A Conjecture of L. Carleson and Applications

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1. Introduction

Let m be the area measure on $\mathbf{C}$. For a meromorphic function f in the unit disc U, let

$$A(r,f)=\int_{\{|z|\le r\}}\frac{|f'|^2}{(1+|f|^2)^2}\,dm,\quad 0\le r<1, \tag{1.1}$$

be the spherical area of the image of $\{|z|\le r\}$ by f, counting multiplicities. In his thesis Carleson [6] considered the classes T_α, $0\le\alpha<1$, of meromorphic functions f in U satisfying

$$|f|_\alpha=\int_0^1 A(r,f)(1-r)^{-\alpha}\,dr<\infty, \tag{1.2}$$

and the class T_1 of meromorphic functions f in U with the property that $A(r,f)$ remains bounded when r tends to 1, that is,

$$|f|_1=\sup_{r<1}A(r,f)<\infty. \tag{1.3}$$

We obviously have $T_1\subset T_\alpha\subset T_\beta\subset T_0$ for all $\alpha,\beta\in(0,1)$ with $\alpha>\beta$. The class T_0 coincides with the class of functions with bounded characteristic, and a well-known theorem of F. and R. Nevanlinna asserts that each $f\in T_0$ is the quotient of two bounded analytic functions in U. In [6, p. 39] Carleson proved an analogue of this theorem for the classes T_α just defined, namely, the fact that each function in T_α is the quotient of two bounded functions, each of which is in T_β for all $\beta<\alpha$, and conjectured that one cannot take $\beta=\alpha$, that is, not every function in T_α is the quotient of two bounded functions in T_α. For all $\alpha\in[0,1]$, T_α contains the weighted Dirichlet space $D_{1-\alpha}$ of analytic functions f in U satisfying

$$\int_U|f'(z)|^2(1-|z|)^{1-\alpha}\,dm<\infty \tag{1.4}$$

Recently, in their paper [11] on invariant subspaces of the multiplication operator on the Dirichlet space D_0, Richter and Shields found a partial "negative" answer to Carleson's conjecture for $\alpha=1$ by showing that every function in

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D_0 is the quotient of two bounded functions in D_0; their result was extended to all spaces D_α, $0<\alpha<1$, in [3]. The aim of the present paper is to give a complete answer to this conjecture. Using a similar method to the one in [3], we shall prove that each function in T_α, $0<\alpha\le 1$, is the quotient of two bounded functions in T_α.

The proof occupies the next three sections of this paper. In Sections 5 and 6 we discuss some applications of the main theorem. First of all, this result holds also for other classes of meromorphic functions between T_1 and T_0 that are defined by means of growth restrictions of the form

$$|f|_\omega=\int_0^1 A(r,f)\,\omega(r)\,dr<\infty, \tag{1.5}$$

where $\omega\in C^1[0,1)$ is a positive increasing function with $\int_0^1\omega(r)\,dr<\infty$. Similar to T_α, the class T_ω of meromorphic functions in U satisfying (1.5) contains the Hilbert space H_w of analytic functions f in U with the property that

$$\int_U |f'(z)|^2 w(|z|)\,dm<\infty, \tag{1.6}$$

where w is defined on $[0,1)$ by $w(r)=\int_r^1\omega(\rho)\,d\rho$. In Section 5 we show that the outer factor of a function in T_ω or H_w belongs to the same class, and following the ideas in [1], [2], and [6] we prove some characterizations of the inner functions in T_ω. As was pointed out in [11] (see also [3]), results of this type have interesting consequences concerning the properties of the multiplication operator on the spaces H_w or on the Dirichlet space $D=D_0$. In Section 6 we prove that every invertible function in H_w is a cyclic vector for this operator, that is, its polynomial multiples are dense in the space. The result answers affirmatively Question 4 in [5] (see also [13]) for the spaces H_w. For the Dirichlet space it was recently proved by Brown [4].

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In order to prove the main theorem, our first purpose is to obtain a suitable equivalent expression for $|f|_\alpha$. We begin with a simple observation. Let f be a meromorphic function in U. Then $A(r,f)$ is an increasing function of r; hence for all $\alpha\in[0,1)$ it satisfies the inequality

$$(1-r)^{1-\alpha}A(r,f)\le\int_r^1 A(\rho,f)(1-\rho)^{-\alpha}\,d\rho,\quad r\in[0,1). \tag{2.1}$$

If $f\in T_\alpha$ it follows that $\lim_{r\to1}(1-r)^{1-\alpha}A(r,f)=0$, so that, integrating by parts in (1.2), we obtain in this case

$$|f|_\alpha=\frac{1}{1-\alpha}\int_U\frac{|f'(z)|^2}{(1+|f(z)|^2)^2}(1-|z|)^{1-\alpha}\,dm. \tag{2.2}$$

