Inner Ideals in Exceptional JBW*-Triples[1]

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1. Introduction

This paper is concerned with the study of the structure of JBW*-triples, which have interested many authors in recent years ([2], [4], [5], [7], [10], [12], [13], [14], [16], [21]). In particular, the investigation of the inner ideal structure of JBW*-triples begun in [8] and [9] is continued here.

A JBW*-triple A is said to be *special* if it is isomorphic to a subtriple of a W*-algebra, and it is said to be *exceptional* if every homomorphism from A onto a subtriple of a W*-algebra is zero. In [2] it is shown that every JBW*-triple has a unique decomposition as an M-sum of a special JBW*-triple and an exceptional JBW*-triple. In this paper the weak* closed inner ideals in an exceptional JBW*-triple are characterized. Exceptional JBW*-triples provide examples of JBW*-triples of Type I, and indeed the methods of proof used here also apply to those JBW*-triples $C(\Omega, B)$ of Type I which consist of all continuous functions from a hyperstonian space Ω into a finite-dimensional Cartan factor B. Since this class does not exhaust the class of all Type I JBW*-triples, the attention of this paper is restricted to the exceptional case.

Let $\mathbb{O}$ be the complex octonian algebra and let M_3^8 be the 27-dimensional vector space of 3×3 hermitian matrices with entries in $\mathbb{O}$. Let $B_{1,2}^8$ be the 16-dimensional vector space of 1×2 matrices with entries in $\mathbb{O}$. When endowed with the triple product

$$\{a\,b\,c\}=(a\circ b^{\wedge t})\circ c+(c\circ b^{\wedge t})\circ a-(a\circ c)\circ b^{\wedge t}$$

(where $b\to b^{\wedge}$ is the conjugate linear involution in $\mathbb{O}$ applied pointwise to b, b^t is the transpose of b, and $a\circ b=(ab+ba)/2$) and the spectral norm as defined by Loos [18], M_3^8 is an exceptional JBW*-triple. Similarly, when endowed with the triple product

$$\{a\,b\,c\}=(a(b^{\wedge t}c)+c(b^{\wedge t}a))/2$$

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and the spectral norm, $B_{1,2}^8$ is also an exceptional JBW*-triple. Indeed these two examples exhaust the class of exceptional JBW*-triples which cannot be decomposed nontrivially as an M-sum, the exceptional JBW*-factors [19]. For hyperstonian spaces Ω_1 and Ω_2 the spaces $C(\Omega_1, M_3^8)$ and $C(\Omega_2, B_{1,2}^8)$ of continuous functions from Ω_1 and Ω_2 into M_3^8 and $B_{1,2}^8$, respectively, endowed with the pointwise triple product and the supremum norm, are exceptional JBW*-triples. It follows from the results of Barton, Dang and Horn [2] that every exceptional JBW*-triple is isomorphic to an M-sum of two such JBW*-triples.

The paper is organized as follows. In Section 2 definitions are given, notation is established and certain important preliminary results are proved. In particular it is shown that in order to obtain the weak* closed inner ideals in an M-sum of JBW*-triples it is sufficient to obtain those in its summands. In Section 3 the inner ideal structure of M_3^8 and $B_{1,2}^8$ is investigated. McCrimmon [20] showed that the nontrivial inner ideals in M_3^8 are either Peirce 2-spaces or point spaces. Here it is shown that the same applies to $B_{1,2}^8$, a result which has been proved recently by Neher [22] using different methods. In Section 4, the results of Section 3 are used to investigate the weak* closed inner ideals in $C(\Omega_1, M_3^8)$ and $C(\Omega_2, B_{1,2}^8)$. It is shown that these are essentially the same in each case. A weak* closed inner ideal J provides a unique M-decomposition of the JBW*-triple into weak* closed ideals I_1, I_2, I_3 and I_4 and a corresponding M-decomposition of J such that $J \cap I_1$ is zero, I_2 is contained in J, $J \cap I_3$ is an M-sum of Peirce 2-spaces and $J \cap I_4$ is an M-sum of weak* closed ideals $J \cap I_{4j}$, which is a sum of Peirce 2-spaces of j collinear tripotents in J. A combination of the two parts provides a description of a weak* closed inner ideal in any exceptional JBW*-triple.

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2. Preliminaries

A unital Jordan*-algebra A which is also a complex Banach space such that for elements a and b in A, $\|a^*\| = \|a\|$, $\|a \circ b\| \leq \|a\| \|b\|$ and $\|\{a\,a\,a\}\| = \|a\|^3$, where

(1) $$\{a\,b\,c\} = a \circ (b^* \circ c) + (a \circ b^*) \circ c - b^* \circ (a \circ c)$$

is the Jordan triple product on A, is said to be a *Jordan C*-algebra* [26] or *JB*-algebra* [27]. A Jordan C*-algebra which is the dual of a Banach space is said to be a *Jordan W*-algebra* [6] or a *JBW*-algebra* [27]. Examples of Jordan C*-algebras and Jordan W*-algebras are C*-algebras equipped with the Jordan product

(2) $$a \circ b = (ab + ba)/2.$$

A complex vector space A equipped with a triple product $(a,b,c) \to \{a\,b\,c\}$ from $A \times A \times A$ to A which is symmetric and linear in the first and third variables, conjugate linear in the second variable and satisfies the identity

$$[D(a,b), D(c,d)] = D(\{a\,b\,c\}, d) - D(c, \{d\,a\,b\}) \\ = D(a, \{b\,c\,d\}) - D(\{c\,d\,a\}, b),$$

where [,] denotes the commutator and D is the mapping from $A \times A$ to the space of linear operators on A defined by $D(a,b)c = \{a\,b\,c\}$, is said to be a *Jordan*-triple.* When A is also a Banach space such that D is continuous from $A \times A$ to the Banach space $B(A)$ of bounded linear operators on A and, for each element a in A, $D(a,a)$ is hermitian with nonnegative spectrum and satisfies $\|D(a,a)\| = \|a\|^2$, then A is said to be a *JB*-triple.* A JB*-triple A which is the Banach space dual of a Banach space is said to be a *JBW*-triple.* A linear subspace J of a JBW*-triple A is said to be an *inner ideal* if $\{J\,A\,J\}$ is contained in J, and is said to be an *ideal* if $\{A\,A\,J\} + \{A\,J\,A\}$ is contained in J.

