A Characterization for 3-Spheres

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Introduction

Let M be an n-dimensional compact smooth manifold. Jacobowitz [3] proved that if $\psi: M \to R^{n+k}$ $(k \leq n-1)$ is a smooth immersion with $\psi(M)$ contained in a closed ball of radius λ, then the sectional curvature of M with respect to the induced metric satisfies $\sup K \geq \lambda^{-2}$. Coghlan and Itokawa [1] proved a pinching theorem for sectional curvature of a compact hypersurface $\psi: M \to R^{n+1}$. They proved that if $\psi(M)$ is contained in a closed ball of radius λ and the sectional curvature satisfies $\sup K = \lambda^{-2}$, then $\psi(M)$ is the boundary of the closed ball.

The purpose of the present paper is to consider the pinching of the Ricci curvature for the hypersurface $\psi: M \to R^{n+1}$. In the case of the standard embedding $\psi: S^n(c) \to R^{n+1}$ of the sphere $S^n(c)$, for each unit vector field X on $S^n(c)$ we have $\|\psi\|^2 \operatorname{Ric}(X,X) = n-1$, where Ric is the Ricci tensor of $S^n(c)$. Suppose that $\psi: M \to R^{n+1}$ is an arbitrary compact connected immersed hypersurface such that $\|\psi\|^2 \operatorname{Ric}(X,X) = n-1$ for all unit vector fields X on M. We consider the question: Must $\psi(M)$ be a Euclidean sphere?

In the present paper we answer this question in the affirmative for $n=3$. In fact, we prove the following theorem.

Theorem. *Let $\psi: M \to R^4$ be an orientable, compact and connected hypersurface. If $0 < \|\psi\|^2 \operatorname{Ric}(X,X) \leq 2$ for each unit vector field X on M, then $\psi(M)$ is a Euclidean sphere in R^4.*

Preliminaries

Let $\langle\ ,\ \rangle$ be the inner product on R^4 and let $\bar{\nabla}$ be the Euclidean connection on R^4. We denote by J_1, J_2, and J_3 the complex structures on R^4 which define the quaternion structure on R^4. Then we have

$$(1.1) \qquad J_1J_2 = -J_2J_1 = J_3, \quad J_2J_3 = -J_3J_2 = J_1, \quad J_3J_1 = -J_1J_3 = J_2;$$

$$(1.2) \qquad \bar{\nabla}J_i = 0, \quad \langle J_i, J_i \rangle = \langle\ ,\ \rangle, \quad i = 1,2,3.$$

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Let $\psi: M \to R^4$ be an orientable hypersurface of R^4, and let N be a unit normal vector field globally defined on M. Let g and ∇ be the induced metric and the Riemannian connection on M, respectively. Then we have

$$\bar{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \bar{\nabla}_X N = -AX, \quad X, Y \in \mathfrak{X}(M), \tag{1.3}$$

where A is the shape operator of M and $\mathfrak{X}(M)$ is the Lie algebra of vector fields on M.

Define the unit vector fields ξ_1, ξ_2, ξ_3 on M by $J_i \xi_i = N$, $i = 1, 2, 3$. Also define three $(1,1)$ tensor fields ϕ_i on M by setting $J_i X = \phi_i X + \eta_i(X)N$, $i = 1, 2, 3$, $X \in \mathfrak{X}(M)$, where the 1-forms η_i are respective duals of ξ_i. It can be verified that ϕ_i, ξ_i, η_i satisfy

$$\phi_i^2 = -I + \eta_i \otimes \xi_i, \quad \phi_i \xi_i = 0, \quad \eta_i \circ \phi_i = 0, \tag{1.4}$$

$$g(\phi_i X, \phi_i Y) = g(X, Y) - \eta_i(X)\eta_i(Y), \quad X, Y \in \mathfrak{X}(M), \ i = 1, 2, 3. \tag{1.5}$$

Now, using (1.2) and (1.3), it is easy to obtain

$$\nabla_X \xi_i = \phi_i AX, \quad X \in \mathfrak{X}(M). \tag{1.6}$$

For a local unit vector field e on M satisfying $g(e, \xi_i) = 0$ for a fixed i, $\{e, \phi_i e, \xi_i\}$ is a local orthonormal frame on M; such a frame will be referred to as an *adapted* frame. From the first equation in (1.6), using an adapted frame, we get $\operatorname{div} \xi_i = 0$, $i = 1, 2, 3$.

