Excision of Equivariant Cyclic Cohomology of Topological Algebras

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1. Introduction

One of the fundamental theorems in cyclic (co-)homology is the excision theorem, which was developed by Wodzicki [19; 21] and generalized to the bivariant case by Kassel [11]. In [11; 18; 20] the excision theorem was used to construct the Chern character from KK-theory to bivariant cyclic theory, and to obtain the vanishing of cyclic (co-)homology of stable **C***-algebras. In this paper we define equivariant cyclic (co-)homology of topological algebras with compact group actions and study its excision property. Section 2 is devoted to the basic definitions of equivariant Hochschild and cyclic (co-)homologies, which are motivated by twisted cyclic (co-)homology [8]. This section can be considered as an improvement of Brylinski's equivariant Hochschild homology [3] and a supplement of Klimek–Kondracki–Lesniewski's equivariant entire cyclic cohomology for finite group actions [12]. Our definitions are slightly different from those in [12]. In Sections 3 and 4 we prove our main results, the excision theorems of equivariant Hochschild and cyclic (co-)homologies, by introducing equivariant H-unitality, which is inspired by Wodzicki's theorem [21]. The main point is to deal with the twisting of group actions. As a corollary of the excision theorem, we obtain the six-term exact sequence of equivariant periodic cyclic cohomology. In Section 5 we show the existence of Mayer–Vietoris sequences of equivariant cyclic (co-)homology which were used in the ordinary case to prove that the periodic cyclic homology $PHC_{\mathrm{ev}}(C^\infty(G,M))$ of the algebra $C^\infty(G,M)$ for a compact smooth manifold M is isomorphic to K-theory $K_G^0(M)\otimes_{\mathbf{Z}}\mathbf{C}$ [2; 3]. Finally, we discuss the equivariant H-unitality in Section 6, which is much more difficult than that in [21]. We do not know in general for what algebras we will have the excision property of equivariant cyclic cohomology, although the equivariant H-unitality is quite understandable. The Chern character in equivariant cyclic (co-)homology will be constructed in [9]. The motivation of the present paper is the possible applications to the equivariant Novikov conjecture [5; 6; 10; 15] by means of equivariant cyclic cohomology, which we hope to study subsequently.

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2. Equivariant Cyclic (Co-)homology

Let $\mathfrak{A}$ be a complete locally multiplicatively-convex algebra over $\mathbf{C}$ with an increasing sequence of seminorms $\{\rho_n\}_{n=1}^{\infty}$ defining the topology. Let G be a compact group acting on $\mathfrak{A}$ by continuous automorphisms $\alpha: G \to \mathrm{Aut}(\mathfrak{A})$ such that if $\mathfrak{A}$ is unital, then each α_g is unital for $g \in G$. Throughout, such an algebra will be called a *G-algebra* and locally convex complete spaces (l.c.c.s) with G-actions will be said to be *G-spaces*. In this section we will define various equivariant (co-)homologies of the G-algebra $\mathfrak{A}$. To this aim, let $C(G)$ be the space of all continuous functions on G and let $\tilde{\otimes}$ denote an admissible topological tensor product [2; 18]. Define morphisms d_n^i: $\mathfrak{A}^{\tilde{\otimes}(n+1)} \tilde{\otimes} C(G) \to \mathfrak{A}^{\tilde{\otimes}(n)} \tilde{\otimes} C(G)$ and $t_n: \mathfrak{A}^{\tilde{\otimes}(n+1)} \tilde{\otimes} C(G) \to \mathfrak{A}^{\tilde{\otimes}(n+1)} \tilde{\otimes} C(G)$ by

$$d_n^i(a_0, a_1, \ldots, a_n, f)(g) = \begin{cases} (a_0 \alpha_g(a_1), a_2, \ldots, a_n) f(g), & i = 0, \\ (a_0, \ldots, a_i a_{i+1}, \ldots, a_n) f(g), & 1 \leq i \leq n-1, \\ (a_n a_0, a_1, \ldots, a_{n-1}) f(g), & i = n, \end{cases}$$

and

$$t_n(a_0, a_1, \ldots, a_n, f)(g) = (a_n, \alpha_g^{-1}(a_0), a_1, \ldots, a_{n-1}) f(g),$$

where $(a_0, a_1, \ldots, a_n, f)$ stands for $(a_0 \tilde{\otimes} a_1 \tilde{\otimes} \cdots \tilde{\otimes} a_n \tilde{\otimes} f)$ in $\mathfrak{A}^{\tilde{\otimes}(n+1)} \tilde{\otimes} C(G)$. Let G act on $\mathfrak{A}^{\tilde{\otimes}(n+1)} \tilde{\otimes} C(G)$ by

$$\rho_h(a_0, a_1, \ldots, a_n, f)(g) = (\alpha_h^{-1}(a_0), \ldots, \alpha_h^{-1}(a_n), h \cdot f)(g), \quad g \in G.$$

Here $(h \cdot f)(g) = f(hgh^{-1})$. Note that d_n^i and t_n differ from those in [12]. All maps defined so far on $\mathfrak{A}^{\tilde{\otimes}(n+1)} \tilde{\otimes} C(G)$ are continuous. It is straightforward to check that

$$d_{n-1}^i d_n^j = d_{n-1}^{j-1} d_n^i, \quad i < j; \tag{1}$$

$$d_n^i t_n = \begin{cases} t_{n-1} d_n^{i-1}, & 1 \leq i \leq n, \\ d_n^n, & i = 0. \end{cases} \tag{2}$$

Moreover, $t_n \rho_h = \rho_h t_n$ and $d_n^i \rho_h = \rho_h d_n^i$ for $h \in G$. For instance,

$$\begin{aligned} t_n \rho_h(a_0, a_1, \ldots, a_n, f)(g) &= t_n(\alpha_h^{-1}(a_0), \ldots, \alpha_h^{-1}(a_n), h \cdot f)(g) \\ &= (\alpha_h^{-1}(a_n), \alpha_g^{-1}(\alpha_h^{-1}(a_0)), \ldots, \alpha_h^{-1}(a_{n-1})) f(hgh^{-1}) \\ &= \alpha_h^{-1}(a_n, \alpha_{hgh^{-1}}^{-1}(a_0), a_1, \ldots, a_{n-1}) f(hgh^{-1}) \\ &= \rho_h(a_n, \alpha_{(\cdot)}^{-1}(a_0), a_1, \ldots, a_{n-1}, f(\cdot))(g) \\ &= \rho_h t_n(a_0, a_1, \ldots, a_n, f)(g). \end{aligned}$$

Furthermore, if $\mathfrak{A}$ is unital, then we define

$$s_n^i: \mathfrak{A}^{\tilde{\otimes}(n+1)} \tilde{\otimes} C(G) \to \mathfrak{A}^{\tilde{\otimes}(n+2)} \tilde{\otimes} C(G)$$

by

$$s_n^i(a_0, a_1, \ldots, a_n, f)(g) = (a_0, \ldots, a_i, 1, a_{i+1}, \ldots, a_n) f(g).$$

s_n^i commutes also with ρ_h and satisfies the following identities:

$$s_{n+1}^i s_n^j = s_{n+1}^{j+1} s_n^i, \quad i \le j; \tag{3}$$

$$d_{n+1}^i s_n^j = \begin{cases} s_{n-1}^{j-1} d_n^i, & i < j, \\ 1, & i = j, i = j+1, \\ s_{n-1}^j d_n^{i-1}, & i > j+1; \end{cases} \tag{4}$$

and

$$s_n^i t_n = \begin{cases} t_{n+1} s_n^{i-1}, & 1 \le i \le n, \\ t_{n+1}^2 s_n^n, & i = 0. \end{cases} \tag{5}$$

Let b, b', and T be the morphisms defined on $\mathfrak{A}^{\tilde{\otimes}(n+1)} \tilde{\otimes} C(G)$ by

$$b(a_0, a_1, \ldots, a_n, f) = \sum_{i=0}^{n} (-1)^i d_n^i (a_0, a_1, \ldots, a_n, f),$$

$$b'(a_0, a_1, \ldots, a_n, f) = \sum_{i=0}^{n-1} (-1)^i d_n^i (a_0, a_1, \ldots, a_n, f),$$

and

$$T(a_0, a_1, \ldots, a_n, f) = (-1)^n t_n (a_0, a_1, \ldots, a_n, f).$$

Similarly, we can consider the space of all continuous multilinear maps from $\mathfrak{A}^{\tilde{\otimes}(n+1)}$ to $C(G)$, $\mathrm{Hom}(\mathfrak{A}^{\tilde{\otimes}(n+1)}, C(G))$, and the dual of maps d_n^i, t_n, b, and b'; for example,

$$(t_n f)(a_0, a_1, \ldots, a_n)(g) = f(a_n, \alpha_g^{-1}(a_0), a_1, \ldots, a_{n-1})(g).$$

But G acts on $\mathrm{Hom}(\mathfrak{A}^{\tilde{\otimes}(n+1)}, C(G))$ by

$$(\tilde{\rho}_h f)(a_0, a_1, \ldots, a_n)(g) = f(\alpha_h^{-1}(a_0), \alpha_h^{-1}(a_1), \ldots, \alpha_h^{-1}(a_n))(h^{-1} g h).$$

d_n^i and t_n also commute with $\tilde{\rho}_h$.

Since $t_n^{n+1} \ne \mathrm{Id}$, we pass now to the quotient space of $\mathfrak{A}^{\tilde{\otimes}(n+1)} \tilde{\otimes} C(G)$. To this end, let us define a map ρ on $\mathfrak{A}^{\tilde{\otimes}(n+1)} \tilde{\otimes} C(G)$ by

$$\rho(a_0, a_1, \ldots, a_n, f)(g) = \alpha_g^{-1}(a_0, a_1, \ldots, a_n) f(g).$$

Let $(\mathfrak{A}^{\tilde{\otimes}(n+1)} \tilde{\otimes} C(G)) \tilde{\otimes}_{G,\rho} \mathbf{C}$ denote the usual equivariant space

$$(\mathfrak{A}^{\tilde{\otimes}(n+1)} \tilde{\otimes} C(G)) \tilde{\otimes}_G \mathbf{C}$$

added with another relation ρ, namely the quotient space of $\mathfrak{A}^{\tilde{\otimes}(n+1)} \tilde{\otimes} C(G)$ by the closure of the images of all maps $\mathrm{Id} - \rho$ and $\mathrm{Id} - \rho_h$, $h \in G$. We define equivariant cyclic modules $\{C_*^G(\mathfrak{A}), b, t\}$ and $\{C_G^*(\mathfrak{A}), b, t\}$ by

$$C_n^G(\mathfrak{A}) = (\mathfrak{A}^{\tilde{\otimes}(n+1)} \tilde{\otimes} C(G)) \tilde{\otimes}_{G,\rho} \mathbf{C};$$

$$\begin{aligned} C_G^n(\mathfrak{A}) &= \mathrm{Hom}_G(\mathfrak{A}^{\tilde{\otimes}(n+1)}, C(G)) \\ &= \{ f \in \mathrm{Hom}(\mathfrak{A}^{\tilde{\otimes}(n+1)}, C(G)) \colon \tilde{\rho}_h(f) = f, h \in G \}. \end{aligned}$$

The following is a basic lemma [4; 13].