An element u in a JB*-triple A is said to be a *tripotent* if $\{u\,u\,u\}$ is equal to u. The set of tripotents in A is denoted by $\mathcal{U}(A)$. For each tripotent u, the operators $Q(u), P_j^A(u)$, $j = 0, 1, 2$, are defined by

$$Q(u)a = \{u\,a\,u\}, \quad P_2^A(u) = Q(u)^2,$$
$$P_1^A(u) = 2(D(u,u) - Q(u)^2), \quad P_0^A(u) = I - 2D(u,u) + Q(u)^2.$$

The linear operators $P_j^A(u)$, $j = 0, 1, 2$, are projections onto the eigenspaces $A_j(u)$ of $D(u,u)$ corresponding to the eigenvalues $j/2$, and

(3) $$A = A_0(u) \oplus A_1(u) \oplus A_2(u)$$

is the *Peirce decomposition* of A relative to u. For $i, j, k = 0, 1, 2$, $A_i(u)$ is a sub-JB*-triple such that $\{A_i(u)\,A_j(u)\,A_k(u)\} \subseteq A_{i-j+k}(u)$ when $i - j + k = 0$, 1 or 2 and $\{0\}$ otherwise, and, as in [10],

(4) $$\{A_2(u)\,A_0(u)\,A\} = \{A_0(u)\,A_2(u)\,A\} = \{0\}.$$

Let A be a JBW*-triple. Then the operators $D(a,b), Q(u), P_j^A(u)$, $j = 0, 1, 2$, are weak* continuous and $A_j(u)$, $j = 0, 1, 2$, are sub-JBW*-triples of A. Moreover, $A_0(u)$ and $A_2(u)$ are weak* closed inner ideals in A and $A_2(u)$ is a JBW*-algebra with product $(a,b) \to \{a\,u\,b\}$, unit u, and involution $a \to \{u\,a\,u\}$. The cone of positive elements in $A_2(u)$ is denoted by $A_2(u)^+$.

A pair u, v of elements in $\mathcal{U}(A)$ is said to be *orthogonal* if one of the following (equivalent) conditions holds:

$$D(u,v) = 0; \qquad \{u\,u\,v\} = 0; \qquad u \in A_0(u);$$
$$D(v,u) = 0; \qquad \{v\,v\,u\} = 0; \qquad v \in A_0(u).$$

For two elements u and v in $\mathcal{U}(A)$ we write $u \leq v$ if one of the following (equivalent) conditions is satisfied:

$$\{u\,u\,v\} = u; \qquad D(u,v) = D(u,u); \qquad P_2^A(u) = v;$$
$$\{u\,v\,u\} = u; \qquad D(v,u) = D(u,u).$$

This defines a partial ordering on $\mathcal{U}(A)$ with respect to which $\mathcal{U}(A)$ with a greatest element adjoined forms a complete lattice. An element u in $\mathcal{U}(A)$ is *maximal* if and only if $A_0(u)$ is zero or, equivalently, if and only if u is an extreme point of the unit ball A_1 in A. Notice that when u is maximal then $A_1(u)$ is a weak* closed inner ideal in A. An element u in $\mathcal{U}(A)$ is said to be *unitary* provided that $A_2(u)$ coincides with A. A pair u, v of elements in $\mathcal{U}(A)$ is said to be *collinear* if u lies in $A_1(v)$ and v lies in $A_1(u)$. For details of the results described above the reader is referred to [1], [7], [10], [15], [16], and [21].

Let I be a weak* closed ideal in the JBW*-triple A and let

$$I^\perp = \bigcap\{A_0(u) : u \in \mathcal{U}(I)\}.$$

Then $I^\perp$ is a weak* closed ideal in A such that

$$A = I \oplus_M I^\perp.$$

Moreover, for all elements a in I and b in $I^\perp$, $D(a,b)$ is zero ([13], [21]). Indeed the M-summands in a JBW*-triple are precisely its weak* closed ideals ([1], [2], [3], [13]).

LEMMA 2.1. *Let I be a weak* closed ideal in the JBW*-triple A and let J be a weak* closed inner ideal in A. Then $J \cap I$ and $J \cap I^\perp$ are weak* closed inner ideals in I and $I^\perp$, respectively, and*

$$J = (J \cap I) \oplus_M (J \cap I^\perp).$$

Proof. Let P be the M-projection onto I and let u be a tripotent. Simple calculations show that Pu is a tripotent in I and $(1-P)u$ is a tripotent in $I^\perp$. Now let u lie in J. Then, since J is an inner ideal, $A_2(u)$ is contained in J. But $Pu \le u$, and it follows from [7, Lemma 2.4] that Pu lies in $A_2(u)$ and, hence, in J. Since the linear span of $\mathcal{U}(J)$ is weak* dense in J and the M-projection P is weak* continuous, it follows that for all a in J, Pa lies in J. Similarly, $(1-P)a$ lies in J. Clearly $J \cap I$ and $J \cap I^\perp$ are weak* closed inner ideals in I and $I^\perp$, respectively, and this completes the proof. □

Recall that a JBW*-triple A is said to be a *factor* if there does not exist a nontrivial weak* closed ideal in A.