Using ψ as the position vector field in R^4 of the hypersurface, we define a smooth function ρ on M by $\rho = \langle \psi, N \rangle$, which is commonly known as the support function of the hypersurface. Using the complex structures J_i, we define three smooth functions ρ_i on M by $\rho_i = \langle J_i \psi, N \rangle$. Also define three vector fields $t_i \in \mathfrak{X}(M)$ by setting $J_i \psi = t_i + \rho_i N$. Then, using (1.2), (1.3), and $\bar{\nabla}\psi = I$ in $J_i \psi = t_i + \rho_i N$, we obtain

$$\nabla_X t_i = \phi_i X + \rho_i AX, \quad d\rho_i(X) = -g(AX, t_i) + \eta_i(X), \quad X \in \mathfrak{X}(M). \tag{1.7}$$

We also have

$$g(t_i, \xi_i) = \langle J_i \psi, \xi_i \rangle = -\langle \psi, J_i \xi_i \rangle = -\langle \psi, N \rangle = -\rho. \tag{1.8}$$

Proof of the Theorem

Using the equations in (1.7), we compute the Hessian of the function ρ_i as

$$\begin{aligned} H_{\rho_i}(X, Y) = &-g((\nabla_X A)(Y), t_i) - g(AX, \phi_i Y) \\ &- g(AY, \phi_i X) - \rho_i g(AX, AY). \end{aligned} \tag{1.9}$$

From the definition of mean curvature α and the Codazzi equation for the hypersurface M in R^4, we have

$$3X.\alpha = \sum_{i=1}^{3} g((\nabla_{e_i} A)(e_i), X), \quad X \in \mathfrak{X}(M), \tag{1.10}$$

where $\{e_1, e_2, e_3\}$ is a local orthonormal frame on M. Thus, using the adapted frame $\{e, \phi_i e, \xi_i\}$ and equation (1.10) in (1.9) to compute the Laplacian of ρ_i, we get

$$\Delta \rho_i = -3t_i.\alpha - \rho_i \operatorname{tr}.A^2. \tag{1.11}$$

From (1.7) we find $\operatorname{div} t_i = 3\rho_i\alpha$ and consequently $\operatorname{div} \alpha t_i = t_i.\alpha + 3\rho_i\alpha^2$. Thus equation (1.11) takes the form

$$\Delta \rho_i = \rho_i S - 3 \operatorname{div} \alpha t_i, \tag{1.12}$$

where $S = 9\alpha^2 - \operatorname{tr}.A^2$ is the scalar curvature of M.

Next we use the second equation in (1.7) to find $\operatorname{div}(\rho_i \alpha t_i)$ and $\operatorname{grad} \rho_i$ as

$$\begin{aligned} \operatorname{div}(\rho_i\alpha t_i) &= \alpha t_i.\rho_i + \rho_i \operatorname{div}(\alpha t_i) \\ &= -\alpha g(At_i, t_i) + \alpha g(t_i, \xi_i) + \rho_i \operatorname{div}(\alpha t_i); \end{aligned} \tag{1.13}$$

$$\operatorname{grad} \rho_i = -At_i + \xi_i. \tag{1.14}$$

Using (1.12), (1.13), and (1.14) in $\Delta\rho_i^2 = 2\rho_i\Delta\rho_i + 2\|\operatorname{grad}\rho_i\|^2$, we obtain

$$\begin{aligned} \Delta\rho_i^2 = {} & 2\rho_i^2 S - 6\alpha g(At_i, t_i) + 6\alpha g(t_i, \xi_i) - 6\operatorname{div}(\rho_i\alpha t_i) \\ & + 2\|At_i\|^2 - 4g(At_i, \xi_i) + 2, \end{aligned}$$

which in light of (1.7) and (1.8) can be rearranged as

$$\Delta\rho_i^2 = 2\rho_i^2 S - 2\operatorname{Ric}(t_i, t_i) - 6\alpha\rho + 4(d\rho_i(\xi_i) - 1) + 2 - 6\operatorname{div}(\rho_i\alpha t_i).$$