LEMMA 1. *Let* $N=\sum_{i=0}^{n} T^i$ *and* $J\colon C_n^G(\mathfrak{A}) \to C_{n-1}^G(\mathfrak{A})$ *be induced by*

$$J(a_0, a_1, \ldots, a_n, f) = (-1)^n (a_n a_0, a_1, \ldots, a_{n-1}, f).$$

Then:

(a) $t^{n+1} = \mathrm{Id}$ *and* $T^{n+1} = \mathrm{Id}$ *on* $C_n^G(\mathfrak{A})$;
(b) $b = \sum_{i=0}^{n} T^i J T^{-(i+1)}$ *and* $b' = \sum_{i=0}^{n-1} T^i J T^{-(i+1)}$;
(c) $b(1-T) = (1-T)b'$ *and* $b'N = Nb$;
(d) $b^2 = (b')^2 = 0$.

The dual version of these results holds for $C_G^*(\mathfrak{A})$.

Proof. (a) follows from

$$t_n^{n+1}(a_0, a_1, \ldots, a_n, f)(g) = \rho(a_0, a_1, \ldots, a_n, f)(g)$$

in $\mathfrak{A}^{\tilde{\otimes}(n+1)} \tilde{\otimes} C(G)$. (b) is from

$$T^i J T^{(i+1)n}(a_0, a_1, \ldots, a_n, f)(g) = (-1)^i \rho^{i+1}(d_n^i(a_0, a_1, \ldots, a_n, f)(g)).$$

(c) and (d) are obvious by (a) and (b) and formulas (1) and (2). □

As in the ordinary case [4; 13], the double complexes $C_{**}^G(\mathfrak{A})$ and $C_G^{**}(\mathfrak{A})$ from $C_*^G(\mathfrak{A})$ and $C_G^*(\mathfrak{A})$ are defined by

$$C_{m,n}^G(\mathfrak{A}) = C_n^G(\mathfrak{A}) \quad \text{and} \quad C_G^{m,n}(\mathfrak{A}) = C_G^n(\mathfrak{A}), \quad m, n \geq 0,$$

with vertical and horizontal differentials δ_1 and δ_2 (resp. δ^1 and δ^2) given by

$$\delta_1(x_n q^m) = \begin{cases} b(x_n) q^m, & m \text{ even}, \\ -b'(x_n) q^m, & m \text{ odd}; \end{cases}$$

$$\delta_2(x_n q^m) = \begin{cases} N(x_n) q^{m-1}, & m \text{ even}, \\ (1-T)(x_n) q^{m-1}, & m \text{ odd}; \end{cases}$$

and δ^i are dual to δ_i. Here an element in $C_{m,n}^G(\mathfrak{A})$ is written as $x_n q^m$ with $x_n \in C_n^G(\mathfrak{A})$ and q an indeterminate of degree 1. For convenience we set $q^m = 0$ for $m < 0$. By Lemma 1, $(\delta_i)^2 = 0 = (\delta^i)^2$ and $\delta_1\delta_2 + \delta_2\delta_1 = 0 = \delta^1\delta + \delta^2\delta^1$. Let $T_*(C_*^G(\mathfrak{A}))$ and $T^*(C_G^*(\mathfrak{A}))$ be the total complexes of $C_{**}^G(\mathfrak{A})$ and $C_G^{**}(\mathfrak{A})$, respectively. Define a morphism S in $C_{**}^G(\mathfrak{A})$ and $C_G^{**}(\mathfrak{A})$ by

$$S(x_n q^m) = x_n q^{m-2} \quad \text{and} \quad S(x^n q^m) = x^n q^{m+2}, \quad m, n \geq 0.$$

Clearly, S commutes with the differentials δ_i and δ^i. Let

$$\operatorname{Ker}(S, T_*(C_*^G(\mathfrak{A}))) = \operatorname{Ker}\{T_*(C_*^G(\mathfrak{A})) \xrightarrow{S} T_*(C_*^G(\mathfrak{A}))[2]\}$$

and

$$\operatorname{Coker}(S, T^*(C_G^*(\mathfrak{A}))) = \operatorname{Coker}\{T^*(C_G^*(\mathfrak{A}))[2] \xrightarrow{S} T^*(C_G^*(\mathfrak{A}))\}.$$

Here the shifted complex $M_*[n]$ of a differential complex $\{M_*, d\}$ is given by $M_i[n] = M_{i-n}$ with differential $(-1)^n d$.

We now come to the main definitions of this paper. Let $B_*^G(\mathfrak{A}) = C_{*-1}^G(\mathfrak{A})$ and $B_G^*(\mathfrak{A}) = C_G^{*-1}(\mathfrak{A})$ with differential b', $* \geq 1$.

DEFINITION 1. Let $\mathfrak{A}$ be a G-algebra.

(a) The equivariant Bar (co-)homology of $\mathfrak{A}$ is

$$BH_*^G(\mathfrak{A}) = H_*(B_*^G(\mathfrak{A}), b');$$
$$BH_G^*(\mathfrak{A}) = H^*(B_G^*(\mathfrak{A}), b'), \quad * \geq 1.$$

(b) The equivariant $\mathcal{H}$-(co-)homology of $\mathfrak{A}$ is

$$\mathcal{H}H_*^G(\mathfrak{A}) = H_*(C_*^G(\mathfrak{A}), b);$$
$$\mathcal{H}H_G^*(\mathfrak{A}) = H^*(C_G^*(\mathfrak{A}), b), \quad * \geq 0.$$

(c) The equivariant Hochschild (co-)homology of $\mathfrak{A}$ is

$$HH_*^G(\mathfrak{A}) = H_*(\mathrm{Ker}(S, T_*(C_*^G(\mathfrak{A}))), \delta_1 + \delta_2);$$
$$HH_G^*(\mathfrak{A}) = H^*(\mathrm{Coker}(S, T^*(C_G^*(\mathfrak{A}))), \delta^1 + \delta^2), \quad * \geq 0.$$

(d) The equivariant cyclic (co-)homology of $\mathfrak{A}$ is

$$HC_*^G(\mathfrak{A}) = H_*(T_*(C_*^G(\mathfrak{A})), \delta_1 + \delta_2);$$
$$HC_G^*(\mathfrak{A}) = H^*(T^*(C_G^*(\mathfrak{A})), \delta^1 + \delta^2), \quad * \geq 0.$$

REMARK 1. (a) This definition is the generalization of ordinary (co-)homologies; that is, if G is trivial then all various homologies above are the ordinary ones.

(b) The following trivial but useful long exact sequences hold:

$$\cdots \longrightarrow \mathcal{H}H_n^G(\mathfrak{A}) \longrightarrow HH_n^G(\mathfrak{A}) \longrightarrow BH_n^G(\mathfrak{A}) \longrightarrow \mathcal{H}H_{n-1}^G(\mathfrak{A}) \longrightarrow \cdots, \tag{6}$$

$$\cdots \longrightarrow BH_G^n(\mathfrak{A}) \longrightarrow HH_G^n(\mathfrak{A}) \longrightarrow \mathcal{H}H_G^n(\mathfrak{A}) \longrightarrow BH_G^{n+1}(\mathfrak{A}) \longrightarrow \cdots, \tag{7}$$

which follow from the short exact sequences

$$0 \longrightarrow C_*^G(\mathfrak{A}) \longrightarrow \mathrm{Ker}(S, T_*(C_*^G(\mathfrak{A}))) \xrightarrow{-P} B_*^G(\mathfrak{A}) \longrightarrow 0, \tag{8}$$

$$0 \longrightarrow B_G^*(\mathfrak{A}) \xrightarrow{-I} \mathrm{Coker}(S, T^*(C_G^*(\mathfrak{A}))) \longrightarrow C_G^*(\mathfrak{A}) \longrightarrow 0, \tag{9}$$

where P and I are the projection and inclusion on the corresponding spaces, respectively.

(c) The equivariant Connes long exact sequences take the following forms:

$$\cdots \longrightarrow HH_n^G(\mathfrak{A}) \longrightarrow HC_n^G(\mathfrak{A}) \xrightarrow{S} HC_{n-2}^G(\mathfrak{A}) \longrightarrow HH_{n-1}^G(\mathfrak{A}) \longrightarrow \cdots. \tag{10}$$

$$\cdots \longrightarrow HC_G^{n-2}(\mathfrak{A}) \xrightarrow{S} HC_G^n(\mathfrak{A}) \longrightarrow HH_G^n(\mathfrak{A}) \longrightarrow HC_G^{n-1}(\mathfrak{A}) \longrightarrow \cdots, \tag{11}$$

which are from the short exact sequences

$$0 \longrightarrow \mathrm{Ker}(S, T_*(C_*^G(\mathfrak{A}))) \longrightarrow T_*(C_*^G(\mathfrak{A})) \xrightarrow{S} T_*(C_*^G(\mathfrak{A}))[2] \longrightarrow 0, \tag{12}$$

$$0 \longrightarrow T^*(C_G^*(\mathfrak{A}))[2] \xrightarrow{S} T^*(C_G^*(\mathfrak{A})) \longrightarrow \mathrm{Coker}(S, T^*(C_G^*(\mathfrak{A}))) \longrightarrow 0. \tag{13}$$

Using the morphism S, we can define equivariant periodic cyclic (co-)homology.

DEFINITION 2. Let $\mathfrak{A}$ be a G-algebra. We call for $* = \mathrm{ev}$ and $i=0$ or $* = \mathrm{odd}$ and $i=1$;

$$PHC_*^G(\mathfrak{A}) = \lim_{\leftarrow S} HC_{i+2n}^G(\mathfrak{A}) \quad \text{and} \quad PHC_G^*(\mathfrak{A}) = \lim_{\rightarrow S} HC_G^{i+2n}(\mathfrak{A})$$

are the equivariant periodic cyclic (co-)homology.