Assume that f is not constant. For $0<\beta<1$ and $\zeta\in f(U)$ we consider the generalized counting functions

$$N_\beta(f,\zeta)=\sum_{f(z)=\zeta}(1-|z|)^\beta, \tag{2.3}$$

and for $0\le r<1$ the usual Nevanlinna counting function of f,

$$N(r,f,\zeta)=\sum_{\substack{f(z)=\zeta\\|z|<r}}\log\frac{r}{|z|},\quad \zeta\in f(U)\setminus\{f(0)\}, \tag{2.4}$$

where multiplicities are counted in the above sums (that may not converge). From the following general change-of-variable formula,

$$\int_U (u\circ f)v|f'|^2\,dm=\int_{f(U)}u(\zeta)\Big(\sum_{f(z)=\zeta}v(z)\Big)dm(\zeta), \tag{2.5}$$

valid for any two nonnegative measurable functions u, v on $\mathbf{C}$ and for meromorphic nonconstant functions f in U, using (2.2) we obtain that if $f\in T_\alpha$, $0\le\alpha<1$, then

$$|f|_\alpha=\frac{1}{1-\alpha}\int_{f(U)}(1+|\zeta|^2)^{-2}N_{1-\alpha}(f,\zeta)\,dm, \tag{2.6}$$

A proof of formula (2.5) may be found in [12] or [3] in the case when f is analytic. For meromorphic f the proof is identical, and may be obtained by dividing the disc U into a set of planar measure zero and a countable disjoint union of open sets R_n such that $f\,|\,R_n$ is analytic and injective. Then (2.5) follows with the usual change-of-variable formula.

LEMMA 1. *Let f be nonconstant and meromoprhic in U. For $z,\lambda\in U$ let $\varphi_z(\lambda)=(z+\lambda)/(1+\bar z\lambda)$. Then for $0<\beta<1$ and $\zeta\in f(U)$,*

$$N_\beta(f,\zeta)=-\frac{1}{2\pi}\int_U\Delta(1-|z|)^\beta N(1,f\circ\varphi_z,\zeta)\,dm(z), \tag{2.7}$$

where Δ denotes the Laplace operator.

Proof. By Green's formula we have, for every $\lambda\in U$,

$$(1-|\lambda|)^\beta=-\frac{1}{2\pi}\int_U\Delta(1-|z|)^\beta\log\left|\frac{1-\bar z\lambda}{\lambda-z}\right|dm(z). \tag{2.8}$$

Since $\Delta(1-|z|)^\beta<0$ in U, by the monotone convergence theorem

$$N_\beta(f,\zeta)=-\frac{1}{2\pi}\int_U\Delta(1-|z|)^\beta\Big(\sum_{f(\lambda)=\zeta}\log\left|\frac{1-\bar z\lambda}{\lambda-z}\right|\Big)dm(z). \tag{2.9}$$

We have $(\lambda-z)/(1-\bar z\lambda)=\varphi_z{}^{-1}(\lambda)$ and

$$\sum_{f(\lambda)=\zeta}\log\left|\frac{1-\bar z\lambda}{\lambda-z}\right|=N(1,f\circ\varphi_z,\zeta)\quad m\text{-a.e. on } U \tag{2.10}$$

which proves (2.7). □

Every function f in T_0 with $f \neq 0$ has a nontangential limit at $e^{i\theta}$ a.e. on $[0, 2\pi]$, denoted by $f(e^{i\theta})$, and $\log|f(e^{i\theta})|$ belongs to $L^1[0, 2\pi]$. Further, f may be written as $f = IF/J$, where I, J are inner functions whose greatest common divisor (I, J) is the constant function 1 and where F is an outer function. Up to some unimodular constants, the functions I, J, F are uniquely determined in this case. The next lemma is derived from a formula of Ahlfors and Shimizu.

LEMMA 2. *Let $f \in T_0$, $f \neq 0$, and $f = IF/J$, with I, J inner functions satisfying $(I, J) = 1$ and F outer. For $z, \lambda \in U$ and $\theta \in [0, 2\pi]$, let $\varphi_z(\lambda) = (z+\lambda)/(1+\bar{z}\lambda)$ and $P_z(\theta) = \mathrm{Re}((e^{i\theta}+z)/(e^{i\theta}-z))$ be the Poisson kernel. If $z \in U$ is not a pole of f then*

$$\frac{2}{\pi}\int_U \frac{|(f\circ\varphi_z)'(\lambda)|^2}{(1+|f\circ\varphi_z(\lambda)|^2)^2}\log\frac{1}{|\lambda|}\,dm(\lambda)$$

$$= -\log(1+|f(z)|^2) - 2\log|J(z)| + \frac{1}{2\pi}\int_0^{2\pi} P_z(\theta)\log(1+|f(e^{i\theta})|^2)\,d\theta. \tag{2.11}$$