For each element a in the JBW*-triple A there exists a smallest element $r(a)$ in $\mathcal{U}(A)$, called the *support* of a, such that a is a positive element in the JBW*-algebra $A_2(r(a))$. It follows that $r(a)$ is the unit element in the smallest sub-JBW*-triple of A containing a which is in fact a sub-JBW*-algebra of $A_2(r(a))$ ([7], [10]). The full force of the following result is not needed in this paper, but it is included since it is of independent interest.

LEMMA 2.2. *Let A be a JBW*-triple and let a be an element of A with support $r(a)$. Then $A_2(r(a))$ is the weak* closure of $\{a\,A\,a\}$.*

Proof. Let J be the weak* closure of the linear subspace $\{aAa\}$. Since $A_2(r(a))$ is a weak* closed inner ideal and a belongs to $A_2(r(a))$, it follows that $\{aAa\}$ and therefore J are subsets of $A_2(r(a))$.

The element a is positive in the JBW*-algebra $A_2(r(a))$, and $r(a)$ coincides with the support projection of a in this JBW*-algebra. Therefore there exists a sequence $\{q_n)$ of real odd polynomials with zero linear summand such that $r(a)$ is the weak* limit of the sequence $(q_n(a))$. Notice that the $(2n+1)$th power of a, $n \geq 0$, computed in the Jordan algebra $A_2(r(a))$ coincides with the $(2n+1)$th power of a computed in the Jordan triple A. Since the elements a^{2n+1}, $n \geq 1$, belong to $\{aAa\}$ it follows that $r(a)$ is an element in J. Now, J is a weak* closed inner ideal and therefore $r(a)Ar(a)$ is a subset of J. □

Let A be a JBW*-triple. By [13, Thm. 3.13] or [7, Thm. 4.6], the supremum of any set of pairwise orthogonal tripotents exists in the partially ordered set $(\mathcal{U}(A), \leq)$. A set F of pairwise orthogonal nonzero minimal tripotents is said to be a *frame* if the supremum of F is a maximal tripotent. If A is finite-dimensional then A possesses a frame and all the frames have the same finite cardinality, the *rank* of A [18]. In such a JBW*-triple A a tripotent u is said to be of *rank n* provided that $A_2(u)$ is of rank n.

Let $\mathfrak{D}_{\mathbb{R}}$ be the real quaternion algebra with basis $1, i_1, i_2, i_3$, where for $j, k, r = 1, 2, 3$,

$$i_j^2 = -1, \qquad i_j i_k = \epsilon_{jkr} i_r.$$

Then $\mathfrak{D}_{\mathbb{R}}$ has a natural real involution $a \to a^-$ defined by

$$1^- = 1, \qquad i_j^- = -i_j.$$

Let $\mathcal{O}_{\mathbb{R}}$ be the (non-associative) real octonian algebra consisting of elements $a + bl$ with a and b in $\mathfrak{D}_{\mathbb{R}}$ and l an element such that, for all a in $\mathfrak{D}_{\mathbb{R}}$,

$$l^2 = -1, \qquad al = la^-.$$

The real involution may be extended to $\mathcal{O}_{\mathbb{R}}$ by defining $l^- = -l$, in which case

$$(a + bl)^- = a^- + l^- b^- = a^- - bl.$$

Let $\mathcal{O}$ be the complex octonian algebra $\mathcal{O}_{\mathbb{R}} + i\mathcal{O}_{\mathbb{R}}$ and let $a \to a^-$ and $a \to a^\wedge$, respectively, be the linear and conjugate linear involutions on $\mathcal{O}$ defined, for elements a and b in $\mathcal{O}_{\mathbb{R}}$, by

$$(a + ib)^- = a^- + ib^-$$

and

$$(a + ib)^\wedge = a^- - ib^-.$$

Let M_3^8 be the space of 3×3 matrices (a_{jk}) with entries in $\mathcal{O}$ such that $(a_{jk}) = (a_{kj})^-$. When endowed with the spectral norm [18, Thm. 3.17] and the triple product

$$\{abc\} = (a \circ b^{\wedge t}) \circ c + (c \circ b^{\wedge t}) \circ a) - (a \circ c) \circ b^{\wedge t},$$

where $(a_{jk})^{\wedge}=(a_{jk}{}^{\wedge})$ and $(a_{jk})^t=(a_{kj})$, M_3^8 is a JBW*-triple factor. The space $B_{1,2}^8$ of all 1×2 matrices with entries in $\mathcal{O}$ endowed with the triple product

$$\{[a_1\,a\,a_2][b_1\,b_2][a_1\,a_2]\}=[a_1\,a_2]\left(\begin{bmatrix} b_1{}^{\wedge} \\ b_2{}^{\wedge}\end{bmatrix}[a_1\,a_2]\right)$$

and the spectral norm is a second JBW*-triple factor. Notice that for each minimal element u of $\mathcal{U}(M_3^8)$, $A_2(u)$ coincides with $\mathbb{C}u$ and is of dimension 1, $A_0(u)$ is isomorphic to M_2^8 and is of dimension 10 and $A_1(u)$ is of dimension 16. Since all tripotents orthogonal to u lie in $A_0(u)$, it is clear that M_3^8 is of rank 3. Similarly, as can be seen from the following lemma (see [17, Thm. 17.9] for the proof), the rank of $B_{1,2}^8$ is 2.