The following lemma will provide an important example of equivariant H-unital algebras in Section 6.

LEMMA 2. *Let $\mathfrak{A}$ be a unital G-algebra. Define* $s' : \mathfrak{A}^{\tilde{\otimes}(n-1)} \tilde{\otimes} C(G) \to \mathfrak{A}^{\tilde{\otimes}(n+2)} \tilde{\otimes} C(G)$ *by*

$$s'(a_0, a_1, \ldots, a_n, f)(g) = (1, \alpha_g^{-1}(a_0), a_1, \ldots, a_n) f(g).$$

Then $s'\rho_h = \rho_h s'$ and $b's' + s'b' = \mathrm{Id}$. Hence $BH_*^G(\mathfrak{A}) = 0$, $* \geq 1$.

Similar results hold for equivariant Bar-cohomology.

Proof. We have

$$\begin{aligned} s'\rho_h(a_0, a_1, \ldots, a_n, f)(g) &= s'(\alpha_h^{-1}(a_0), \ldots, \alpha_h^{-1}(a_n), h\cdot f)(g) \\ &= (1, \alpha_g^{-1}(\alpha_h^{-1}(a_0)), \alpha_h^{-1}(a_1), \ldots, \alpha_h^{-1}(a_n))(h\cdot f)(g) \\ &= \alpha_h^{-1}(1, \alpha_{hgh^{-1}}^{-1}(a_0), a_1, \ldots, a_n) f(hgh^{-1}) \\ &= \rho_h(1, \alpha_{(\cdot)}^{-1}(a_0), a_1, \ldots, a_n, f(\cdot))(g) \\ &= \rho_h s'(a_0, a_1, \ldots, a_n, f)(g). \end{aligned}$$

The rest of the proof also follows from an easy computation. □

We close this section by the following example.

EXAMPLE 1. Let $\mathfrak{A}$ be $\mathbf{C}$. Then G acts trivially on $\mathfrak{A}$. Denote by $R(G)$ the subspace of $C(G)$ consisting of all central functions on G; that is, $f \in R(G)$ if $f \in C(G)$ and $f(h^{-1}gh) = f(g)$ for any $h \in G$. $C_n^G(\mathfrak{A})$ is isomorphic to $\mathbf{C}(\underbrace{1 \tilde{\otimes} \cdots \tilde{\otimes} 1}_{n+1}) \tilde{\otimes} R(G)$.

$$b(\underbrace{1, 1, \ldots, 1}_{n+1}, f)(g) = \begin{cases} 0, & n \text{ odd}; \\ (\underbrace{1, 1, \ldots, 1}_{n}) f(g), & n \text{ even}. \end{cases}$$

Hence $\mathcal{H}H_*^G(\mathfrak{A}) = R(G)$ for n even and 0 for n odd. Since $BH_*^G(\mathfrak{A}) = 0$, $* \geq 1$, by Lemma 2, $HH_*^G(\mathfrak{A}) = \mathcal{H}H_*^G(\mathfrak{A})$ in view of (6). It follows from (10) that $HC_n^G(\mathfrak{A}) = R(G)$ for n even and 0 for n odd. The results hold also for $HH_G^*(\mathfrak{A})$ and $HC_G^*(\mathfrak{A})$.

3. Excision in Equivariant Cyclic Homology

In this section we will prove an excision theorem for equivariant cyclic homology. Under a mild condition it associates a long exact sequence of equivariant cyclic homology with the following equivariant **C**-split short exact sequence of G-algebras:

(14) $$0 \to \mathcal{A} \xrightarrow{i} \mathcal{E} \underset{j}{\overset{\pi}{\rightleftarrows}} \mathcal{R} \to 0.$$

Here $\mathcal{E}$ and $\mathcal{R}$ are G-algebras, i and π are equivariant continuous homomorphisms, and j is an equivariant continuous linear map such that $\pi j = \mathrm{Id}$. In general, j may not be a homomorphism. We can consider $\mathcal{E} = \mathcal{A} \tilde{\oplus} \mathcal{R}$ as a topological space. To get the excision property, let us first consider the Bar homology as in [21]. Let

$$BK_*^G = \mathrm{Ker}\{B_*^G(\mathcal{E}) \xrightarrow{\pi_*} B_*^G(\mathcal{R})\}.$$

Define a filtration of the complex $\{BK_*^G, b'\}$ by

$$F_p BK_{p+q}^G = \overline{\mathrm{Span}}\{[e_1, e_2, \ldots, e_{p+q}, f] \in B_{p+q}^G(\mathcal{E}) : \text{at least } q\ e_i's \in \mathcal{A}\}.$$

Here $\overline{\mathrm{Span}}$ means the closure of Span and $\{e_1, \ldots, e_n, f]$ denotes the element in $B_n^G(\mathcal{E})$ with the representative $(e_1, \ldots, e_n, f)$. Then

$$0 \subset B_n^G(\mathcal{A}) = F_0 BK_n^G \subset \cdots \subset F_{n-1} BK_n^G = BK_n^G.$$

The spectral sequence corresponding to this filtration converges to the homology of $\{BK_*^G, b'\}$ [14]. Its E_0-terms are

$$BE_{p,q}^{G,0} = \overline{\mathrm{Span}}\{[e_1, \ldots, e_{p+q}, f] \in B_{p+q}^G(\mathcal{E}) : \text{exactly } q\ e_i's \in \mathcal{A}\},$$

with differential $d_{p,q}^0$ induced by b'. Write the homogeneous elements of $BE_{p,q}^{G,0}$ in the following two forms: Case I,

$$[a_{(k_1)}, r_{(n_2)}, \ldots, a_{(k_l)}, r_{(n_{l+1})}, f], \quad n_j, k_i \in \mathbf{Z}^+, 2 \le j \le l, \ 1 \le i \le l, \ n_{l+1} \in \mathbf{N},$$

and Case II,

$$[r_{(n_1)}, a_{(k_1)}, \ldots, a_{(k_l)}, r_{(n_{l+1})}, f], \quad n_j, k_i \in \mathbf{Z}^+, \ 1 \le i, j \le l, \ n_{l+1} \in \mathbf{N},$$

where

$$a_{(k_i)} = (a_{k_{i-1}+1}, a_{k_{i-1}+2}, \ldots, a_{k_{i-1}+k_i}, f), \quad 1 \le i \le l, \ a_i \in \mathcal{A};$$

$$r_{(n_j)} = (r_{n_{j-1}+1}, r_{n_{j-1}+2}, \ldots, r_{n_{j-1}+n_j}, f), \quad 1 \le j \le l+1, \ r_j \in \mathcal{R}.$$

The differential $d_{p,q}^0$ acts on these two kinds of elements as follows: Case I,

$$d_{p,q}^0[a_{(k_1)}, r_{(n_2)}, \ldots, a_{(k_l)}, r_{(n_{l+1})}, f] = [b'(a_{(k_1)}), r_{(n_2)}, \ldots, a_{(k_l)}, r_{(n_{l+1})}, f]$$
$$+ \sum_{j=2}^{l} (-1)^{\sum_{i=1}^{j-1}(k_i + n_{i+1})} [a_{(k_1)}, r_{(n_2)}, \ldots, \tilde{b}'(a_{(k_j)}), r_{(n_{j+1})}, \ldots, r_{(n_{l+1})}, f]$$

and Case II,

$$d_{p,q}^0[r_{(n_1)}, a_{(k_1)}, \ldots, a_{(k_l)}, r_{(n_{l+1})}, f]$$
$$= \sum_{j=1}^{l} (-1)^{\sum_{i=1}^{j-1}(n_i + k_i) + n_j} [r_{(n_1)}, a_{(k_1)}, \ldots, r_{(n_j)}, \tilde{b}'(a_{(k_j)}), \ldots, a_{(k_l)}, r_{(n_{l+1})}, f].$$

Here $\tilde{b}'(a_0, a_1, \ldots, a_n) = \sum_{i=0}^{n-1}(-1)^i(a_0, \ldots, a_i a_{i+1}, \ldots, a_n)$, and $[b'(a_{(k_i)}), \ldots]$ and $[\ldots, \tilde{b}'(a_{(k_j)}), \ldots]$ denote the parts of element $b'[\ldots]$ involving b' action on $a_{(k_1)}$ and $a_{(k_j)}$, respectively.

To further analyze $\{BE_{**}^{G,0}, d_{**}^0\}$, we introduce the following two complexes, modifications of those in [21], to deal with the group twisting.

Case I: For $l \in \mathbf{Z}^+$ let $\mathcal{O}_1 = (n_2, n_3, \ldots, n_{l+1})$ with $|\mathcal{O}_1| = \sum_{i=2}^{l+1} n_i$ and $l(\mathcal{O}_1) = l$ for $n_i > 0$, $2 \le i \le l$, and $n_{l+1} \ge 0$. Let $BT_*^G(\mathcal{O}_1)$ be the total complex of the $(l+1)$-tuple of the complexes

$$(B_*(\mathcal{A}) \tilde{\otimes} \tilde{B}_*(\mathcal{A})[n_2] \tilde{\otimes} \cdots \tilde{\otimes} \tilde{B}_*(\mathcal{A})[n_l] \tilde{\otimes} \mathcal{R}^{\tilde{\otimes}|\mathcal{O}_1|}[-|\mathcal{O}_1| + n_{l+1}] \tilde{\otimes} C(G)) \tilde{\otimes}_{G,\rho} \mathbf{C},$$

where $B_*(\mathcal{A})$ and $\tilde{B}_*(A)$ are equal to $\mathcal{A}^{\tilde{\otimes}(*)}$ with differentials b' and $\tilde{b}'$, respectively, and where $\mathcal{R}^{\tilde{\otimes}|\mathcal{O}_1|}[-|\mathcal{O}_1| + n_{l+1}]$ concentrates in dimension $-|\mathcal{O}_1| + n_{l+1}$ with trivial differential.