Proof. Let $f \in T_0$ with $f \neq 0$ and $f = IF/J$, where I, J, F are as in the statement of the lemma. Let $I = B_I S_I$ and $J = B_J S_J$, where B_I, B_J are Blaschke products and S_I, S_J are singular inner functions. Also let

$$v_f(z) = \frac{2}{\pi}\int_U \frac{|(f\circ\varphi_z)'(\lambda)|^2}{(1+|f\circ\varphi_z(\lambda)|^2)^2}\log\frac{1}{|\lambda|}\,dm(\lambda) + \log(1+|f(z)|^2)$$

$$+2\log|B_J(z)| - \frac{1}{2\pi}\int_0^{2\pi} P_z(\theta)\log(1+|f(e^{i\theta})|^2)\,d\theta. \tag{2.12}$$

We prove first the following double inequality. If 0 is not a pole of f then

$$0 \le v_f(0) \le -2\log|S_J(0)|. \tag{2.13}$$

Indeed, integrating by parts we obtain

$$\int_U \frac{|f'|^2}{(1+|f|^2)^2}\log\frac{1}{|\lambda|}\,dm = \int_0^1 \log\frac{1}{r}\left[\frac{d}{dr}A(r,f)\right]dr$$

$$= \int_0^1 A(r,f)\,\frac{dr}{r}, \tag{2.14}$$

and by the formula mentioned previously (see [10, p. 11]) for $0 \le \rho < 1$,

$$\frac{2}{\pi}\int_0^\rho A(r,f)\,\frac{dr}{r} = \frac{1}{2\pi}\int_0^{2\pi}\log(1+|f(\rho e^{i\theta})|^2)\,d\theta$$

$$-\log(1+|f(0)|^2) + 2N(\rho, 1/f, 0). \tag{2.15}$$

Since $\lim_{\rho\to 1} N(\rho, 1/f, 0) = -\log|B_J(0)|$, the first inequality follows by Fatou's lemma. We have also

$$\limsup_{\rho\to 1} \frac{1}{2\pi}\int_0^{2\pi} \log(1+|f(\rho e^{i\theta})|^2)\,d\theta$$

$$\le \limsup_{\rho\to 1}\left[\frac{1}{2\pi}\int_0^{2\pi}\log(1+|F(\rho e^{i\theta})|^2)\,d\theta - \frac{1}{2\pi}\int_0^{2\pi}\log|J(\rho e^{i\theta})|^2\,d\theta\right] \tag{2.16}$$

$$= \limsup_{\rho\to 1}\frac{1}{2\pi}\int_0^{2\pi}\log(1+|F(\rho e^{i\theta})|^2)\,d\theta - 2\log|S_J(0)|.$$

Using the fact that F is outer and $|F(e^{i\theta})| = |f(e^{i\theta})|$ a.e. on $[0,2\pi]$, we apply Jensen's inequality to the convex function $x\mapsto \log(1+e^x)$ to obtain

$$\begin{aligned}\int_0^{2\pi}\log(1+|F(\rho e^{i\theta})|^2)\,d\theta &\le \int_0^{2\pi}\int_0^{2\pi} P_{\rho e^{i\theta}}(t)\log(1+|F(e^{it})|^2)\frac{dt}{2\pi}\,d\theta\\ &= \int_0^{2\pi}\log(1+|f(e^{it})|^2)\,dt,\end{aligned} \tag{2.17}$$

and the second inequality in (2.13) follows by (2.15) and (2.16). We shall use the double inequality to prove that $v_f(z) = -2\log|S_J(z)|$, $z\in U$. The first step is to show that v_f is harmonic in U, as follows. The function

$$\log(1+|f|^2) + 2\log|B_J| = \log(|B_J|^2 + |fB_J|^2) \tag{2.18}$$

is twice continuously differentiable on U, because B_J and fB_J have no common zeros there and if $z\in U$ is not a pole of f then

$$\begin{aligned}\Delta(\log(1+|f|^2)+2\log|B_J|)(z) &= \Delta(\log(1+|f|^2))(z)\\ &= \frac{4|f'(z)|^2}{(1+|f(z)|^2)^2}.\end{aligned} \tag{2.19}$$

For every compactly supported function $u\in C^\infty(U)$ we have, by Green's formula,

$$u(\lambda) = -\frac{1}{2\pi}\int_U \log\frac{1}{|\varphi_z^{-1}(\lambda)|}\Delta u(z)\,dm(z), \quad \lambda\in U. \tag{2.20}$$

If we denote by $\mathcal{I}(z)$ the area integral in (2.12) and use the substitution $\varphi_z(\lambda) = \zeta$, then

$$\begin{aligned}\int_U \mathcal{I}\Delta u\,dm &= \int_U \Delta u(z)\left(\frac{2}{\pi}\int_U \frac{|f'|^2}{(1+|f|^2)^2}\log\frac{1}{|\varphi_z^{-1}|}\,dm\right)dm(z)\\ &= -\int_U \frac{4|f'|^2}{(1+|f|^2)^2}u\,dm = -\int_U u\Delta(\log(1+|f|^2)+2\log|B_J|^2)\,dm\\ &= -\int_U (\log(1+|f|^2)+2\log|B_J|^2)\,\Delta u\,dm.\end{aligned} \tag{2.21}$$

From (2.12) and another application of Green's formula, we obtain

$$\int_U v_f\Delta u\,dm = \int_U (\mathcal{I} + \log(1+|f|^2) + 2\log|B_J|^2)\,\Delta u\,dm = 0, \tag{2.22}$$

which shows that v_f is harmonic in U.