LEMMA 2.3. *Let the elements c_1 and c_2 of $\mathcal{O}$ be defined by*

$$c_1=\tfrac{1}{2}(1+il), \qquad c_2=\tfrac{1}{2}(1-il),$$

and let $u_j=[c_j,0]$, $j=1,2$, be elements of $B_{1,2}^8$. Then

(i) *u_1 and u_2 are minimal elements of $\mathcal{U}(B_{1,2}^8)$ such that*

$$A_0(u_1)=[\mathbb{C}c_2\,c_2\mathcal{O}], \qquad A_1(u_1)=[c_1(\mathcal{O}c_2)+c_2(\mathcal{O}c_1)c_1\mathcal{O}];$$

(ii) *(u_1,u_2) is a frame for $B_{1,2}^8$.*

Recall that a JBW*-triple A is said to be *exceptional* if every homomorphism from A onto a subtriple of a W*-algebra is zero. The proof of the next lemma can be found in [2], [12], and [19].

LEMMA 2.4. (i) *An exceptional JBW*-triple factor is isomorphic to either M_3^8 or $B_{1,2}^8$.*

(ii) *Let A be an exceptional JBW*-triple. Then there exist hyperstonian spaces Ω_1 and Ω_2 such that A is isomorphic to the JBW*-triple*

$$C(\Omega_1,M_3^8)\oplus_M C(\Omega_2,B_{1,2}^8).$$

For further details about the algebraic theory of Jordan algebras and Jordan triples the reader is referred to [15], [17], and [18]; for the theory of Jordan C*-algebras and JB*-triples, see [11], [21], and [24].

3. Inner Ideals in Exceptional JBW*-Triple Factors

Recall that a subspace J of the JBW*-triple A is said to be a *point space* if, for each element a in J,

$$\{a\,A\,a\}=\mathbb{C}a.$$

Notice that every point space is an inner ideal and a subspace of a point space is a point space.

The following result is essentially due to McCrimmon [20].

THEOREM 3.1. *Let J be an inner ideal in the JBW*-factor M_3^8. Then one of the following possibilities occurs:*

(i) $J=\{0\}$;
(ii) $J=M_3^8$;
(iii) $J=(M_3^8)_2(u)$ *for some tripotent u in* M_3^8 *of rank* 1 *or* 2;
(iv) *J is a point space of dimension not less than* 2.

Proof. This is immediate from [20] and Lemma 2.2. □

In order to identify the inner ideals in $B_{1,2}^8$ some preliminary results are needed.

LEMMA 3.2. *Let A be a Jordan*-triple, let u be a tripotent in A, and let K be a subspace of* $A_1(u)$. *Then the subspace* $A_2(u)+K$ *is an inner ideal in A if and only if the subspaces* $\{K\, A_2(u)\, A_2(u)\}$ *and* $\{K\, A_1(u)\, K\}$ *are contained in K and the subspace* $\{K\, A_2(u)\, K\}$ *is zero.*

Proof. Let J be the subspace $A_2(u)+K$ and suppose that J is an inner ideal. Then, using the Peirce relations and the defining property of an inner ideal, it is clear that

$$\{K\, A_2(u) A_2(u)\} \subseteq J \cap A_1(u) = K,$$

$$\{K\, A_1(u) K\} \subseteq J \cap A_1(u) = K,$$

$$\{K\, A_2(u) K\} \subseteq J \cap A_0(u) = \{0\}.$$

Conversely, if the three conditions hold, then, using the Peirce relations,

$$\begin{aligned}\{J\,A\,J\} &= \{A_2(u)+K\, A\, A_2(u)+K\}\\ &= \{A_2(u)\, A\, A_2(u)\} + \{K\, A\, A_2(u)\} + \{K\, A\, K\}\\ &= A_2(u) + \{K\, A_0(u) A_2(u)\} + \{K\, A_1(u) A_2(u)\}\\ &\quad + \{K\, A_2(u)\, A_2(u)\} + \{K\, A_0(u)\, K\}\\ &\quad + \{K\, A_1(u)\, K\} + \{K\, A_2(u)\, K\}\\ &\subseteq A_2(u) + \{0\} + A_2(u) + K + A_2(u) + K + \{0\} = J,\end{aligned}$$

and J is an inner ideal. □

LEMMA 3.3. *Let A be the triple factor* $B_{1,2}^8$ *and let u be a tripotent in A of rank* 2. *Then, when* $\mathbb{O}$ *is endowed with its natural triple product, the Jordan*-triples* $A_1(u)$ *and* $\mathbb{O}$ *are isomorphic.*

Proof. Since both A and u are of rank 2, u is maximal and $A_0(u)$ is zero. By [17, Thm. 17.1], without loss of generality u can be chosen as in Lemma 2.3. Then $A_2(u)$ and $A_1(u)$ can be identified with $[\mathbb{O}\,0]$ and $[0\,\mathbb{O}]$, respectively. For elements a and b in $\mathbb{O}$,

$$\begin{aligned}\{[0\,a][0\,b][0\,a]\} &= [0\,a]\left(\begin{bmatrix}0\\ b^\wedge\end{bmatrix}[0\,a]\right)\\ &= [0\,a(b^\wedge a)].\end{aligned}$$

Therefore the mapping $a \to [0\,a]$ is an isomorphism from $\mathbb{O}$ onto $A_1(u)$. □

LEMMA 3.4. *Let A be the triple factor* $B_{1,2}^8$, *and let J be an inner ideal in A containing a tripotent u of rank* 2. *Then J is equal to either* $A_2(u)$ *or A.*