Case II: Let $\mathcal{O}_2 = (n_1, n_2, \ldots, n_{l+1})$ for $n_i > 0$, $1 \le i \le l$, and $n_{l+1} \ge 0$. Let $BT_*(\mathcal{O}_2)$ be the total complex of

$$(\tilde{B}_*(\mathcal{A})[n_1] \tilde{\otimes} \tilde{B}_*(\mathcal{A})[n_2] \tilde{\otimes} \cdots \tilde{\otimes} \tilde{B}_*(\mathcal{A})[n_l] \\ \tilde{\otimes} \mathcal{R}^{\tilde{\otimes}|\mathcal{O}_2|}[-|\mathcal{O}_2| + n_{l+1}] \tilde{\otimes} C(G)) \tilde{\otimes}_{G,\rho} \mathbf{C}.$$

Note that b' in $B_*(\mathcal{A})$ involves the group twisting. The complexes are defined so that the following map is a morphism of the complexes for $p \ge 1$:

$$\{BE_{p,*}^{G,0}, d_{p,*}^0\} \xrightarrow{F} \tilde{\bigoplus_{\substack{\mathcal{O}_1 : |\mathcal{O}_1| = p \\ l(\mathcal{O}_1) \ge 1}}} BT_*^G(\mathcal{O}_1) \tilde{\oplus} \tilde{\bigoplus_{\substack{\mathcal{O}_2 : |\mathcal{O}_2| = p \\ l(\mathcal{O}_2) \ge 1}}} BT_*^G(\mathcal{O}_2), \tag{15}$$

where

$$F[r_{(n_1)}, a_{(k_1)}, \ldots, a_{(k_l)}, r_{(n_{l+1})}, f] = [a_{(k_1)}, a_{(k_2)}, \ldots, a_{(k_l)}, r_{(n_1)}, r_{(n_2)}, \ldots, r_{(n_{l+1})}, f]$$

and

$$F[a_{(k_1)}, \ldots, a_{(k_l)}, r_{(n_{l+1})}, f] = [a_{(k_1)}, a_{(k_2)}, \ldots, a_{(k_l)}, r_{(n_2)}, \ldots, r_{(n_{l+1})}, f].$$

In fact, the differentials of $BT_*^G(\mathcal{O}_i)$ have the same formulas as d_{**}^0 [14]. Thus, F commutes with the differentials of the complexes. Obviously, F is one-to-one and onto. Hence F is an isomorphism of the complexes. Therefore, we obtain the next lemma.

LEMMA 3. *The E_1-terms $BE_{p,*}^{G,1}$ of the spectral sequence associated with the filtration $F_*BK_*^G$ are isomorphic to*

$$\bigoplus_{all\, \mathcal{O}_1 : |\mathcal{O}_1| = p} H_*(BT_*^G(\mathcal{O}_1), d_{p,*}^0) \oplus \bigoplus_{all\, \mathcal{O}_2 : |\mathcal{O}_2| = p} H_*(BT_*(\mathcal{O}_2), d_{p,*}^0), \quad p \ge 1.$$

Proof. This is a consequence of the definition of E_1-terms of the spectral sequence together with (15). □

The following is an excision theorem for equivariant Bar homology which also gives a condition that guarantees $BE_{p,*}^{G,1} = 0$ for $p \ge 1$. The G-algebras satisfying this condition will be called equivariant H-unital in Section 6, where we will focus on this question.

THEOREM 1. *Let $\mathfrak{A}$ be a G-algebra. For each G-space X, let* $B_n^G(\mathfrak{A},X)=(\mathfrak{A}^{\tilde{\otimes}(n)}\tilde{\otimes}X\tilde{\otimes}C(G))\tilde{\otimes}_{G,\rho}\mathbf{C}$ *and* $\tilde{B}_n^G(\mathfrak{A},X)=(\mathfrak{A}^{\tilde{\otimes}(n)}\tilde{\otimes}X)\tilde{\otimes}_{G,\rho}\mathbf{C}$ *for* $n\geq 1$. *If* $H_*(B_*^G(\mathfrak{A},X),b')=0$ *and* $H_*(\tilde{B}_*^G(\mathfrak{A},X),\tilde{b}')=0$, $*\geq 1$, *for each G-space X, then the inclusion* $B_*^G(\mathfrak{A})\xrightarrow{i_*}BK_*^G$ *induces an isomorphism in homology, and there is a long exact sequence of equivariant Bar homology associated with* (14):

$$\text{(16)} \qquad \cdots\longrightarrow BH_n^G(\mathfrak{A})\xrightarrow{i_*}BH_n^G(\mathcal{E})\xrightarrow{\pi_*}BH_n^G(\mathfrak{R})\xrightarrow{\partial}BH_{n-1}^G(\mathfrak{A})\longrightarrow\cdots.$$

Proof. Note that if $BE_{p,*}^{G,1}=0$ for $p\geq 1$, then the spectral sequence degenerates at the E_2-terms and

$$BH_*^G(\mathfrak{A})\simeq BEE_{0,*}^{G,1}\overset{i_*}{\simeq}BE_{0,*}^{G,\infty}\simeq H_*(BK_*^G,b') \quad \text{for } *\geq 1.$$

The long exact sequence (16) is an immediate consequence of this result and the short exact sequence $0\to BK_*^G\to B_*^G(\mathcal{E})\to B_*^G(\mathfrak{R})\to 0$. Hence it suffices to prove $BE_{p,*}^{G,1}=0$ for $p\geq 1$. By Lemma 3 and induction on the number l, it is enough to show that:

(a) $H_*(T_*((B_*(\mathfrak{A})\tilde{\otimes}\tilde{B}_*(\mathfrak{A})[n]\tilde{\otimes}X\tilde{\otimes}C(G))\tilde{\otimes}_{G,\rho}\mathbf{C}))=0$; and
(b) $H_*(T_*((\tilde{B}_*(\mathfrak{A})[m]\tilde{\otimes}\tilde{B}_*(\mathfrak{A})[n]\tilde{\otimes}X)\tilde{\otimes}_{G,\rho}\mathbf{C}))=0$ for $*\geq 1$ and $m,n>0$.

To this end, consider the spectral sequence associated with the filtration of the double complex $(B_*(\mathfrak{A})\tilde{\otimes}\tilde{B}_*(\mathfrak{A})[n]\tilde{\otimes}X\tilde{\otimes}C(G))\tilde{\otimes}_{G,\rho}\mathbf{C}$ by columns. This spectral sequence converges to

$$H_*(T_*((B_*(\mathfrak{A})\tilde{\otimes}\tilde{B}_*(\mathfrak{A})[n]\tilde{\otimes}X\tilde{\otimes}C(G))\tilde{\otimes}_{G,\rho}\mathbf{C}))$$

and has the E_1-terms

$$E_1^{p,q}=H_p((B_p(\mathfrak{A})\tilde{\otimes}\tilde{B}_q(\mathfrak{A})[n]\tilde{\otimes}X\tilde{\otimes}C(G))\tilde{\otimes}_{G,\rho}\mathbf{C}),$$

which are all zero by our assumption. The spectral sequence thus degenerates at the E_2-terms and has all vanishing E_p-terms. This implies (a). The same reasoning proves (b). □

THEOREM 2. *Under the assumptions of Theorem 1, there is a long exact sequence of equivariant $\mathcal{H}$-homology associated with* (14):

$$\cdots\longrightarrow\mathcal{H}H_n^G(\mathfrak{A})\xrightarrow{i_*}\mathcal{H}H_n^G(\mathcal{E})\xrightarrow{\pi_*}\mathcal{H}H_n^G(\mathfrak{R})\xrightarrow{\partial}\mathcal{H}H_{n-1}^G(\mathfrak{A})\longrightarrow\cdots.$$

Proof. This amounts to verifying that the inclusion

$$C_*^G(\mathfrak{A})\xrightarrow{i_*}\mathcal{H}H_*^G=\operatorname{Ker}\{C_*^G(\mathcal{E})\xrightarrow{\pi_*}C_*^G(\mathfrak{R})\}$$

induces an isomorphism in homology. The proof is similar to that of Theorem 1. We only indicate it briefly.

Filter $\mathcal{H}K_*^G$ by the following:

$$F_p\mathcal{H}K_{p+q}^G=\overline{\operatorname{Span}}\{[e_0,e_1,\dots,e_{p+q},f]\in C_{p+q}^G(\mathcal{E})\text{: at least } q+1\ e_i's\in\mathfrak{A}\};$$

$$0\subset C_n^G(\mathfrak{A})=F_0\mathcal{H}K_n^G\subset\cdots\subset F_n\mathcal{H}K_n^G=\mathcal{H}K_n^G.$$

The spectral sequence associated with this filtration converges to the homology of $\mathcal{H}K_*^G$. Its E_0-terms are

$$\mathcal{H}E^{G,0}_{p,q} \simeq \overline{\mathrm{Span}}\{[e_0,\ldots,e_{p+q},f]\in C^G_{p+q}(\mathcal{E})\colon \text{exactly } q+1\ e_i\text{'s} \in \mathcal{A}\}.$$

Each element of $\mathcal{H}E^{G,0}_{**}$ can be written as either Case I or Case II as before. The differential $d^{G,0}_{p,q}$ has the same formula as that of $BE^{G,0}_{**}$. Using the notation of Lemma 3, we can form the following complexes: in Case I, $\mathcal{H}T^G_*(\mathcal{O}_1)$ is the total complex of the $(l+1)$-tuple

$$\begin{aligned}(B_*(\mathcal{A})\tilde{\otimes}\tilde{B}_*(\mathcal{A})[n_2]\tilde{\otimes}\cdots\tilde{\otimes}\tilde{B}_*(\mathcal{A})[n_l]&\\ \tilde{\otimes}\mathcal{R}^{\tilde{\otimes}|\mathcal{O}_1|}[-|\mathcal{O}_1|-1+n_{l+1}]\tilde{\otimes}C(G))\tilde{\otimes}_{G,\rho}\mathbf{C};&\end{aligned}$$

in Case II, $\mathcal{H}T_*(\mathcal{O}_2)$ is the total complex of the $(l+1)$-tuple

$$\begin{aligned}(\tilde{B}_*(\mathcal{A})[n_1]\tilde{\otimes}\tilde{B}_*(\mathcal{A})[n_2]\tilde{\otimes}\cdots\tilde{\otimes}\tilde{B}_*(\mathcal{A})[n_l]&\\ \tilde{\otimes}\mathcal{R}^{\tilde{\otimes}|\mathcal{O}_2|}[-|\mathcal{O}_2|-1+n_{l+1}]\tilde{\otimes}C(G))\tilde{\otimes}_{G,\rho}\mathbf{C}.&\end{aligned}$$