Now if $z \in U$ then $F \circ \varphi_z$ is outer, $I \circ \varphi_z$ and $J \circ \varphi_z$ are inner, $(I \circ \varphi_z, J \circ \varphi_z) = 1$, and their Blaschke and singular inner factors are obtained by composing B_I, S_I, B_J, S_J respectively with φ_z. Using some computations with the Poisson kernel, it follows that

$$v_f(z) = v_{f \circ \varphi_z}(0); \tag{2.23}$$

hence if $z \in U$ is not a pole of f then by (2.13) we have

$$0 \le v_f(z) \le -2\log|S_J \circ \varphi_z(0)| = -2\log|S_J(z)|. \tag{2.24}$$

This leads to $v_f = -2\log|S|$ for some singular inner function S, because v_f is harmonic. From (2.24) it follows also that S divides S_J. If in addition we have $f(z) \ne 0$, applying the above argument to $1/f$ after some simple computations we obtain

$$0 \le v_{1/f}(z) = v_f(z) - 2\log\left|\frac{S_I(z)}{S_J(z)}\right|; \tag{2.25}$$

that is, $\log|S_J/S| \ge \log|S_I|$. Thus S_J/S divides both S_J and S_I, which shows that $S = S_J$ and the proof is complete. □

Recalling that $T_\alpha \subset T_0$ for all $\alpha \in (0,1]$, we now put together the preceding results in order to obtain the following.

PROPOSITION 3. *Let $0 < \alpha < 1$ and $f \in T_\alpha$ be nonconstant with $f = IF/J$, where I, J are inner functions satisfying $(I, J) = 1$ and F is outer. For $z \in U$ and $\theta \in [0, 2\pi]$, let $P_z(\theta) = \mathrm{Re}((e^{i\theta}+z)/(e^{i\theta}-z))$ be the Poisson kernel. Then*

$$|f|_\alpha = -\frac{1}{4(1-\alpha)}\int_U \Delta(1-|z|)^{1-\alpha}\left[\frac{1}{2\pi}\int_0^{2\pi} P_z(\theta)\log(1+|f(e^{i\theta})|^2)\,d\theta - \log(1+|f(z)|^2) - 2\log|J(z)|\right] dm(z). \tag{2.26}$$

Proof. Let $E(z,f)$ be the inner bracket in the above integral. From Lemma 2 and (2.5) we have

$$E(z,f) = \frac{2}{\pi}\int_{f(U)} (1+|\zeta|^2)^{-2} N(1, f\circ\varphi_z, \zeta)\, dm(\zeta). \tag{2.27}$$

Using Lemma 1 and Fubini's theorem we obtain

$$-\frac{1}{4}\int_U \Delta(1-|z|)^{1-\alpha} E(z,f)\, dm = \int_{f(U)} (1+|\zeta|^2)^{-2} N_{1-\alpha}(f,\zeta)\, dm, \tag{2.28}$$

and the result follows by (2.6). □

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A final lemma is needed for the proof of our main theorem.

LEMMA 4. *Let (X, μ) be a probability space, and let $f \in L^1(\mu)$ with $f > 0$ μ-a.e. on X and $\log f \in L^1(\mu)$. For $0 \le \gamma \le 1$ let*

$$E_\gamma(f)=\int_X \log(1+f)\,d\mu-\log\left(1+\gamma\exp\int_X \log f\,d\mu\right). \tag{3.1}$$

Then

$$E_\gamma(\min\{1,f\})\le E_\gamma(f). \tag{3.2}$$

Proof. Let $A=\{x\in X,\ f(x)\ge 1\}$ and assume that $a=\mu(A)>0$; otherwise the inequality is trivial. Then (3.2) is equivalent to

$$\int_A \log\left(\frac{1+f}{2}\right)d\mu \ge \log\left(\frac{1+\gamma\exp\int_X \log f\,d\mu}{1+\gamma\ \exp\int_{X\setminus A}\log f\,d\mu}\right). \tag{3.3}$$