Proof. Since u is of rank 2 in A it follows that u is maximal in $\mathcal{U}(A)$ and also in J. Therefore the Peirce decomposition of J relative to u is given by

$$J = J_2(u) \oplus J_1(u).$$

But, since J is an inner ideal,

$$J_2(u) = \{u\{u\,J\,u\}u\} \subseteq \{u\{u\,A\,u\}u\} \subseteq \{u\,J\,u\} = J_2(u),$$
$$J_1(u) \subseteq A_1(u),$$

and it follows that J is of the form $A_2(u)+K$ with K a subspace of $A_1(u)$. Again using [17, Thm. 17.1], without loss of generality u can be chosen as in Lemma 2.3. Then $A_2(u)$ and $A_1(u)$ can be identified with $[\mathcal{O}\,0]$ and $[0\,\mathcal{O}]$, respectively. Let K' be the subspace of $\mathcal{O}$ consisting of elements a in $\mathcal{O}$ such that $[0\,a]$ lies in K. By Lemma 3.2, K' is a subspace of $\mathcal{O}$ such that, for all elements a in K', b and c in $\mathcal{O}$, the element $\{[0\,a][b\,0][c\,0]\}$ lies in K. But

$$2\{[0\,a][b\,0][c\,0]\} = [0\,c(b^\wedge a)],$$

and it follows that K' is a left ideal in $\mathcal{O}$. Since $\mathcal{O}$ possesses no nontrivial left ideals, K' is either zero in which case J coincides with $A_2(u)$, or K' is equal to $\mathcal{O}$ in which case J coincides with A. □

LEMMA 3.5. *Let A be the triple factor $B^8_{1,2}$, and let J be an inner ideal in A which contains no tripotent of rank 2. Then J is a point space.*

Proof. Let a be an element of J and let $r(a)$ be its support in A. By Lemma 2.2, the inner ideal $\{a\,A\,a\}$ coincides with $A_2(r(a))$ and $r(a)$ lies in J. It follows that either $r(a)$ is zero in which case a is zero, or $r(a)$ is of rank 1 and therefore minimal. In this case

$$\{a\,A\,a\} = A_2(r(a)) = \mathbb{C}r(a).$$

But, since a lies in $A_2(r(a))$, a is also a multiple of $r(a)$ and $\{a\,A\,a\}$ coincides with $\mathbb{C}a$ as required. □

A combination of Lemma 3.4 and Lemma 3.5 gives the following result, which has recently been proved by other methods by Neher [22, §3.2(f)].

THEOREM 3.6. *Let J be an inner ideal in the JBW*-factor $B^8_{1,2}$. Then one of the following possibilities occurs:*

(i) $J = \{0\}$;
(ii) $J = B^8_{1,2}$;
(iii) $J = (B^8_{1,2})_2(u)$ *for some tripotent u in $B^8_{1,2}$ of rank* 1 *or* 2;
(iv) *J is a point space of dimension not less than* 2.

4. Inner Ideals in Exceptional JBW*-Triples

By Lemma 2.1 and Lemma 2.4, it is enough to classify the weak* closed inner ideals in $C(\Omega, M^8_3)$ and $C(\Omega, B^8_{1,2})$, where Ω is a hyperstonian space.

It is clear that the next two results hold in greater generality than that in which they are stated. However, in the interests of brevity they are stated for the special case under consideration. The authors are grateful to E. Neher, who supplied the shorter proof of Lemma 4.2 which replaces the authors' original proof.

LEMMA 4.1. *Let B be either M_3^8 or $B_{1,2}^8$, let Ω be a hyperstonian space, let A be the exceptional JBW*-triple $C(\Omega, B)$, let J be a weak* closed inner ideal in A, and let u be a maximal element of $\mathcal{U}(J)$. For ω in Ω define*

$$J(\omega) := \{a(\omega) : a \in J\}.$$

Then $J(\omega)$ is an inner ideal in B and $u(\omega)$ is a maximal element of $\mathcal{U}(J(\omega))$.

Proof. Since, for each element c in B, the constant function $\omega \to c$ is an element of A, it follows that, for each element a in J,

$$\{a\,c\,a\}(\omega) = \{a(\omega)c\,a(\omega)\}$$

and hence that $J(\omega)$ is an inner ideal in B. Since u is maximal in $\mathcal{U}(J)$, for each element a in J,

$$a = 2\{u\,u\,a\} - \{u\{u\,a\,u\}u\}.$$

Hence, for each element ω in Ω and c in $J(\omega)$,

$$c = 2\{u(\omega)u(\omega)c\} - \{u(\omega)\{u(\omega)c\,u(\omega)\}u(\omega)\}.$$

Therefore, $u(\omega)$ is maximal in $\mathcal{U}(J(\omega))$. □

LEMMA 4.2. *Let B be either M_3^8 or $B_{1,2}^8$, let Ω be a hyperstonian space, let A be the exceptional JBW*-triple $C(\Omega, B)$, let u be an element of $\mathcal{U}(A)$, and let Ω_j be the set of points ω in Ω for which $u(\omega)$ is of rank j. Then Ω_j is a clopen subset of Ω.*

Proof. Let $L(B)$ denote the finite-dimensional complex vector space of linear mappings from B to itself, let $b_1, b_2, \ldots, b_r$ be a basis for B, and let e_{kl}, $k, l = 1, 2, \ldots, r$ be the corresponding basis of matrix units for $L(B)$. Let ϕ be a continuous function from Ω to $L(B)$ and suppose that, for each ω in Ω,

$$\phi(\omega) = \sum_{k,l=1}^{r} \phi_{kl}(\omega)e_{kl}.$$

The set $\{\omega : \operatorname{rank}\phi(\omega) \leq j\}$ coincides with the intersection of the zero sets of the continuous functions

$$\omega \to \det(\phi_{kl}(\omega))_{k \neq E, l \in F},$$

where E and F are subsets of $\{1, 2, \ldots, r\}$ of cardinality $j+1$. It is therefore closed.