Hence,

$$\{\mathcal{H}E^{G,0}_{p,*},d^{G,0}_{p,*}\}\xrightarrow{F_1}\tilde{\bigoplus_{\substack{\mathcal{O}_1:|\mathcal{O}_1|=p\\ l(\mathcal{O}_1)\geq 1}}}\mathcal{H}T^G_*(\mathcal{O}_1)\tilde{\oplus}\tilde{\bigoplus_{\substack{\mathcal{O}_2:|\mathcal{O}_2|=p\\ l(\mathcal{O}_2)\geq 1}}}\mathcal{H}T^G_*(\mathcal{O}_2).$$

The isomorphism F_1 has the same formula as F for $BE^{G,0}_{**}$. The rest of the proof is identical to that of Theorem 1. □

THEOREM 3. *Under the assumptions of Theorem 1, there are long exact sequences of equivariant Hochschild and cyclic homologies associated with* (14):

$$\cdots\longrightarrow HH^G_n(\mathcal{A})\xrightarrow{i_*}HH^G_n(\mathcal{E})\xrightarrow{\pi_*}HH^G_n(\mathcal{R})\xrightarrow{\partial}HH^G_{n-1}(\mathcal{A})\longrightarrow\cdots \tag{17}$$

and

$$\cdots\longrightarrow HC^G_n(\mathcal{A})\xrightarrow{i_*}HC^G_n(\mathcal{E})\xrightarrow{\pi_*}HC^G_n(\mathcal{R})\xrightarrow{\partial}HC^G_{n-1}(\mathcal{A})\longrightarrow\cdots. \tag{18}$$

Proof. By Theorems 1 and 2 and the Five Lemma, the middle vertical inclusion in the following commutative diagram induces an isomorphism in homology:

$$\begin{array}{ccccccccc}0\longrightarrow & \mathcal{H}K^G_* & \longrightarrow & \mathrm{Ker}(S,T_*(\mathcal{H}K^G_*)) & \longrightarrow & BK^G_* & \longrightarrow 0\\ & \uparrow i_* & & \uparrow i_* & & \uparrow i_* & \\ 0\longrightarrow & C^G_*(\mathcal{A}) & \longrightarrow & \mathrm{Ker}(S,T_*(C^G_*(\mathcal{A}))) & \longrightarrow & B^G_*(\mathcal{A}) & \longrightarrow 0.\end{array}$$

(17) then follows from the short exact sequence

$$0\longrightarrow\mathrm{Ker}(S,T_*(\mathcal{H}K^G_*))\longrightarrow\mathrm{Ker}(S,T_*(C^G_*(\mathcal{E})))\longrightarrow\mathrm{Ker}(S,T_*(C^G_*(\mathcal{R})))\longrightarrow 0.$$

To show (18), we need only to check that

$$T_*(C^G_*(\mathcal{A}))\xrightarrow{i_*}CK^G_*=\mathrm{Ker}\{T_*(C^G_*(\mathcal{E}))\xrightarrow{\pi_*}T_*(C^G_*(\mathcal{R}))\}$$

induces an isomorphism in homology, since (18) is then an immediate result of the short exact sequence

$$0\longrightarrow CK^G_*\longrightarrow T_*(C^G_*(\mathcal{E}))\longrightarrow T_*(C^G_*(\mathcal{R}))\longrightarrow 0.$$

But the isomorphism of i_* in homology can be easily obtained by the Five Lemma and the following commutative diagram:

$$\begin{array}{ccccccc} 0 \longrightarrow \operatorname{Ker}(S, T_*(\mathcal{HK}_*^G)) & \xrightarrow{I} & CK_*^G & \xrightarrow{S} & CK_*^G[2] & \longrightarrow 0 \\ \uparrow i_* & & \uparrow i_* & & \uparrow i_* & \\ 0 \longrightarrow \operatorname{Ker}(S, T_*(C_*^G(\mathfrak{A}))) & \xrightarrow{I} & T_*(C_*^G(\mathfrak{A})) & \xrightarrow{S} & T_*(C_*^G(\mathfrak{A}))[2] & \longrightarrow 0, \end{array}$$

because of the isomorphism of the first vertical inclusion in homology by the result of the previous paragraph. □

4. Excision in Equivariant Cyclic Cohomology

In this section we will show that there are long exact sequences of equivariant Hochschild and cyclic cohomologies associated with (14). These long exact sequences can be checked to be compatible with equivariant Connes long exact sequence. We then obtain the six-term exact sequence of equivariant periodic cyclic cohomology.

Consider the equivariant $\mathbf{C}$-split short exact sequence (14) of G-algebras. Without loss of generality we can assume $\mathcal{E} = \mathfrak{A} \oplus \mathfrak{R}$ as a topological space. Let $\mathcal{E}^{\tilde{\otimes}(n+1)} = K_n(\mathcal{E}, \mathfrak{R}) \oplus \mathfrak{R}^{\tilde{\otimes}(n+1)}$, where $K_n(\mathcal{E}, \mathfrak{R})$ is the direct sum of the $(n+1)$-tuple tensor products of $\mathfrak{A}$ and $\mathfrak{R}$ containing at least one $\mathfrak{A}$ as a tensor factor, for example, $K_1(\mathcal{E}, \mathfrak{R}) = (\mathfrak{A} \tilde{\otimes} \mathfrak{A}) \oplus (\mathfrak{R} \tilde{\otimes} \mathfrak{A}) \oplus (\mathfrak{A} \tilde{\otimes} \mathfrak{R})$ for $n = 1$. The multiplication in $K_n(\mathcal{E}, \mathfrak{R})$ is well defined and continuous since $\mathfrak{A}$ is an ideal in $\mathcal{E}$. Let $\mathcal{HK}_G^n = \operatorname{Hom}_G(K_n(\mathcal{E}, \mathfrak{R}), C(G))$. Then $C_G^n(\mathcal{E}) = \mathcal{HK}_G^n \oplus C_G^n(\mathfrak{R})$. We have the short exact sequence

$$0 \longrightarrow C_G^n(\mathfrak{R}) \xrightarrow{\pi^*} C_G^n(\mathcal{E}) \longrightarrow \mathcal{HK}_G^n \longrightarrow 0. \tag{19}$$

To analyze the complex $\{\mathcal{HK}_G^*, b\}$, we filter $K_n(\mathcal{E}, \mathfrak{R})$ and $\mathcal{HK}_G^n$ as follows:

$$0 \subset \mathfrak{A}^{\tilde{\otimes}(n+1)} = F_0 K_n(\mathcal{E}, \mathfrak{R}) \subset F_1 K_n(\mathcal{E}, \mathfrak{R}) \subset \cdots \subset F_n K_n(\mathcal{E}, \mathfrak{R}) = K_n(\mathcal{E}, \mathfrak{R}),$$

$$0 \subset F^0 \mathcal{HK}_G^n \subset F^1 \mathcal{HK}_G^n \subset \cdots \subset F^n \mathcal{HK}_g^n = \mathcal{HK}_G^n,$$

where

$$F_p K_n(\mathcal{E}, \mathfrak{R}) = \overline{\operatorname{Span}}\{(e_0, e_1, \dots, e_n) \in K_n(\mathcal{E}, \mathfrak{R}) \colon \text{at least } n-p+1 \ e_i's \in \mathfrak{A}\}$$

and

$$F^p \mathcal{HK}_G^n = \{\varphi \in \mathcal{HK}_G^n \colon \varphi = 0 \text{ on } K_n(\mathcal{E}, \mathfrak{R}) \backslash F_p K_n(\mathcal{E}, \mathfrak{R})\}.$$

The spectral sequence associated with this filtration of $\{\mathcal{HK}_G^*, b\}$ converges to $H^*(\mathcal{HK}_G^*, b)$. Its E^0-terms are given by

$$\mathcal{HE}_{G,0}^{p,q} \simeq \{\varphi \in \mathcal{HK}_G^{p+q} \colon \varphi(e_0, e_1, \dots, e_{p+q}) \neq 0 \text{ if exactly } q+1 \ e_i's \in \mathfrak{A}\}.$$

Writing down the elements of $K_n(\mathcal{E}, \mathfrak{R})$ as Case I and Case II in Section 3, we have

$$K_n(\mathcal{E}, \mathfrak{R}) = K_n^a(\mathcal{E}, \mathfrak{R}) \oplus K_n^r(\mathcal{E}, \mathfrak{R})$$

and

$$\mathcal{H}E_{G,0}^{p,q}=\mathcal{H}E_{G,0}^{p,q,a}\oplus\mathcal{H}E_{G,0}^{p,q,r},$$

where $K_n^a(\mathcal{E},\mathcal{R})$ (resp. $K_n^r(\mathcal{E},\mathcal{R})$) consists of all elements in $K_n(\mathcal{E},\mathcal{R})$ whose first factor is in $\mathcal{A}$ (resp. $\mathcal{R}$), and

$$\mathcal{H}E_{G,0}^{p,q,a(r)}=\{\varphi\in\mathcal{H}E_{G,0}^{p,q}\colon \varphi\neq 0 \text{ only in } K_n^a(\mathcal{E},\mathcal{R}) \text{ (resp. } K_n^r(\mathcal{E},\mathcal{R}))\}.$$

The coboundary operator $d_{G,0}^{p,q}$ of $\mathcal{H}E_{G,0}^{p,q,a(r)}$ induced by b is dual to that in Section 3. We can also use $\mathcal{O}_i$ in Section 3 to define $\mathcal{H}T_G^*(\mathcal{O}_1)$ as the total complex of the $(l+1)$-tuple

$$\begin{aligned}\operatorname{Hom}_G(B_*(\mathcal{A})\tilde{\otimes}\tilde{B}_*(\mathcal{A})[n_2]\tilde{\otimes}\cdots\tilde{\otimes}\tilde{B}_*(\mathcal{A})[n_l]\\ \tilde{\otimes}\mathcal{R}^{\tilde{\otimes}|\mathcal{O}_1|}[-|\mathcal{O}_1|-1+n_{l+1}], C(G))\end{aligned}$$

for $(n_2,n_3,\dots,n_{l+1})\in\mathcal{O}_1$; for $(n_1,n_2,\dots,n_{l+1})\in\mathcal{O}_2$, define $\mathcal{H}T_G^*(\mathcal{O}_2)$ as the total complex of the $(l+1)$-tuple

$$\begin{aligned}\operatorname{Hom}_G(\tilde{B}_*(\mathcal{A})[n_1]\tilde{\otimes}\tilde{B}_*(\mathcal{A})[n_2]\tilde{\otimes}\cdots\tilde{\otimes}\tilde{B}_*(\mathcal{A})[n_l]\\ \tilde{\otimes}\mathcal{R}^{\tilde{\otimes}|\mathcal{O}_2|}[-|\mathcal{O}_2|-1+n_{l+1}], C(G)).\end{aligned}$$

