Examples of Nonproper Affine Actions

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0. Introduction

Margulis ([Ma1], [Ma2]) was the first to show that complete affinely flat manifolds with free fundamental groups of rank ≥ 2 exist. These "Margulis space-times" correspond to free subgroups of the affine group that act properly discontinuously on $\mathbf{R}^3$. These subgroups of the affine group are conjugate to free subgroups of $\mathcal{V} \rtimes \mathbf{G} = \mathbf{H}$, where $\mathcal{V}$ is the group of parallel translations in $\mathbf{R}^3$ and $\mathbf{G} = SO^0(2,1)$ [FG].

Margulis also presented in [Ma1] and [Ma2] a test that identifies some free subgroups of $\mathbf{H}$ that do not act properly discontinuously on $\mathbf{R}^3$. Although Margulis's constraints are necessary to ensure that a free subgroup of $\mathbf{H}$ acts properly discontinuously on $\mathbf{R}^3$, it will be shown here that they are not sufficient. Margulis's proof that these constraints are necessary will also be presented here, to correct the many mistakes that are in the translation and to isolate and clarify the ideas in the proof as they appear in [Ma1] and [Ma2].

1. Geometry of $\mathbf{R}^{2,1}$

Consider subgroups of the affine group of the form $\Gamma = \langle h_1, h_2 \rangle \subset \mathbf{H}$. Γ acts on $\mathcal{E} = \mathbf{R}^{2,1}$ with its inner product $\mathbf{B}(x,y) = x_1 y_1 + x_2 y_2 - x_3 y_3$ invariant under the action of $\mathbf{G}$. For the null cone $C = \{x \in \mathcal{E} \mid \mathbf{B}(x,x) = 0\}$, let $W = \{x \in C \mid x_3 > 0\}$. Also, the Euclidean length of a vector is

$$\|x\| = ({x_1}^2 + x_2^2 + x_3^2)^{1/2}$$

and the Euclidean distance between two vectors is $\rho(x,y) = \|x-y\|$ or (more generally) the Euclidean distance between two sets

$$\rho(A,B) = \min\{\rho(a,b) \mid a \in A \text{ and } b \in B\}.$$

Note that if z is the vector $\{x_1, x_2, -x_3)$ then $\mathbf{B}(x,y)$ is equal to the Euclidean inner product of z and y and the Lorentzian Schwartz inequality, $\mathbf{B}(x,y) \leq \|z\| \|y\| = \|x\| \|y\|$, holds.

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Let $l: \mathbf{H} \to \mathbf{G}$ be the projection of $\mathbf{H}$ onto the linear part of elements of $\mathbf{H}$. If a free group Γ acts properly discontinuously on $\mathcal{E}$, then $l(\Gamma)$ is free. l is injective, for otherwise the kernel would be an abelian normal subgroup. The subgroups of $\mathbf{G}$ that will be of interest are purely hyperbolic, that is, those in which every nonidentity element of the group is hyperbolic.

g in $\mathbf{G}$ is hyperbolic if it has three distinct positive eigenvalues, $\lambda(g) < 1 < \lambda(g)^{-1}$. Corresponding repelling, fixed, and attracting eigenvectors x_g^-, x_g^0, and x_g^+, respectively, are defined so that x_g^- and x_g^+ are in $W \cap S^2$, where S^2 is the Euclidean unit sphere and $\mathbf{B}(x_g^0, x_g^0) = 1$ with $\{x_g^0, x_g^-, x_g^+\}$ a right-handed basis for $\mathcal{E}$. Further, g is said to be ϵ-hyperbolic if $\rho(x_g^-, x_g^+) > \epsilon$, and hyperbolic g and f are said to be ϵ-transversal if $\rho(\{x_g^-, x_g^+\}, \{x_f^-, x_f^+\}) > \epsilon$ for $0 < \epsilon < 2^{1/2}$.

v in $\mathcal{E}$ is said to be spacelike if $\mathbf{B}(v, v) > 0$. For spacelike v, define unique vectors x_v^+ and x_v^- in $W \cap S^2$ so that $\mathbf{B}(v, x_v^{\pm}) = 0$ and $\{v, x_v^-, x_v^+\}$ is a right-handed basis for $\mathcal{E}$. v is said to be an ϵ-spacelike vector if $\rho(x_v^-, x_v^+) > \epsilon$. Note that if $v = x_h^0$ then $x_v^{\pm} = x_h^{\pm}$.

h in $\mathbf{H}$ is hyperbolic only if $l(h) = g$ is hyperbolic. Let $x_h^0 = x_g^0$ and $x_h^{\pm} = x_g^{\pm}$, and let C_h be the unique h-invariant 1-dimensional subspace that is parallel to x_h^0. For x in C_h, let $d_h = hx - x$ and $\alpha(h) = \langle d_h, x_h^0 \rangle$. Define the sign of h to be the sign of $\alpha(h)$. Since $\mathbf{B}(x_h^0, x_h^-) = \mathbf{B}(x_h^0, x_h^+) = 0$, $\alpha(h) = \mathbf{B}(hx - x, x_h^0)$ for each x in $\mathcal{E}$. Also, define $E_h^{\pm}$ to be the plane containing C_h that is parallel to $\langle x_h^0, x_h^{\pm} \rangle$, and define $B_h^{\pm}(v)$ to be the line that contains v and is parallel to $x_h^{\pm}$.