Since ξ_i is divergence-free (cf. preliminaries), we have

$$\operatorname{div}(\rho_i\xi_i) = \xi_i.\rho_i + \rho_i \operatorname{div}\xi_i = d\rho_i(\xi_i).$$

Thus we have

$$\Delta\rho_i^2 = 2\rho_i^2 S - 2\operatorname{Ric}(t_i, t_i) - 6(1 + \rho\alpha) + 4 + 4\operatorname{div}(\rho_i\xi_i) - 6\operatorname{div}(\rho_i\alpha t_i).$$

Integrating this equation over M and using Minkowski's formula (cf. [2]), we get

$$\int_M \{\rho_i^2 S - \operatorname{Ric}(t_i, t_i) + 2\}\, dv = 0, \quad i = 1, 2, 3. \tag{1.15}$$

Define the unit vector fields $\hat{t}_i$ by $t_i = \|t_i\|\hat{t}_i$. Then, using $\|\psi\|^2 = \|J_i\psi\|^2 = \|t_i\|^2 + \rho_i^2$ in (1.15), we have

$$\int_M \{\rho_i^2(S + \operatorname{Ric}(\hat{t}_i, \hat{t}_i)) + (2 - \|\psi\|^2 \operatorname{Ric}(\hat{t}_i, \hat{t}_i))\}\, dv = 0.$$

From the hypothesis of the theorem it follows that the Ricci curvature is nonnegative and so is the scalar curvature S, and thus from the above integral we get

$$\rho_i^2(S + \operatorname{Ric}(\hat{t}_i, \hat{t}_i)) = 0 \quad \text{and} \quad \|\psi\|^2 \operatorname{Ric}(\hat{t}_i, \hat{t}_i) = 2.$$

The second equation gives $\operatorname{Ric}(\hat{t}_i, \hat{t}_i) > 0$, and thus from the first equation we have $\rho_i = 0$, $i = 1, 2, 3$. Now, using $\rho_i = 0$ in the second equation in (1.7), we get $At_i = \xi_i$. Since the ξ_i are globally defined unit vector fields on M and A is a linear operator, it follows that the t_i are nowhere zero on M and thus the $\hat{t}_i$ are defined everywhere on M. From (1.1) and (1.2), it can be easily deduced that $\{\hat{t}_1, \hat{t}_2, \hat{t}_3\}$ is an orthonormal frame on M. Then equation (1.11) together with $\rho_i = 0$ ensures that α is a constant.

Our next aim is to show that the support function ρ is nowhere zero on M. For this, from $\|\psi\|^2 \operatorname{Ric}(\hat{t}_1, \hat{t}_1) = 2$ we observe that $\operatorname{Ric}(t_1, t_1) > 0$ and consequently, as $At_1 = \xi_1$, $3\alpha g(At_1, t_1) - \|At_1\|^2 = -3\alpha\rho - 1 > 0$. This proves ρ is nowhere zero on M.

We write $\psi = t + \rho N$ for some $t \in \mathfrak{X}(M)$. Then it is easy to get

$$\nabla_X t = X + \rho AX, \quad d\rho(X) = -g(AX, t), \quad X \in \mathfrak{X}(M).$$

Using these equations and the fact that α is a constant, we compute the Laplacian of ρ as $\Delta\rho = -3\alpha - \rho \operatorname{tr}. A^2$. Thus we have

$$\int_M \{3\alpha + \rho \operatorname{tr}. A^2\}\, dv = 0. \tag{1.16}$$

As α is constant, from Minkowski's formula we have

$$\int_M 3\alpha\, dv = -\int_M 3\rho\alpha^2\, dv. \tag{1.17}$$

Equations (1.16) and (1.17) give

$$\int_M \rho(3\alpha^2 - \operatorname{tr}. A^2)\, dv = 0. \tag{1.18}$$

The Schwarz inequality states that $3\alpha^2 \le \operatorname{tr}. A^2$, with equality holding at a point if any only if it is an umbilic point. Since M is connected and $\rho \neq 0$, the integral (1.18) gives $3\alpha^2 = \operatorname{tr}. A^2$, proving that M is totally umbilic; our theorem then follows from [4, Thm. 5.1, p. 30]. □

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