Then, as in the proof of Theorem 2, for $p\geq 1$ we have

$$\{\mathcal{H}E_{G,0}^{p,*}\}\xleftarrow{F_1^*}\tilde{\bigoplus_{\substack{\mathcal{O}_1:|\mathcal{O}_1|=p\\ l(\mathcal{O}_1)\geq 1}}}\mathcal{H}T_G^*(\mathcal{O}_1)\tilde{\oplus}\tilde{\bigoplus_{\substack{\mathcal{O}_2:|\mathcal{O}_2|=p\\ l(\mathcal{O}_2)\geq 1}}}\mathcal{H}T_G^*(\mathcal{O}_2).$$

Here the isomorphism F_1^* is induced by the map F_1 in Section 3; that is,

$$\begin{aligned}(F_1^*f)(r_{(n_1)},a_{(k_1)},\dots,a_{(k_l)},r_{(n_{l+1})})\\ =f(a_{(k_1)},a_{(k_2)},\dots,a_{(k_l)},r_{(n_1)},r_{(n_2)},\dots,r_{(n_{l+1})}).\end{aligned}$$

It follows from this identification of $\mathcal{H}E_{G,0}^{*,*}$ that the E^1-terms of the spectral sequence are given for $p\geq 1$ by

$$\mathcal{H}E_{G,1}^{p,*}\xleftarrow{F_1^*}\tilde{\bigoplus_{\substack{\mathcal{O}_1:|\mathcal{O}_1|=p\\ l(\mathcal{O}_1)\geq 1}}}H^*(\mathcal{H}T_G^*(\mathcal{O}_1))\tilde{\oplus}\tilde{\bigoplus_{\substack{\mathcal{O}_2:|\mathcal{O}_2|=p\\ l(\mathcal{O}_2)\geq 1}}}H^*(\mathcal{H}T_G^*(\mathcal{O}_2)).\tag{20}$$

To summarize, we have the excision theorem of equivariant $\mathcal{H}$-cohomology.

THEOREM 4. *Let $\mathcal{A}$ be a G-algebra. Suppose for each G-space X that*

$$H^*(\operatorname{Hom}_G(B_*(\mathcal{A})\tilde{\otimes}X, C(G)), b')=0$$

and

$$H^*(\operatorname{Hom}_G(\tilde{B}_*(\mathcal{A})\tilde{\otimes}X, C(G)), \tilde{b}')=0$$

for $*\geq 1$. Then the inclusion $C_G^*(\mathcal{A})\xrightarrow{i^*}\mathcal{H}K_G^*$ induces an isomorphism in cohomology, and there is a long exact sequence of equivariant $\mathcal{H}$-cohomology associated with (14):

$$\cdots\to\mathcal{H}H_G^n(\mathcal{R})\xrightarrow{\pi^*}\mathcal{H}H_G^n(\mathcal{E})\xrightarrow{i^*}\mathcal{H}H_G^n(\mathcal{A})\xrightarrow{\partial}\mathcal{H}H_G^{n+1}(\mathcal{R})\to\cdots.\tag{21}$$

Proof. The inclusion $i^*\colon C_G^*(\mathcal{A})\to\mathcal{H}K_G^*$ means that $f\in C_G^*(\mathcal{A})$ extends trivially to $C_G^*(\mathcal{E})$. The proof of Theorem 1 shows that $H^*(\mathcal{H}T_G^*(\mathcal{O}_i))=0$ for

$*\geq 1$ by our assumption. It follows from (20) that $\mathcal{H}E_{G,1}^{p,*}=0$ for $p\geq 1$. Thus, the spectral sequence degenerates at the E^2-term and

$$\mathcal{H}H_G^*(\mathcal{A})=\mathcal{H}E_{G,2}^{0,*}\overset{i_*}{\cong}\mathcal{H}E_{G,\infty}^{0,*}\simeq H^*(\mathcal{H}K_G^*).$$

The long exact sequence (21) then follows from the short exact sequence (19). $\square$

Similarly, we can prove the excision theorem of Bar cohomology under the assumption of Theorem 4 by verifying the inclusion $i^*: B_G^*(\mathcal{A})\to BK_G^*=\mathrm{Hom}_G(K_{n-1}(\mathcal{E},\mathcal{R}),C(G))$ to be an isomorphism in cohomology. We leave the details to the reader.

THEOREM 5. *With the assumptions of Theorem 4, there are long exact sequences of equivariant Hochschild and cyclic cohomologies associated with* (14):

$$\text{(22)}\qquad \cdots\longrightarrow HH_G^n(\mathcal{R})\xrightarrow{\pi^*}HH_G^n(\mathcal{E})\xrightarrow{i^*}HH_G^n(\mathcal{A})\xrightarrow{\partial}HH_G^{n+1}(\mathcal{R})\longrightarrow\cdots;$$

$$\text{(23)}\qquad \ldots\longrightarrow HC_G^n(\mathcal{R})\xrightarrow{\pi^*}HC_G^n(\mathcal{E})\xrightarrow{i^*}HC_G^n(\mathcal{A})\xrightarrow{\partial}HC_G^{n+1}(\mathcal{R})\longrightarrow\cdots.$$

Proof. By the excision properties of equivariant Bar and $\mathcal{H}$-cohomologies, the Five Lemma, and the following commutative diagrams,

(24)

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & C_G^*(\mathcal{R}) \arrow[u] \arrow[r] & C_G^*(\mathcal{E}) \arrow[u] \arrow[r] & \mathcal{H}K_G^* \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Coker}(S,T^*(C_G^*(\mathcal{R}))) \arrow[u] \arrow[r] & \mathrm{Coker}(S,T^*(C_G^*(\mathcal{E}))) \arrow[u] \arrow[r] & \mathrm{Coker}(S,T^*(\mathcal{H}K_G^*)) \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & B_G^*(\mathcal{R}) \arrow[u] \arrow[r] & B_G^*(\mathcal{E}) \arrow[u] \arrow[r] & \mathcal{H}K_G^* \arrow[u] \arrow[r] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

and

$$\text{(25)}\qquad \begin{tikzcd}
0 \arrow[r] & BK_G^* \arrow[r] & \mathrm{Coker}(S,T^*(\mathcal{H}K_G^*)) \arrow[r] & \mathcal{H}K_G^* \arrow[r] & 0 \\
0 \arrow[r] & B_G^*(\mathcal{A}) \arrow[u] \arrow[r] & \mathrm{Coker}(S,T^*(C_G^*(\mathcal{A}))) \arrow[u] \arrow[r] & C_G^*(\mathcal{A}) \arrow[u] \arrow[r] & 0,
\end{tikzcd}$$

we get that the middle vertical inclusion in (25) induces an isomorphism in cohomology. (22) is then a consequence of the middle exact row in (24).

The proof of (23) is similar to that of (18) in Theorem 3. We omit the details. $\square$

COROLLARY 1. *With the assumptions of Theorem 4, the long exact sequence* (23) *in Theorem 5 commutes with the map S, and there is a six-term exact sequence of equivariant periodic cyclic cohomology*

$$
\begin{array}{ccccc}
PHC_G^{\mathrm{ev}}(\mathfrak{R}) & \xrightarrow{\pi^*} & PHC_G^{\mathrm{ev}}(\mathcal{E}) & \xrightarrow{i^*} & PHC_G^{\mathrm{ev}}(\mathfrak{A}) \\
\uparrow\partial & & & & \downarrow\partial \\
PHC_G^{\mathrm{odd}}(\mathfrak{A}) & \xleftarrow{i^*} & PHC_G^{\mathrm{odd}}(\mathcal{E}) & \xleftarrow{\pi^*} & PHC_G^{\mathrm{odd}}(\mathfrak{R}).
\end{array} \tag{26}
$$

Proof. Since the map S in the cochain level commutes with π^*, i^*, b, b', T, and N, it is clear that S commutes with all maps in (23). (26) then follows from (23) and the fact that the direct limit preserves the exactness. □

5. Mayer-Vietoris Sequences

An immediate application of Theorems 3 and 5 is the existence of Mayer-Vietoris sequences of equivariant Hochschild and cyclic (co-)homologies associated with equivarant Cartesian squares of G-algebras

$$
\begin{array}{ccc}
\mathfrak{A} & \xrightarrow{\phi_1} & \mathfrak{A}_1 \\
\downarrow\phi_2 & & \downarrow\psi_1 \\
\mathfrak{A}_2 & \xrightarrow{\psi_2} & \mathfrak{R};
\end{array} \tag{27}
$$

that is, $\mathfrak{A}=\{(a_1,a_2): a_i\in\mathfrak{A}_i,\ \psi_1(a_1)=\psi_2(a_2)\}$. ϕ_i and ψ_i are equivariant continuous homomorphisms such that either ψ_1 or ψ_2 is surjective. We fix ψ_2 to be surjective. Suppose that ψ_2 is also equivariant $\mathbf{C}$-split; that is, suppose

$$0\longrightarrow \mathrm{Ker}(\psi_2)\longrightarrow \mathfrak{A}_2\xrightarrow{\psi_2}\mathfrak{R}\longrightarrow 0$$

is an equivariant $\mathbf{C}$-split short exact sequence. Then we have the commutative diagram with equivariant $\mathbf{C}$-split short exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ker}(\phi_1) & \longrightarrow & \mathfrak{A} & \xrightarrow{\phi_1} & \mathfrak{A}_1 & \longrightarrow & 0 \\
 & & =\downarrow\phi_2 & & \downarrow\phi_2 & & \downarrow\psi_1 & & \\
0 & \longrightarrow & \mathrm{Ker}(\psi_2) & \longrightarrow & \mathfrak{A}_2 & \xrightarrow{\psi_2} & \mathfrak{R} & \longrightarrow & 0.
\end{array} \tag{28}
$$

THEOREM 6. *Let ψ_2 be surjective and equivariant $\mathbf{C}$-split in* (27).