If Γ acts properly discontinuously on $\mathcal{E}$, define π to be the projection from $\mathcal{E}$ onto $\mathcal{E}/\Gamma$. $\pi(C_h)$ is seen to be the simple closed geodesic in $\mathcal{E}/\Gamma$. The direction and "Lorentzian length" of this geodesic are given by $\mathbf{B}(d_h, x_h^0) = \alpha(h)$. In addition, $\alpha(h)$ is related to $\min_x \rho(hx, x)$ for x in $\mathcal{E}$. Let P_x be the plane parallel to x_h^-, x_h^+ and containing x. Note that $h(P_x) = P_{hx}$ and that $\rho(hx, x) \geq \rho(P_x, P_{hx})$. Let $y = P_x \cap C_h$, which may be chosen to be the origin modulo conjugation of h by a translation. Choose $x_h^- = \beta[\cos\theta, \sin\theta, 1]$, $x_h^+ = \beta[\cos\theta, -\sin\theta, 1]$, and $x_h^0 = [\csc\theta, 0, \cot\theta]$, with $\beta = 2^{-1/2}$ and $0 < \theta < \pi$. For ϵ-hyperbolic h, $2^{1/2}\sin\theta > \epsilon$, and by elementary plane geometry we have

$$\begin{aligned} \rho(hx, x) \geq \rho(P_x, P_{hx}) &= \left| \frac{\alpha(h)\sin\theta}{(1+\cos^2\theta)^{1/2}} \right| \\ &\geq |\alpha(h)| \frac{\epsilon}{2}. \end{aligned} \tag{1}$$

Define a "conical neighborhood" $A \subset W$ of v in W to be an open connected subset containing v, such that if $w \in A$ then $kw \in A$ for all $k > 0$. Conical neighborhoods A and B in W are ϵ-separated if $\rho(a/\|a\|, b/\|b\|) > \epsilon$ for all $a \in A$ and $b \in B$. The closure of the neighborhood A in the usual topology is denoted $\mathrm{cl}(A)$.

For hyperbolic g_i, conical neighborhoods $A_i^{\pm}$ of $x_{g_i}^{\pm}$, respectively, can be chosen so that $\mathrm{cl}(g_i(A_i^-)) = (W - A_i^+)$. Note that for $g_k = g_i^3$, A_k^+ and A_k^- can be chosen to be $g_i(A_i^+)$ and $g_i^{-1}(A_i^-)$, respectively. A group in $\mathbf{G}$ generated by hyperbolic elements $g_1, g_2, \ldots, g_n$ acts as a *Schottky group* on W if there exist $2n$ conical neighborhoods $A_i^{\mp}$, as above, which are mutually distinct.

2. Generators with Opposite Signs

THEOREM 1. *Given* 3ϵ*-hyperbolic and* 3ϵ*-transversal* h_1 *and* h_2 *in* $\mathbf{H}$ *with opposite signs,* $\Gamma = \langle h_1, h_2 \rangle$ *does not act properly discontinuously on* $\mathcal{E}$.

To prove Theorem 1, inequalities from Margulis's Basic Lemma ([Ma1], [Ma2]) will be established in Lemmas 1 and 2.

LEMMA 1. *For* 3ϵ*-hyperbolic and* 3ϵ*-transversal* $h, k \in \mathbf{H}$ *such that* hk *is* ϵ*-hyperbolic,*

$$(2)\qquad |\alpha(hk) - \alpha(h) - \alpha(k)| < \frac{4}{\epsilon^3}[\rho(C(h), C(k))\|x_{hk}^0\| + (\|d_h\|\|x_{hk}^+ - x_h^+\| + \|d_k\|\|x_{hk}^- - x_k^-\|)].$$

Proof. First, note that

$$\mathbf{B}(d_h + d_k, x_{hk}^0) = \mathbf{B}(d_h, x_{hk}^0) + \mathbf{B}(d_k, x_{hk}^0)$$

due to the bilinearity of the inner product. The left side of (2) can then be written as

$$(3)\qquad |(\mathbf{B}(d_h, x_{hk}^0) - \alpha(h)) + (\mathbf{B}(d_k, x_{hk}^0) - \alpha(k)) + (\alpha(hk) - \mathbf{B}(d_h + d_k, x_{hk}^0)|.$$