When v and w are tripotents in B having the same rank, by [18] there exists an automorphism ψ of B such that, for $j = 0, 1, 2$, $\psi B_j(v) = B_j(w)$. Therefore, for $0 \leq s \leq r$, there exist nonnegative integers $f(s)$ and $g(s)$ such that v is of rank s if and only if $P_2(v)$ is of rank $f(s)$ and if and only if $P_0(v)$ is of rank $g(s)$. Clearly the mapping f is increasing and the mapping g is decreasing.

Applying the first part of the proof to the function ϕ defined for ω in Ω by

$$\phi(\omega)=P_0(u(\omega))=(P_0(u))(\omega)$$

shows that, for each nonnegative integer s, the set $\{\omega : \operatorname{rank} u(\omega)\geq s\}$ is closed. Applying the same result to the function ϕ defined for ω in Ω by

$$\phi(\omega)=P_2(u(\omega))=(P_2(u))(\omega)$$

shows that, for each positive integer s, the set $\{\omega : \operatorname{rank} u(\omega)\leq s-1\}$ is closed, which implies that the set $\{\omega : \operatorname{rank} u(\omega)\geq s\}$ is open. It follows that, for each nonnegative integer s, the set

$$\Sigma_s=\{\omega : \operatorname{rank} u(\omega)\geq s\}$$

is clopen. Therefore $\Omega_j=\Sigma_j\setminus\Sigma_{j+1}$ is also clopen, and the proof is complete. □

The main result of the paper describes the weak* closed inner ideals in $C(\Omega,B)$ where B is either M_3^8 or $B_{1,2}^8$. Whilst the statement of the result is the same in both cases, the proofs deviate at several points.

THEOREM 4.3. *Let B be either M_3^8 or $B_{1,2}^8$, let Ω be a hyperstonian space, and let A be the exceptional JBW*-triple $C(\Omega,B)$. Let J be a weak* closed inner ideal in A. Then there exist weak* closed ideals $I_1,I_2,I_{31},I_{32},I_{42},I_{43},\dots,I_{4m}$ in A, maximal tripotents u_{31} and u_{32} in $J\cap I_{31}$ and $J\cap I_{32}$ respectively, and tripotents $z_1,z_2,\dots,z_m$ in J such that*

(i) $A=I_1\oplus_M I_2\oplus_M I_{31}\oplus_M I_{32}\oplus_M I_{42}\oplus_M I_{43}\oplus_M\cdots\oplus_M I_{4m}$;

(ii) $J\cap I_1=\{0\}$, $I_2\subseteq J$, $J\cap I_{31}=A_2(u_{31})$, $J\cap I_{32}=A_2(u_{32})$ *and if, for $j=2,3,\dots,m$, R_{4j} denotes the M-projection from A onto I_{4j} then $R_{4j}z_1$, $R_{4j}z_2$, $R_{4j}z_3$, $\dots$, $R_{4j}z_j$ are pairwise collinear tripotents in $J\cap I_{4j}$ such that*

$$J\cap I_{4j}=A_2(R_{4j}z_1)+A_2(R_{4j}z_2)+\cdots+A_2(R_{4j}z_j);$$

(iii) $J=I_2\oplus_M(J\cap I_{31})\oplus_M(J\cap I_{32})\oplus_M(J\cap I_{42})\oplus_M(J\cap I_{43})\oplus_M\cdots\oplus_M(J\cap I_{4m})$;

(iv) *there exist pairwise disjoint clopen sets $\Omega_1,\Omega_2,\Omega_{31},\Omega_{32},\Omega_{42},\Omega_{43},\dots,\Omega_{4m}$ in Ω with union Ω such that $I_1,I_2,I_{31},I_{32},I_{42},I_{43},\dots,I_{4m}$ are respectively isomorphic to $C(\Omega_1,B)$, $C(\Omega_2,B)$, $C(\Omega_{31},B)$, $C(\Omega_{32},B)$, $C(\Omega_{42},B)$, $C(\Omega_{43},B),\dots,C(\Omega_{4m},B)$; and*

(v) *for ω in Ω_{3k}, $k=1,2$, $(J\cap I_{3k})(\omega)$ is of rank k and for ω in Ω_{4j}, $j=2,3,\dots,m$, $(J\cap I_{4j})(\omega)$ is a point space in B of dimension j with basis $\{z_1(\omega),z_2(\omega),z_3(\omega),\dots,z_j(\omega)\}$.*

Proof. By Theorems 3.1 and 3.6 and Lemma 4.1, four possibilities occur. Let Ω_1, Ω_2, Ω_3, and Ω_4, respectively, be the sets of points in Ω such that $J(\omega)=\{0\}$, $J(\omega)=B$, $J(\omega)=A_2(v)$ for some tripotent v in B of rank 1 or 2, and $J(\omega)$ is a point space of dimension not less than 2. Let u be a maximal element of $\mathcal{U}(J)$. Then, by Lemma 4.1, $u(\omega)$ is a maximal element of $\mathcal{U}(J(\omega))$.

Since u is maximal in $\mathcal{U}(J)$, J has the Peirce decomposition

$$J=J_2(u)\oplus J_1(u)$$

relative to u. Since $J_1(u)$ is a JBW*-triple and, in fact, an inner ideal in J, it possesses a maximal tripotent w; hence, for each ω in Ω, $w(\omega)$ is a maximal tripotent in $J_1(u)(\omega)$.

First observe that ω lies in Ω_1 if and only if $u(\omega)$ is zero. Since u is a continuous function it follows that Ω_1 is closed. But since $\|u(\omega)\|$ is either zero or one, it is clear that Ω_1 is also open.