Let $b=\exp\int_{X\setminus A}\log f\,d\mu$. We have $0<b<1$ and

$$\begin{aligned}\log\left(\frac{1+\gamma b\exp\int_A \log f\,d\mu}{1+\gamma b}\right)&\le\log\left(\frac{1+\exp\int_A\log f\,d\mu}{2}\right)\\ &\le a\log\left(\frac{1+\exp(1/a)\int_A\log f\,d\mu}{2}\right),\end{aligned} \tag{3.4}$$

where the last inequality is just

$$\left(\frac{1+x}{2}\right)^{1/a}\le\frac{1+x^{1/a}}{2},\quad x>0,\ a\le 1.$$

We claim that

$$a\log\left(\frac{1+\exp(1/a)\int_A\log f\,d\mu}{2}\right)\le\int_A\log\left(\frac{1+f}{2}\right)d\mu. \tag{3.5}$$

Indeed, this inequality is equivalent to

$$\exp\left(-\frac{1}{a}\int_A\log(1+f)\,d\mu\right)+\exp\left(\frac{1}{a}\int_A\log\left(\frac{f}{1+f}\right)d\mu\right)\le 1, \tag{3.6}$$

which follows by Jensen's inequality

$$\begin{aligned}&\exp\left(\frac{1}{a}\int_A\log\left(\frac{1}{1+f}\right)d\mu\right)+\exp\left(\frac{1}{a}\int_A\log\left(\frac{f}{1+f}\right)d\mu\right)\\ &\le\frac{1}{a}\int_A\frac{1}{1+f}\,d\mu+\frac{1}{a}\int_A\frac{f}{1+f}\,d\mu=1.\end{aligned} \tag{3.7}$$

□

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For a meromorphic function $f\in T_0$, $f\not\equiv 0$, let ϕ_f be the outer function in U satisfying $|\phi_f(e^{i\theta})|=\min\{1,1/|f(e^{i\theta})|\}$ a.e. on $[0,2\pi]$; that is, let

$$\phi_f(z)=\exp\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\log\min\{1,1/|f(e^{i\theta})|\}\,d\theta. \tag{4.1}$$

The main result of this paper is the following.

THEOREM 1. *Let $0<\alpha\le 1$ and $f\in T_\alpha$ be nonconstant with $f=IF/J$, where I,J are inner functions such that $(I,J)=1$ and F is outer. Then $J\phi_f$ and $fJ\phi_f$ are in T_α and satisfy*

$$|J\phi_f|_\alpha\le|f|_\alpha \quad \text{and} \quad |fJ\phi_f|_\alpha\le|f|_\alpha. \tag{4.2}$$

Proof. Assume first that $0<\alpha<1$. Since $|fJ\phi_f(e^{i\theta})|^2=\min\{1,|F(e^{i\theta})|^2\}$ a.e. on $[0,2\pi]$, an application of Lemma 4 with $X=[0,2\pi]$ and $d\mu=(1/2\pi)P_z\,d\theta$, where $z\in U$ is fixed, yields

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}&P_z(\theta)\log(1+|fJ\phi_f(e^{i\theta})|^2)\,d\theta-\log(1+|fJ\phi_f(z)|^2)\\ &=E_{|I(z)|^2}(\min\{1,|F|^2\})\le E_{|I(z)|^2}(|F|^2)\\ &\le\frac{1}{2\pi}\int_0^{2\pi}P_z(\theta)\log(1+|f(e^{i\theta})|^2)\,d\theta-\log(1+|f(z)|^2)-2\log|J(z)|,\end{aligned}\tag{4.3}$$

and the second inequality in (4.2) follows by Proposition 3. We have also $|J\phi_f(e^{i\theta})|^2=\min\{1,1/|F(e^{i\theta})|^2\}$ a.e. on $[0,2\pi]$, and another application of Lemma 4 gives

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}&P_z(\theta)\log(1+|J\phi_f(e^{i\theta})|^2)\,d\theta-\log(1+|J\phi_f(z)|^2)\\ &=E_{|J(z)|^2}(\min\{1,1/|F|^2\})\le E_{|J(z)|^2}(1/|F|^2)\\ &\le\frac{1}{2\pi}\int_0^{2\pi}P_z(\theta)\log(1+1/|f(e^{i\theta})|^2)\,d\theta\\ &\quad-\log(1+1/|f(z)|^2)|^2)-2\log|I(z)|.\end{aligned}\tag{4.4}$$

Using again Proposition 3 we obtain $|J\phi_f|_\alpha\le|1/f|_\alpha$, and from (1.2) we have $|1/f|_\alpha=|f|_\alpha$, which finishes the proof in the case $\alpha<1$. For the limit case $\alpha=1$ we simply observe that for any increasing nonnegative function u on $[0,1)$,

$$\lim_{r\to 1}u(r)=\limsup_{\alpha\to 1}(1-\alpha)\int_0^1 u(r)(1-r)^{-\alpha}\,dr. \tag{4.5}$$

If we apply this equality to the functions $A(r,J\phi_f)$ and $A(r,f)$, then

$$|J\phi_f|_1=\limsup_{\alpha\to 1}(1-\alpha)|J\phi_f|_\alpha\le\limsup_{\alpha\to 1}(1-\alpha)|f|_\alpha=|f|_1. \tag{4.6}$$

Analogously we obtain the second inequality, and the proof is now complete. □

For any nonconstant $f\in T_\alpha$ with $f=IF/J$, where I,J are inner functions satisfying $(I,J)=1$ and F is outer, we have that $J\phi_f$ and $fJ\phi_f$ are bounded in U and $f=fJ\phi_f/J\phi_f$. Thus we obtain the following corollary.