(a) *If* $\mathrm{Ker}(\psi_2)$ *satisfies the conditions in Theorem 1, then the following sequences are exact:*

$$\cdots\longrightarrow HH_n^G(\mathfrak{A})\xrightarrow{(\phi_1)_*\oplus(\phi_2)_*}HH_n^G(\mathfrak{A}_1)\oplus HH_n^G(\mathfrak{A}_2)$$
$$\xrightarrow{(\psi_1)_*-(\psi_2)_*}HH_n^G(\mathfrak{R})\xrightarrow{\partial}HH_{n-1}^G(\mathfrak{A})\longrightarrow\cdots;$$

$$\cdots\longrightarrow HC_n^G(\mathfrak{A})\xrightarrow{(\phi_1)_*\oplus(\phi_2)_*}HC_n^G(\mathfrak{A}_1)\oplus HC_n^G(\mathfrak{A}_2)$$
$$\xrightarrow{(\psi_1)_*-(\psi_2)_*}HC_n^G(\mathfrak{R})\xrightarrow{\partial}HC_{n-1}^G(\mathfrak{A})\longrightarrow\cdots.$$

(b) *If* $\mathrm{Ker}(\psi_2)$ *satisfies the conditions in Theorem 4, then there are the following exact sequences associated with* (27):

$$\cdots\longrightarrow HH_G^n(\mathfrak{R})\xrightarrow{(\psi_1)^*\oplus(\psi_2)^*}HH_G^n(\mathfrak{A}_1)\oplus HH_G^n(\mathfrak{A}_2)$$
$$\xrightarrow{(\phi_1)^*-(\phi_2)^*}HH_G^n(\mathfrak{A})\xrightarrow{\partial}HH_G^{n+1}(\mathfrak{R})\longrightarrow\cdots;$$

$$\cdots \longrightarrow HC_G^n(\mathfrak{R}) \xrightarrow{(\psi_1)^* \oplus (\psi_2)^*} HC_G^n(\mathfrak{A}_1) \oplus HC_G^n(\mathfrak{A}_2) \xrightarrow{(\phi_1)^* - (\phi_2)^*} HC_G^n(\mathfrak{A}) \xrightarrow{\partial} HC_G^{n+1}(\mathfrak{R}) \longrightarrow \cdots;$$

and

$$\begin{array}{ccccc} PHC_G^{\mathrm{ev}}(\mathfrak{R}) & \xrightarrow{(\psi_1)^* \oplus (\psi_2)^*} & PHC_G^{\mathrm{ev}}(\mathfrak{A}_1) \oplus PHC_G^{\mathrm{ev}}(\mathfrak{A}_2) & \xrightarrow{(\phi_1)^* - (\phi_2)^*} & PHC_G^{\mathrm{ev}}(\mathfrak{A}) \\ \uparrow\partial & & & & \downarrow\partial \\ PHC_G^{\mathrm{odd}}(\mathfrak{A}) & \xleftarrow{(\phi_1)^* - (\phi_2)^*} & PHC_G^{\mathrm{odd}}(\mathfrak{A}_1) \oplus PHC_G^{\mathrm{odd}}(\mathfrak{A}_2) & \xleftarrow{(\psi_1)^* \oplus (\psi_2)^*} & PHC_G^{\mathrm{odd}}(\mathfrak{R}). \end{array}$$

Proof. The long exact sequence of Hochschild homology in (a) follows from the diagram chase and the following long exact sequences of Hochschild homology associated with (28):

$$\begin{array}{ccccccccc} \cdots \xrightarrow{(\phi_1)_*} & HH_n^G(\mathfrak{A}_1) & \xrightarrow{\partial} & HH_{n-1}^G(\mathrm{Ker}(\phi_1)) & \xrightarrow{I} & HH_{n-1}^G(\mathfrak{A}) & \xrightarrow{(\phi_1)_*} & HH_{n-1}^G(\mathfrak{A}_1) & \longrightarrow \cdot \\ & \downarrow(\psi_1)_* & & \cong \downarrow(\phi_2)_* & & \downarrow(\phi_2)_* & & \downarrow(\psi_1)_* & \\ \cdots \xrightarrow{(\psi_2)_*} & HH_n^G(\mathfrak{R}) & \xrightarrow{\partial} & HH_{n-1}^G(\mathrm{Ker}(\psi_2)) & \xrightarrow{I} & HH_{n-1}^G(\mathfrak{A}_2) & \xrightarrow{(\psi_2)_*} & HH_{n-1}^G(\mathfrak{R}) & \longrightarrow \cdot \end{array}$$

Similar arguments work for the other long exact sequences in (a) and (b). The six-term exact sequence in (b) is also obtained by the proof of Corollary 1. The details are left to the reader. □

6. Equivariant H-unitality

We have noted that the assumption in Theorem 1 is crucial in deriving the excision theorems of equivariant Hochschild and cyclic homologies. In this section we will verify that some algebras satisfy this assumption and hence are equivariant H-unital according to the following definition.

DEFINITION 3. Let $\mathfrak{A}$ be a G-algebra. $\mathfrak{A}$ is said to be *equivariant H-unital* if for any G-space X,

$$H_*(B_*^G(\mathfrak{A}, X), b') = 0 \quad \text{and} \quad H_*(\tilde{B}_*^G(\mathfrak{A}, X), \tilde{b}') = 0$$

for $* \geq 1$. Here the notation is as in Theorem 1. By Lemma 2, if $\mathfrak{A}$ is unital then $\mathfrak{A}$ is equivariant H-unital. The equivariant H-unitality is the generalization of H-unitality to the equivariant case [21]. As we will see in Theorem 7, the important examples of equivariant H-unital algebras are the G-algebras with uniformly bounded left (right) approximate identity (UBL(R)AI). Recall that a G-algebra $\mathfrak{A}$ has UBL(R)AI [17] if there is a left (right) approximate identity $\{e_\iota : \iota \in \mathfrak{I}\} \subset \mathfrak{A}$ such that

$$\sup_{1 \leq n \leq \infty} \{\rho_n(e_\iota) : \iota \in \mathfrak{I}\} = \nu_0 < \infty. \tag{29}$$

Every $\mathbf{C}^*$-algebra has UBL(R)AI. The useful fact of UBL(R)AI is that $\mathfrak{A}$ has left (right) sequence factorization property (L(R)SFP) [17], since $\mathfrak{A}$ is already Fréchet by our convention of G-algebras. This means that for any null sequence $\{x_n\} \subset \mathfrak{A}$ there exist $a \in \mathfrak{A}$ and null sequence $\{y_n\} \subset \mathfrak{A}$ such that

$ay_n = x_n$ ($y_n a = x_n$) and $y_n = \lim_{m\to\infty} a_m x_n$ uniformly for $n = 1, 2, \ldots$, where $a_m \in \mathfrak{A}$ [7]. In fact, we can say a little more about a and a_m for G-algebras. Let $\tilde{\mathfrak{A}}$ be the algebra obtained by adjoining an identity e to $\mathfrak{A}$. G acts on $\tilde{\mathfrak{A}}$ by $g(x+\lambda e) = g(x) + \lambda e$, and $\tilde{\mathfrak{A}}$ has seminorms $\{\tilde{\rho}_n\}$ where $\tilde{\rho}_n(x+\lambda e) = \rho_n(x) + |\lambda|$.

LEMMA 4. *Let $\mathfrak{A}$ be a G-algebra with UBL(R)AI. Suppose $\rho_n(\alpha_g(a)) \le \rho_n(a)$ for $g \in G$, $a \in \mathfrak{A}$, and $n \ge 1$. Then, with the above notation, $a_m \in \tilde{\mathfrak{A}}$, and a and a_m are equivariant.*

Proof. Let $\{e_\iota : \iota \in \mathcal{I}\}$ be the UBL(R)AI of $\mathfrak{A}$. Then $e_\iota x \to x$ uniformly on each compact subset of $\mathfrak{A}$ [17]. Let ν_0 be as in (29) and $\{x_n\} \subset \mathfrak{A}$ be a null sequence. The following is proved in [17]: Choose $0 < \lambda < (1+\nu_0)^{-1}$. Then there is a sequence $\{u_m\}_{m=0}^{\infty} \subset \mathfrak{A}$ such that

(a) $u_m - (1-\lambda)^m e \in \mathfrak{A}$ and $u_m^{-1} \in \tilde{\mathfrak{A}}$,
(b) $\tilde{\rho}_m(u_m - u_{m-1}) \le (1-\lambda)^{m-1}$, and
(c) $\rho_m((u_m^{-1} - u_{m-1}^{-1})x_n) \le 2^{-m}$ for all n.

Let $\tilde{u}_m = \int_G \alpha_g(u_m)\,dg$. Then all $\tilde{u}_m$ are equivariant and assertions (a)–(c) above hold also for $\{\tilde{u}_m\}$. It follows then that $\{\tilde{u}_m\}$ and $(\tilde{u}_m^{-1} x_n\}$ are Cauchy sequences in $\tilde{\mathfrak{A}}$ for all $n \ge 0$. Let

$$a = \lim_{m\to\infty} \tilde{u}_m \in \mathfrak{A} \quad \text{and} \quad y_n = \lim_{m\to\infty} \tilde{u}_m^{-1} x_n \in \mathfrak{A}.$$

The rest of the proof is clear. □

Let us from now on fix $\tilde{\otimes}$ to be the projective tensor product. Lemma 4 plays an important role in the proof of the following theorem.

THEOREM 7. (a) *Let $\mathfrak{A}$ be a G-algebra with L(R)SFP. Then, for any unital G-algebra $\mathfrak{B}$ and G-space X,*

$$H_n((\mathfrak{A}\tilde{\otimes}\mathfrak{B})^{\tilde{\otimes}(n)}\tilde{\otimes}X\tilde{\otimes}C(G), b') = 0 \tag{30}$$

and

$$H_n((\mathfrak{A}\tilde{\otimes}\mathfrak{B})^{\tilde{\otimes}(n)}\tilde{\otimes}X, \tilde{b}') = 0 \tag{31}$$

for $n \ge 1$.

(b) *Suppose that $\mathfrak{A}$ is a G-algebra with UBL(R)AI such that $\rho_n(\alpha_g(x)) \le \rho_n(x)$ for all $g \in G$ and $n \ge 1$. Then, for any unital G-algebra $\mathfrak{B}$ and G-space X,*

$$H_n(((\mathfrak{A}\tilde{\otimes}\mathfrak{B})^{\tilde{\otimes}(n)}\tilde{\otimes}X\tilde{\otimes}C(G))\tilde{\otimes}_{G,\rho}\mathbf{C}, b') = 0$$

and

$$H_n(((\mathfrak{A}\tilde{\otimes}\mathfrak{B})^{\tilde{\otimes}(n)}\tilde{\otimes}X)\tilde{\otimes}_{G,\rho}\mathbf{C}, \tilde{b}') = 0$$

for $n \ge 1$. In particular, $\mathfrak{A}$ is equivariant H-unital.