Notice that if $B=M_3^8$ then ω lies in Ω_2 if and only if $u(\omega)$ is of rank 3. Then Ω_2 is clopen, by Lemma 4.2. If $B=B_{1,2}^8$ then Ω_2 is the set of points in Ω for which $u(\omega)$ and $w(\omega)$ are both of rank 2. It follows from Lemma 4.2 that Ω_2, being the intersection of two clopen sets, is itself clopen.

Next observe that Ω_3 consists of the set of points in the clopen set $\Omega_3 \cup \Omega_4$ for which $w(\omega)$ is zero, and that consequently Ω_3 is clopen. It follows that Ω_4 is also clopen. Let Ω_{31} and Ω_{32} be the sets of points in Ω_3 for which $u(\omega)$ is of rank 1 and 2, respectively. These are clopen, by Lemma 4.2.

Notice that, for each element ω in Ω_4,

$$\begin{aligned} J(\omega) &= B_2(u(\omega)) \oplus (J_1(u))(\omega) \\ &= \mathbb{C}u(\omega) \oplus \mathbb{C}w(\omega) \oplus (J_1(u)(\omega))_1(w(\omega)), \end{aligned}$$

and that $u(\omega)$ and $w(\omega)$ are linearly independent. Let $z_1, z_2, \ldots, z_m$ be tripotents in J chosen as follows:

$$z_1 = u, \qquad z_2 = w;$$

having chosen $z_1, z_2, \ldots, z_r$, if possible choose a maximal nonzero tripotent z_{r+1} in the sub-triple $(\ldots((J_1(z_1))_1(z_2))_1 \ldots)_1(z_r)$. Then $z_1, z_2, \ldots, z_{r+1}$ are linearly independent and, for each ω in Ω_4, $z_{r+1}(\omega)$ is a maximal tripotent in $(\ldots((J_1(z_1(\omega)))_1(z_2(\omega)))_1 \ldots)_1(z_r(\omega))$. This process terminates when $z_1, z_2, \ldots$, z_m have been selected. Moreover, for each ω in Ω_4 there exists j such that $2 \le j \le m$ with $\{z_1(\omega), z_2(\omega), \ldots, z_j(\omega)\}$ a basis for $J(\omega)$ and

$$z_{j+1}(\omega) = z_{j+2}(\omega) = \cdots = z_m(\omega) = 0.$$

It follows from Lemma 4.2 that the set Ω_{4j} of points ω for which $J(\omega)$ is a point space of dimension j is clopen, being the intersection of the two clopen sets

$$\{\omega : \operatorname{rank} z_r(\omega) = 1,\ r = 1, 2, \ldots, j\}$$

and

$$\{\omega : \operatorname{rank} z_r(\omega) = 0,\ r = j+1, j+2, \ldots, m\}.$$

Define the projections $R_1 R_2, R_{31}, R_{32}, R_{42}, R_{43}, \ldots, R_{4m}$ on A, for each element a in A and ω in Ω, by

$$(R_k a)(\omega) = \chi_{\Omega_k}(\omega) a(\omega),$$

where χ_{Ω_k} denotes the characteristic function of the set Ω_k, and let their ranges be $I_1, I_2, I_{31}, I_{32}, I_{42}, I_{43}, \ldots, I_{4m}$, respectively. These are weak* closed ideals in A with I_k isomorphic to $C(\Omega_k, B)$. The proof of parts (i) and (iv) are now complete.

To prove (ii) first notice that, for each element a in $J \cap I_1$ and ω in Ω,

$$a(\omega) = \chi_{\Omega_1}(\omega) a(\omega) \in \chi_{\Omega_1}(\omega) J(\omega) = \{0\}$$

and $a = 0$. Hence $J \cap I_1 = \{0\}$.

To show that I_2 is contained in J, first let $B = M_3^8$. Then $R_2 u$ is a maximal tripotent in $J \cap I_2$ and since $J \cap I_2$ is an inner ideal in I_2, $(J \cap I_2)_2(R_2 u)$ coincides with $(I_2)_2(R_2 u)$. But for all ω in Ω_2, $(J \cap I_2)_2(R_2 u)(\omega)$ coincides with M_3^8 and hence $(I_2)_2(R_2(u))(\omega)$ coincides with M_3^8. It follows that $R_2 u$ is maximal in $\mathcal{U}(I_2)$. For each element a in I_2,

$$\begin{aligned} a &= P_2^{I_2}(R_2 u) a + P_1^{I_2}(R_2 u) a \\ &= P_2^{I_2}(R_2 u) \in (I_2)_2(R_2 u) = (J \cap I_2)_2(R_2 u) \in J \cap I_2. \end{aligned}$$

Hence $I_2 \subseteq J$. Now let $B = B_{1,2}^8$. Again $R_2 u$ is a maximal tripotent in $J \cap I_2$ and again $(J \cap I_2)_2(R_2 u)$ and $(I_2)_2(R_2 u)$ coincide. Of course $(J \cap I_2)_1(R_2 u)$ is contained in $(I_2)_1(R_2 u)$. But for all ω in Ω_2, $(J \cap I_2)_2(R_2 u)(\omega)$ and $(J \cap I_2)_1(R_2 u)(\omega)$ are both of dimension 8, and it follows that $(J_2)_1(R_2 u)(\omega)$ is of dimension 8. It follows that $R_2 u$ is maximal in $\mathcal{U}(I_2)$. Let v be a maximal tripotent in $(J \cap I_2)_1(R_2 u)$. Then, from above, for each ω in Ω_2, $(I_2(\omega))_1(R_2 u(\omega))$ coincides with $((J \cap I_2)(\omega))_1(R_2 u(\omega))$ which, by Lemma 3.3, coincides with $((J \cap I_2)(\omega))_2(v(\omega))$. Since this is equal to $(I_2(\omega))_2(v(\omega))$, for each element b in $(I_2)_1(R_2 u)$ and ω in Ω_2, $b(\omega)$ lies in $(J_2(\omega))_2(v(\omega))$. Hence b is an element of $(I_2)_2(v)$ which is contained in $J \cap I_2$. Therefore, using the Peirce decomposition of I_2 relative to $R_2 u$, for each element a in I_2 we have

$$a = P_2^{I_2}(R_2 u) a + P_1^{I_2}(R_2 u) a \in J \cap I_2.$$

It follows that $I_2 \subseteq J$ as required. Notice that (iii) now follows from Lemma 2.1.