COROLLARY 1. *For $0<\alpha\le 1$, every function in T_α is the quotient of two bounded functions in T_α.*

It follows easily from (2.2) that each bounded function in T_α, $0 \le \alpha \le 1$, belongs actually to the Hilbert space $D_{1-\alpha}$ defined by means of (1.4). Then from the above corollary it turns out that every function in T_α, $\alpha \in [0,1]$, is the quotient of two bounded functions in $D_{1-\alpha}$.

5. Applications. The Spaces T_ω and H_w

We shall continue the investigation of the classes T_α in a slightly more general context obtained by replacing, in the definition of T_α, the function $r \mapsto (1-r)^{-\alpha}$ by any positive increasing function $\omega \in C^1[0,1)$ with $\int_0^1 \omega(r)\,dr < \infty$. More precisely, for such a function ω, let T_ω be the class of meromorphic functions f in U satisfying

$$|f|_\omega = \int_0^1 A(r,f)\,\omega(r)\,dr < \infty. \tag{5.1}$$

We obviously have $T_1 \subset T_\omega \subset T_0$, and if $w \in C^2[0,1)$ is defined by

$$w(r) = \int_r^1 \omega(\rho)\,d\rho \tag{5.2}$$

then an integration by parts shows that, for every $f \in T_\omega$,

$$|f|_\omega = \int_U \frac{|f'(z)|^2}{(1+|f(z)|^2)^2} w(|z|)\,dm. \tag{5.3}$$

It is not difficult to see that Theorem 1 holds for the classes T_ω as well. Indeed, the formulas proved in Section 2 remain true if $\Delta(1-|z|)^{1-\alpha}$ is replaced by $\Delta w(|z|)$, and the inequalities $|J\phi_f|_\omega \le |f|_\omega$ and $|fJ\phi_f|_\omega \le |f|_\omega$ follow as above from Lemma 4. We shall continue to refer to these results even if the more general context is concerned.

The class T_ω contains the spaces H_w of analytic functions f in U with the property that

$$\|f\|_w^2 = |f(0)|^2 + \int_U |f'(z)|^2 w(|z|)\,dm < \infty, \tag{5.4}$$

and a bounded analytic function in U belongs to T_ω if and only if it belongs to H_w. Consequently, every function in T_ω is the quotient of two bounded analytic functions in H_w.

Some simple computations with the Parseval formula show that for every function $f \in H_w$ with $f(z) = \sum_{n=0}^\infty a_n z^n$, $z \in U$, we have

$$\|f\|_w^2 = \sum_{n=0}^\infty w_n |a_n|^2, \tag{5.5}$$

where $w_0 = 1$ and, for $n \ge 1$,

$$w_n = 2\pi n^2 \int_0^1 r^{2n-1} w(r)\,dr. \tag{5.6}$$

It follows that H_w is a separable Hilbert space of analytic functions in U and that polynomials are dense in H_w. Since w is decreasing and concave and since $\lim_{r\to 1} w(r)=0$, we have $w(0)\geq w(r)\geq (1-r)w(0)$, $r\in[0,1)$, which shows that $D\subset H_w\subset H^2$, where $D=D_0$ (see (1.4)), is the usual Dirichlet space and $H^2=D_1$ is the Hardy space on U. The other weighted Dirichlet spaces D_α, $0<\alpha<1$, are obtained by letting $w(r)=(1-r)^\alpha$, $r\in[0,1)$. The spaces T_1 and D appear as a limit case, namely when the function w is constant. As for the classes T_ω, we have the following useful identity for the norm on the Hilbert space H_w.

PROPOSITION 5. *Let $f\in H_w$. Then*

(5.7)
$$\|f\|_w^2=|f(0)|^2-\frac{1}{4}\int_U \Delta w(|z|)\left[\frac{1}{2\pi}\int_0^{2\pi} P_z(\theta)|f(e^{i\theta})|^2\,d\theta-|f(z)|^2\right]dm(z).$$

A proof of this result may be found in [3]. It is very similar to the one of Proposition 3, using (2.5), Lemma 1, and the Littlewood-Paley formula instead of Lemma 2.