Proof. (a) Let $z \in (\mathfrak{A}\tilde{\otimes}\mathfrak{B})^{\tilde{\otimes}(n)}\tilde{\otimes}X$. Write z as [16]

$$z = \sum_{i=1}^{\infty} \lambda_i (a_1^i \tilde{\otimes} b_1^i)\tilde{\otimes}(a_2^i \tilde{\otimes} b_2^i)\tilde{\otimes}\cdots\tilde{\otimes}(a_n^i \tilde{\otimes} b_n^i)\tilde{\otimes}x^i)$$

for $a_j^i \in \mathcal{A}$, $b_j^i \in \mathcal{B}$, $x^i \in X$, and $\sum_{i=1}^{\infty}|\lambda_i| \leq 1$, where $\{a_j^i\}$, $\{b_j^i\}$, and $\{x_i\}$ are null sequences for $j = 1, 2, \ldots, n$. By the assumptions there are $a \in \mathcal{A}$ and a null sequence $\{y_1^i\} \subset \mathcal{A}$ such that $ay_1^i = a_1^i$ and $y_1^i = \lim_{m\to\infty} \tilde{a}_m a_1^i$, with $\tilde{a}_m \in \mathcal{A}$ uniformly for $1 \leq i \leq \infty$. Let

$$\tilde{z} = \sum_{i=1}^{\infty} \lambda_i((y_1^i \tilde{\otimes} b_1^i) \tilde{\otimes} (a_2^i \tilde{\otimes} b_2^i) \tilde{\otimes} \cdots \tilde{\otimes} (a_n^i \tilde{\otimes} b_n^i) \tilde{\otimes} x^i),$$

which is in $(\mathcal{A}\tilde{\otimes}\mathcal{B})^{\tilde{\otimes}(n)}\tilde{\otimes}X$. Using the continuity and $\mathcal{A}$-linearity of $\tilde{b}'$, we have $\tilde{b}'((a\tilde{\otimes}1)\tilde{\otimes}\tilde{z}) + (a\tilde{\otimes}1)\tilde{\otimes}\tilde{b}'(\tilde{z}) = z$. Thus, if $\tilde{b}'(z) = 0$, then from $\tilde{b}'(\tilde{z}) = \lim_{m\to\infty}(\tilde{a}_m\tilde{\otimes}1)\tilde{b}'(z) = 0$ we obtain $\tilde{b}'((a\tilde{\otimes}1)\tilde{\otimes}\tilde{z}) = z$. Hence (31) is verified.

To prove (30), let us define $\tilde{S}(\tilde{z})$ for z and $\tilde{z}$ as in the last paragraph with extra factor f^i by

$$\tilde{S}(\tilde{z})(g) = \sum_{i=1}^{\infty} \lambda_i((a\tilde{\otimes}1)\tilde{\otimes}a_g^{-1}(y_1^i\tilde{\otimes}b_1^i)\tilde{\otimes}(a_2^i\tilde{\otimes}b_2^i)\tilde{\otimes}\cdots\tilde{\otimes}(a_n^i\tilde{\otimes}b_n^i)\tilde{\otimes}x^i)f^i(g).$$

Then $b'\tilde{S}(\tilde{z}) + \tilde{S}(b'(\tilde{z})) = z$. Clearly, $b'(\tilde{z}) = 0$ if $b'(z) = 0$. Thus $b'\tilde{S}(\tilde{z}) = z$ is a coboundary. This implies (30).

Similarly, we can show (30) and (31) for G-algebra $\mathcal{A}$ with RSFP by considering z_1 and $\tilde{S}_1(z)$ instead of $\tilde{z}$ and $\tilde{S}(z)$, where

$$z_1 = \sum_{i=1}^{\infty} \lambda_i((a_1^i\tilde{\otimes}b_1^i)\tilde{\otimes}(a_2^i\tilde{\otimes}b_2^i)\tilde{\otimes}\cdots\tilde{\otimes}(y_n^i\tilde{\otimes}b_n^i)\tilde{\otimes}x^i)$$

for $a_n^i = y_n^i a$, and

$$\tilde{S}_1(z) = (-1)^{n-1}\sum_{i=1}^{\infty} \lambda_i((a_1^i\tilde{\otimes}b_1^i)\tilde{\otimes}(a_2^i\tilde{\otimes}b_2^i)\tilde{\otimes}\cdots\tilde{\otimes}(a_n^i\tilde{\otimes}b_n^i)\tilde{\otimes}(a\tilde{\otimes}1)\tilde{\otimes}x^i).$$

The rest of the proof is the same as above. This proves (a).

The proof of (b) is identical to that of (a) except for the verification of $\tilde{b}'[\tilde{z}] = 0$ if $\tilde{b}'[z] = 0$. But this follows from Lemma 4. □

We now turn to the assumptions of Theorem 4. First, when $G = \{e\}$, standard functional analysis can be used to obtain the following.

PROPOSITION 1. *Let $\mathcal{A}$ be a Fréchet algebra with $L(R)SFP$. Then, for any unital Fréchet algebra $\mathcal{B}$ and l.c.c. topological space X,*

$$H^n(\mathrm{Hom}((\mathcal{A}\tilde{\otimes}\mathcal{B})^{\tilde{\otimes}(n)}\tilde{\otimes}X), \tilde{b}') = 0 \quad \textit{for } n \geq 1.$$

In particular, $\mathcal{A}$ is H-unital.

Proof. Note that

$$\mathrm{Hom}((\mathcal{A}\tilde{\otimes}\mathcal{B})^{\tilde{\otimes}(n)}\tilde{\otimes}X) \xrightarrow{(\tilde{b}')^n} \mathrm{Hom}((\mathcal{A}\tilde{\otimes}\mathcal{B})^{\tilde{\otimes}(n+1)}\tilde{\otimes}X)$$
$$\xrightarrow{(\tilde{b}')^{n+1}} \mathrm{Hom}((\mathcal{A}\tilde{\otimes}\mathcal{B})^{\tilde{\otimes}(n+2)}\tilde{\otimes}X)$$

is the dual of the sequence

$$(\mathfrak{A}\tilde{\otimes}\mathfrak{B})^{\tilde{\otimes}(n+2)}\tilde{\otimes}X \xrightarrow{(\tilde{b}')_{n+1}} (\mathfrak{A}\tilde{\otimes}\mathfrak{B})^{\tilde{\otimes}(n+1)}\tilde{\otimes}X \xrightarrow{(\tilde{b}')_{n}} (\mathfrak{A}\tilde{\otimes}\mathfrak{B})^{\tilde{\otimes}(n)}\tilde{\otimes}X.$$

The latter sequence is exact by Theorem 7. It follows then that

$$\operatorname{Ker}((\tilde{b}')^{n+1}) = (\operatorname{Im}((\tilde{b}')_{n+1}))^{o} \quad \text{and} \quad \operatorname{Im}((\tilde{b}')^{n}) = (\operatorname{Ker}((\tilde{b}')_{n}))^{o}.$$

where $(Y)^o$ stands for the annihilator of set Y. Hence,

$$\operatorname{Ker}((\tilde{b}')^{n+1}) = (\operatorname{Im}((\tilde{b}')_{n+1}))^{o} = (\operatorname{Ker}((\tilde{b}')_{n}))^{o} = \operatorname{Im}((\tilde{b}')^{n}). \qquad \square$$

However, for a general compact group G, $\operatorname{Hom}_G((\mathfrak{A}\tilde{\otimes}\mathfrak{B})^{\tilde{\otimes}(n)}\tilde{\otimes}X, C(G))$ is no longer the dual of $(\mathfrak{A}\tilde{\otimes}\mathfrak{B})^{\tilde{\otimes}(n)}\tilde{\otimes}X\tilde{\otimes}C(G))\tilde{\otimes}_{G,\rho}C$. The situation in this case is quite different. We provide only a special result here.

PROPOSITION 2. *Let $\mathfrak{A}$ be a closed G-subalgebra of unital G-algebra $\mathfrak{A}_1$ such that $\mathfrak{A}$ is an ideal in $\mathfrak{A}_1$ and $\sum_{i=1}^{n} a^i b^i = 1$, with $a^i \in \mathfrak{A}$ and $b^i \in \mathfrak{A}_1$ being equivariant. Then, for any unital G-algebra $\mathfrak{B}$ and G-space X,*

$$H^n(\operatorname{Hom}_G((\mathfrak{A}\tilde{\otimes}\mathfrak{B})^{\tilde{\otimes}(n)}\tilde{\otimes}X, C(G)), \tilde{b}') = 0$$

and

$$H^n(\operatorname{Hom}_G((\mathfrak{A}\tilde{\otimes}\mathfrak{B})^{\tilde{\otimes}(n)}\tilde{\otimes}X, C(G)), b') = 0$$

for $n \geq 1$.

Proof. Let $f \in \operatorname{Hom}_G((\mathfrak{A}\tilde{\otimes}\mathfrak{B})^{\tilde{\otimes}(n)}\tilde{\otimes}X, C(G))$. Define

$$\begin{aligned} &S_2(f)(a_1\tilde{\otimes}b_1, a_2\tilde{\otimes}b_2, \ldots, a_{n-1}\tilde{\otimes}b_{n-1}, x) \\ &\quad = \sum_{i=1}^{m} f(a^i\tilde{\otimes}1, b^i a_1\tilde{\otimes}b_1, \ldots, a_{n-1}\tilde{\otimes}b_{n-1}, x) \end{aligned}$$

and

$$\begin{aligned} &S_2'(f)(a_1\tilde{\otimes}b_1, a_2\tilde{\otimes}b_2, \ldots, a_{n-1}\tilde{\otimes}b_{n-1}, x)(g) \\ &\quad = \sum_{i=1}^{m} f(a^i\tilde{\otimes}1, \alpha_g^{-1}(b^i a_1\tilde{\otimes}b_1), \ldots, a_{n-1}\tilde{\otimes}b_{n-1}, x)(g). \end{aligned}$$

It is easy to check that $b'S_2' + S_2'b' = \mathrm{Id}$ and $\tilde{b}'S_2 + S_2\tilde{b}' = \mathrm{Id}$. The result then follows from these identities. $\square$

One application of Proposition 2 is the case when G acts trivially on $\mathfrak{A}$. Let us point out finally that the equivariant cyclic cohomology defined in this paper can only be used for the equivariant index problem with character-valued index in $R(G)$. For distributive character-valued index [1] it is more convenient to define equivariant cyclic cohomology as the dual of equivariant cyclic homology, and then the excision problem of equivariant cyclic cohomology will be well understood.

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