For $j = 1, 2$, writing u_{3j} for $R_{3j} u$, it is clear that u_{3j} is a maximal tripotent in $J \cap I_{3j}$. For each element a in $J \cap I_{3j}$ and ω in Ω,

$$(P_1^{J \cap I_{3j}}(u_{3j}) a)(\omega) = \chi_{\Omega_{3j}(\omega)} P_1^{J \cap I_{3j}}(u_{3j}(\omega)) a(\omega) = \{0\}.$$

It follows that a lies in $(J \cap I_{3j})_2(u_{3j})$, which coincides with $(I_{3j})_2(u_{3j})$ and $A_2(u_{3j})$ by properties of inner ideals. Since $J \cap I_{3j}$ is an inner ideal containing u_{3j} it follows that $A_2(u_{3j})$ is contained in $J \cap I_{3j}$. Therefore $A_2(u_{3j})$ and $J \cap I_{3j}$ coincide.

Finally, for $j = 2, 3, \ldots, m$, writing u_{4j} for $R_{4j} u$ and for $r = 1, 2, \ldots, m$ writing x_r for $R_{4j} z_r$, it is clear that

$$\begin{aligned} J \cap I_{4j} &= (J \cap I_{4j})_2(u_{4j}) \oplus (J \cap I_{4j})_1(u_{4j}) \\ &= A_2(x_1) + ((J \cap I_{4j})_1(x_1))_2(x_2) + ((J \cap I_{4j})_1(x_1))_1(x_2) \\ &= A_2(x_1) + A_2(x_2) + ((J \cap I_{4j})_1(x_1))_1(x_2), \end{aligned}$$

recalling that, since x_1 is maximal, $(J \cap I_{4j})_1(x_1)$ is an inner ideal in $J \cap I_{4j}$. Proceeding inductively, it follows that

$$J \cap I_{4j} = A_2(x_1) + A_2(x_2) + \cdots + A_2(x_j) + K,$$

where

$$K=(\ldots(((J\cap I_{4j})_1(x_1))_1(x_2))_1\ldots)_1(x_j).$$

For ω in Ω_{4j},

$$A_2(x_j)(\omega)=A_2(R_{4j}z_j)(\omega)=\chi_{\Omega_{4j}}(\omega)\{z_j(\omega)\{z_j(\omega)Az_j(\omega)\}z_j(\omega)\}\neq\{0\},$$

by the choice of Ω_{4j}. But

$$\begin{aligned}K(\omega)&=(\ldots((((J\cap I_{4j})_1(x_1))_1(x_2))_1\ldots)_1(x_j))(\omega)\\&=\chi_{\Omega_{4j}}(\omega)\,\mathrm{lin}\{z_{j+1}(\omega),z_{j+2}(\omega),\ldots,z_m(\omega)\}=\{0\}\end{aligned}$$

and hence

$$J\cap I_{4j}=A_2(x_1)+A_2(x_2)+\cdots+A_2(x_j)$$

as required. Observe that, for each ω in Ω_{4j}, $x_2(\omega),x_3(\omega),\ldots,x_j(\omega)$ lie in $(J\cap I_{4j})(\omega)_1(x_1(\omega))$. Since $(J\cap I_{4j})(\omega)$ is a point space in which all tripotents are minimal, by [4, Prop. 2.1] the tripotents $x_2(\omega),x_3(\omega),\ldots,x_j(\omega)$ are collinear with $x_1(\omega)$. Similarly the tripotents $x_3(\omega),x_4(\omega),\ldots,x_j(\omega)$ are collinear with $x_2(\omega)$. Proceeding inductively, it follows that $x_1(\omega),x_2(\omega),\ldots,$ $x_j(\omega)$ are pairwise collinear. Hence, for all ω in Ω_{4j} and for $1\le r,s\le j$ with $r\neq s$,

$$P_1^{(J\cap I_{4j})(\omega)}(x_s(\omega))\,x_r(\omega)=x_r(\omega),\qquad P_1^{(J\cap I_{4j})(\omega)}(x_r(\omega))\,x_s(\omega)=x_s(\omega).$$

Therefore,

$$P_1^{(J\cap I_{4j})}(x_s)x_r=x_r,\qquad P_1^{(J\cap I_{4j})}(x_r)x_s=x_s,$$

and x_r,x_s are collinear in $J\cap I_{4j}$ and hence in A. This completes the proof of (ii), and (v) follows immediately. □

Notice that recent results of S. Vasilovsky [25] have shown that in the case of $B_{1,2}^8$ the largest possible value for m in this theorem is 5, whilst in the case of M_3^8 it was shown by McCrimmon [20] and Racine [23] that the largest possible value for m is 6.

Finally, it is clear that the ideals I_1, I_2, I_{31}, I_{32} and I_4 in the theorem are uniquely defined by the weak* closed inner ideal J. However, the decomposition of I_4 depends upon the choice of the maximal tripotent w in J.

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