REMARK. An immediate consequence of Proposition 5 is that if $f\in H_w$ (or D) and I_1 is an inner divisor of the inner factor of f then $f/I_1\in H_w$ (or D). This property is called the ($\mathfrak{F}$) property (see [14]) and is shared by the classes T_ω and T_1 as well. Proposition 3 shows this. The fact that D has this property follows also by Carleson's formula for the Dirichlet integral [7]. Consequently, we obtain

COROLLARY 2. *Let $f\in T_\omega$ (or T_1) with $f\neq 0$ and $f=IF/J$, where I,J are inner functions satisfying $(I,J)=1$ and F is outer. Then $F, IF, F/J$ are in T_ω (or T_1). If $f\in H_w$ (or D) then $F\in H_w$ (or D).*

Proof. F/J and $1/IF$ are obtained by dividing respectively f by I and $1/f$ by J. Also $F=IF/I$. The result follows from the above remark.

Proposition 5 may be also used to give a simple proof of the following characterization of inner functions in certain H_w-spaces in terms of the growth restrictions satisfied by their derivatives, or by conditions concerning the distribution of values of such functions. The result is known for the usual weights $w(r)=(1-r)^\alpha$ and may be found in P. Ahern's paper [1] (see also [2] and [6]). Actually, for inner functions in weighted Dirichlet spaces the norms of the form (5.7) were first considered in [1].

PROPOSITION 6. *Assume that there exists a positive constant c such that*

(5.8)
$$-(1-r)^2 w''(r)\geq c\,w(r),\quad r\in[0,1).$$

For an inner function I, the following assertions are equivalent.

(i) $I\in T_\omega$.
(ii) $I\in H_w$.

(iii) $\int_0^1 (1-r)\omega'(r)(\int_0^{2\pi} |I'(re^{i\theta})|\, d\theta)\, dr < \infty$.

(iv) *There is a set A of logarithmic capacity zero such that* $\sum_{I(z)=\zeta} w(|z|) < \infty$ *for every* $\zeta \in U \setminus A$.

Proof. If (ii) holds then (iii) follows immediately from Proposition 5 and the inequality

$$|I'(z)| \le \frac{1-|I(z)|^2}{1-|z|^2} \le \frac{1-|I(z)|^2}{1-|z|} \quad \text{for } z \in U. \tag{5.9}$$

Conversely, we have by (5.2) and (5.8) that $(1-r)\omega'(r) \ge c\omega(r)$ on $[0,1)$. If $W(r) = \omega'(r) + \omega(r)$ then, for all $r \in [0,1)$,

$$\int_0^r W(\rho)\, d\rho \le \omega(r) - \omega(0) + r\omega(r) < \frac{2}{c}(1-r)\omega'(r) \tag{5.10}$$

and $\Delta w(|z|) \le 2W(|z|)$ for $1/2 \le |z| < 1$. Further, the following inequality holds for all $\theta \in [0, 2\pi]$ with $\lim_{r\to 1} |I(re^{i\theta})| = 1$:

$$1 - |I(re^{i\theta})|^2 \le 2 \int_r^1 |I'(\rho e^{i\theta})|\, d\rho. \tag{5.11}$$

Now use (5.10), (5.11), and Fubini's theorem to obtain

$$\begin{aligned} \int_0^1 W(r)(1-|I(re^{i\theta})|^2)\, dr &\le 2\int_0^1 W(r) \int_r^1 |I'(\rho e^{i\theta})|\, d\rho\, dr \\ &= 2\int_0^1 |I'(\rho e^{i\theta})| \int_0^\rho W(r)\, dr\, d\rho \\ &\le \frac{2}{c}\int_0^1 (1-\rho)\omega'(\rho) |I'(\rho e^{i\theta})|\, d\rho \end{aligned} \tag{5.12}$$

a.e. on $[0, 2\pi]$, and this implies (ii).

Let us prove the equivalence of (ii) and (iv). For $\zeta \in U$ consider the inner function $I_\zeta = (I-\zeta)/(1-\bar{\zeta}I)$. Then $I \in H_w$ if and only if $I_\zeta \in H_w$, and $I(z) = \zeta$ if and only if $I_\zeta(z) = 0$. For all $\zeta \in U$ we have from the factorization formula that $-\log|I_\zeta(z)| \ge N(1, I_\zeta \circ \varphi_z, 0)$; hence, by Lemma 1,

$$\int_U \Delta w(|z|) \log|I_\zeta(z)|\, dm \ge 2\pi \sum_{I_\zeta(z)=0} w(|z|). \tag{5.13}$$

If I_ζ is a Blaschke product then (5.13) holds with equality, and by Frostman's theorem [9, p. 117] there exists a set A of capacity zero such that I_ζ is a Blaschke product for all $\zeta \in U \setminus A$.

