ so that $V = \{(z_0, w), |w| < 1\}$, the unit disk, and $g(z_0, w)$ becomes a Blaschke product. In order to arrive at a contradiction, we shall consider the vanishing order of ϕ at z_0. If the Levi determinant Λ vanishes to at most finite order at z_0, then we have a contradiction from Lemma 2.2 in [P1]. Hence it remains to consider the case when ϕ vanishes to infinite order with respect to the z derivative at z_0. Under this condition, we claim that g is independent of the variable z. Indeed, consider the complex tangent vector

$$L = \bar{w}\frac{\partial}{\partial z} - \phi'(z)\frac{\partial}{\partial w}.$$

The properness of F gives

$$g\bar{g}+\phi(f,\bar{f})=0. \tag{1}$$

Taking L of each side of (1), it follows that

$$\bar{w}(g_z\bar{g}+\phi'(f)f_z)-\phi'(z)(g_w\bar{g}+\phi'(f)f_w)=0. \tag{2}$$

Letting $z=z_0$, and using $f(z_0,w)=z_0$ and $\phi'(z_0)=0$, it is easily seen from (2) that $g_z(z_0,w)=0$. Taking L of each side of (2), it follows that

$$\bar{w}^2(g_z^{(2)}\bar{g}+\phi'(f)(f_z)^2+\phi'(f)f_{zz})$$
$$-\bar{w}\{\phi''(z)[g_w\bar{g}+\phi'(f)f_w]+\phi'(z)[g_w\bar{g}+\phi'(f)f_w]_z\}-\phi'(z)A_w=0,$$

where A is the left side of (2). Letting $z=z_0$ and using $\phi'(z_0)=\phi''(z_0)=0$, we have $g_z^{(2)}(z_0,w)=0$. Now it is clear that after taking L successively, we may conclude that g vanishes to infinite order at z_0, and therefore g is independent of the variable z by the uniqueness of holomorphic functions.

Note that $\det[f']=f_zg_w-f_wg_z=f_zg_w$ and that $f_z(z_0,w)=0$, since g is independent of z and $\det[F']$ vanishes on V_F. Since V_F consists of only varieties parallel to the w-axis, we see that $g_w\neq 0$, from which it follows that

$$g(w)=e^{i\theta}\frac{w-a}{1-\bar{a}w},$$

where $|a|<1$. Now we assume $z_0=0$ and $\phi(0)=-1$. Consider $F^2=F\circ F=(f^2(z,w),g^2(w))$. It is easy to verify that

$$\frac{\partial^k f^2}{\partial z^k}(z,w)\,|_{z=0}=0 \quad \text{for } k=0,1,2,3,$$

using the fact that $f(0,w)=f_z(0,w)=0$.

Hence, without loss of generality, we may assume that

$$\frac{\partial^k f}{\partial z^k}(z,w)\,|_{z=0}=0 \quad \text{for } k=0,1,2,3.$$

From this and the fact that $f(z,w)$ is smooth on $\bar{\Omega}$ and holomorphic on Ω, it easily follows that there is a constant $0<\delta<1$ such that for $|z|<\delta$ and $w\in\bar{\Delta}$,

$$|f(z,w)|<|z|^2.$$

CLAIM 1. *There exists a point* (z_0,w_0) *in* $b\Omega$ *such that* $|z_0|<\delta$ *and* $w_0\in\Delta$.

Proof of Claim 1. Seeking a contradiction, we suppose not. Then we have $\phi(z)\le -1$ for $|z|<\delta$; otherwise, if $\phi(z_0)>-1$ for some z_0 and $\phi(0)=-1$, by the mean-value theorem there is an α in $(0,1)$ with $\phi(\alpha z_0)$ in $(-1,0)$ and then $(\alpha z_0,\sqrt{-\phi(\alpha z_0)})$ is in $b\Omega$. Let $\lambda(z)=-\ln(-\phi(z))$. Then λ is subharmonic, since

$$\frac{\partial^2\lambda(z)}{\partial z\partial\bar{z}}=-\frac{\phi_{z\bar{z}}(z)\phi(z)-|\phi_z(z)|^2}{\phi^2(z)}\ge 0$$

by the pseudoconvexity of Ω. We notice λ has a maximal value at 0; by the maximum principle, $\lambda(z)=-1$ for $|z|<\delta$. This means that $b\Omega$ contains a piece of Levi flat hypersurface, which is impossible by our assumption. So Claim 1 is proved. □