Now assume that the sum $\sum_{I(z)=\zeta} w(|z|)$ is finite for some $\zeta \in U \setminus A$. From Proposition 5 and (5.13) (with equality) we obtain

$$\|I_\zeta\|_w^2 - |I_\zeta(0)|^2 \le \frac{1}{2}\int_U \Delta w(|z|) \log|I_\zeta(z)|\, dm = \pi \sum_{I(z)=\zeta} w(|z|); \tag{5.14}$$

that is, $I \in H_w$. The converse is similar to the proof of Frostman's theorem. Let $I \in H_w$ and denote by A_1 the set of points $\zeta \in U$ with $\sum_{I(z)=\zeta} w(|z|) = \infty$.

If A_1 has positive logarithmic capacity then it contains a compact set K with positive capacity; hence there exists a probability measure μ supported on K such that the logarithmic potential defined by $-\int \log|z-\zeta|\,d\mu(\zeta)$ is bounded above on $\mathbf{C}$. Consider the function

$$v(z)=\int \log\left|\frac{1-\bar{\zeta}z}{z-\zeta}\right| d\mu(\zeta). \tag{5.15}$$

Since v is bounded in U and the support of the measure μ does not intersect ∂U, we deduce that there exists a positive constant c_1 such that $v(z)\le c_1(1-|z|^2)$, $z\in U$. Then

$$-\int_U v\circ I(z)\,\Delta w(|z|)\,dm \le 4c_1(\|I\|_w^2-|I(0)|^2), \tag{5.16}$$

and by Fubini's theorem and (5.13) the integral on the left-hand side becomes

$$\int\left(\int_U \Delta w(|z|)\log|I_\zeta(z)|\,dm(z)\right)d\mu(\zeta)\ge 2\pi\int\left(\sum_{I(z)=\zeta} w(|z|)\right)d\mu(\zeta). \tag{5.17}$$

Combining the last two relations, we obtain that $\sum_{I(z)=\zeta} w(|z|)$ belongs to $L^1(\mu)$, in particular, the sum is finite μ-a.e. This contradiction shows that A_1 has capacity zero. ☐

It turns out from the above that the inner factors of a function in T_ω are not necessarily in T_ω. Indeed, if w satisfies (5.8) then we can find a sequence $\{r_n\}$ in $[0,1)$, tending to 1 such that $\sum_{n\ge 1}(1-r_n)$ is finite, but $\sum_{n\ge 1} w(r_n)=\infty$. If B is the Blaschke product with zeros r_n for $n\ge 1$ then B is not in H_w, but the function $f(z)=(1-z)^2B(z)$ belongs to the Dirichlet space [7].

6. Cyclic Vectors in H_w

We consider the multiplication operator M_z defined on the Hilbert spaces H_w by

$$(M_z f)(\zeta)=\zeta f(\zeta) \quad \text{for } \zeta\in U, f\in H_w; \tag{6.1}$$

it is a bounded weighted shift on these spaces. A closed subspace $\mathfrak{M}$ of H_w is called invariant for M_z if $M_z\mathfrak{M}\subset\mathfrak{M}$. For a function $f\in H_w$ we denote by $[f]$ the smallest invariant subspace containing f, and we say that f is a cyclic vector for M_z if $[f]=H_w$; that is, the polynomial multiples of f are dense in H_w. In the case when $H_w=H^2$ (i.e., $w(r)=1-r$ on $[0,1)$), the invariant subspaces and the cyclic vectors are described by Beurling's theorem, but for other H_w-spaces it is considerably more difficult to do this (see e.g. [5]). A useful instrument to attack such problems is the following lemma.

LEMMA 7. *If $f\in H_w$ (or D) and g is a bounded function in U such that $gf\in H_w$ (or D), then $fg\in[f]$.*

A result of this type was first proved for the Dirichlet space in [5] and [11], and a proof of Lemma 7 may be found in [3]; we shall omit the details. Using

Theorem 1 and the above lemma, we obtain the weighted version of Brown's theorem [4] mentioned at the beginning.

COROLLARY 3. *If f, $1/f \in H_w$(or D), then f is a cyclic vector for M_z.*

Proof. For $t>0$ consider the functions $h_t = (f/t)\phi_{f/t}$. We have $|h_t| \le 1$ in U and, by Lemma 7, $h_t \in [f]$. Since H_w and D are contained in H^2, it follows that f is an outer function; hence the h_t are also outer. We obtain $\lim_{t\to 0} h_t(z) = 1$ for $z \in U$ and, by Theorem 1,

$$\|h_t - h_t(0)\|_w^2 \le 4|h_t|_\omega \le 4\left|\frac{f}{t}\right|_\omega \le \int_U \frac{t^2|f'|^2}{(t^2+|f|^2)^2}\, w\, dm \le t^2\left\|\frac{1}{f}\right\|_w^2 \tag{6.2}$$

Then $\lim_{t\to 0} h_t = 1$ in the norm of H_w, and f is cyclic because $1 \in [f]$. □

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