Now, let $F^n = (f_n(z, w), g^n(w))$ be the nth iteration of F, and let (z_0, w_0) be a point as in Claim 1. We then have

$$|f_1(z_0, w_0)| < |z_0|^2 < \delta^2;$$

$$|f_2(z_0, w_0)| = |f(f_1(z_0, w_0), g_1(w_0))| < |f_1(z_0, w_0)|^2 < |z_0|^{2^2} < \delta^{2^2};$$

and, in general, for all n,

$$|f_n(z_0, w_0)| = |f(f_{n-1}(z_0, w_0), g_{n-1}(w_0))| < |z_0|^{2^n} < \delta^{2^n}.$$

On the other hand, by the properness of F^n, it follows that

$$\phi(f_n(z_0, w_0)) + |g^n(w_0)|^2 = 0.$$

Hence

$$|1 + \phi(f_n(z_0, w_0))| = 1 - |g^n(w_0)|^2.$$

Noting that

$$|1 + \phi(f_n(z_0, w_0))| \leq C|f_n(z_0, w_0)| \leq C\delta^{2^n}$$

for some constant C, we obtain for some positive constant C

$$\delta^{2^n} \geq C(1 - |g^n(w_0)|^2).$$

But this contradicts the following claim.

Claim 2. *For some positive constant C and $0 < b < 1$,*

$$(1 - |g^n(w_0)|^2) \geq Cb^n.$$

Proof of Claim 2. Using the basic properties of iteration of linear fractional transformations, we can argue as follows. If g is elliptic then all orbits of interior points are bounded away from the unit circle, so there is nothing to prove. If g is hyperbolic then g has both fixed points on the unit circle; picking a linear fractional transformation ϕ mapping these points to 0 and ∞ we have $g^n = \phi^{-1} \circ d^n \circ \phi$, where d is a dilation map $z \to \lambda z$ with λ a positive real number. If g is parabolic then the unique fixed point again lies on the unit circle and we have a similar formula $g^n = \phi^{-1} \circ t^n \circ \phi$, where t is the translation $z \to z+1$. In both of these cases the desired estimate follows from the explicit formula. The loxodromic case does not occur for automorphisms of the disk. This completes the proof of the claim. From all of the above, the proof of the theorem is complete. □

We make the following observation concerning the argument presented in this note. The key idea in proving that the branch locus of a proper self-map is empty is to show that the map has an invariant piece of branch locus if the branch locus is not empty. In the proof we made use of the compactness property of an analytic variety in the sense that a variety of a domain intersects any compact subset of the domain in only finitely many components of the variety. Now for $n \geq 1$ let us consider a proper holomorphic self-map of a bounded domain Ω in C^n, $f: \Omega \to \Omega$. Consider its iterations and the union of the critical points of iterations, that is,

$$V = \bigcup_{k=1}^{\infty} V_{f^k}.$$

In general, a union of varieties such as V may not be a variety. Our question is whether V has the compactness property mentioned above. Certainly V would have such a property if we could prove that V is actually a variety. To this end let us look at a simple example in the unit disk $D = \{z \in C; |z| < 1\}$.

EXAMPLE. Let $f(z) = z^2$. Then $f^k = z^{2^k}$ and $V_{f^k} = \{0\}$, $V = \{0\}$. Therefore V has the compactness property.

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Xiaojun Huang
Department of Mathematics
Washington University
St. Louis, MO 63130

Yifei Pan
Department of Mathematics
Purdue University
Ft. Wayne, IN 46805