Interestingly, $\mathbf{B}(d_h, x_h^+) = 0$ and $\alpha(h) = \mathbf{B}(d_h, B_h^+(x_h^0))$, since each element of $B_h^+(x_h^0)$ can be expressed as the sum of x_h^0 and a real multiple of x_h^+. (3) is then bounded above by

$$(4)\qquad |\mathbf{B}(d_h, x_{hk}^0 - B_h^+(x_h^0))| + |\mathbf{B}(d_k, x_{hk}^0 - B_k^-(x_k^0))| + |\alpha(hk) - \mathbf{B}(d_h + d_k, x_{hk}^0)|.$$

Of the first two terms in (4), which are similar, only the first will be examined. By the Schwartz inequality, the first term in (4) is bounded above by $\|d_h\|\|x_{hk}^0 - B_h^+(x_h^0)\|$. If r is the x_3-coordinate of x_{hk}^0 and

$$s = B_h^+(x_h^+) \cap \{x \mid x_3 = r\},$$

then

$$\|x_{hk}^0 - s\| \geq \|x_{hk}^0 - B_h^+(x_h^0)\|.$$

Note that x_{hk}^0 and s belong to the circle defined by $\{x \mid x_3 = r\} \cap \{x \mid \mathbf{B}(x,x) = 1\}$ with radius $(r^2+1)^{1/2}$. The triangles

$$\{x_{hk}^0, s, (0,0,r)\} \quad \text{and} \quad \{x_{hk}^+, x_s^+, (0,0,2^{-1/2})\}$$

are similar and

$$(5)\qquad \|x_{hk}^0 - s\| = [(r^2+1)/2]^{1/2}\|x_{hk}^+ - x_s^+\| = [(r^2+1)/2]^{1/2}\|x_{hk}^+ - x_h^+\|.$$

The x_3-coordinate of x_{hk}^0 is bounded above by $2/\epsilon$. Thus

$$(6)\qquad |\mathbf{B}(d_h, x_{hk}^0 - B_h^+(x_h^0))| < 4/\epsilon^3\|d_h\|\|x_{hk}^+ - x_h^+\| \quad \text{and}$$

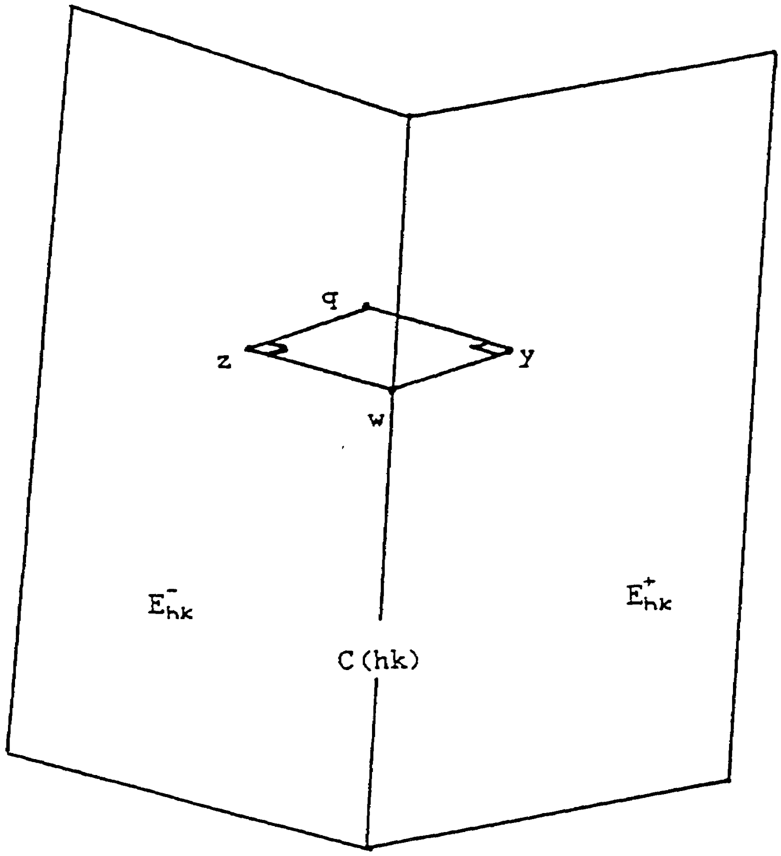

Figure 1

$$|\mathbf{B}(d_k, x^0_{hk} - B^-_k(x^-_k))| < 4/\epsilon^3 \|d_k\| \|x^-_{hk} - x^-_k\|.$$

The third term of (4) can be controlled using $u \in C_h$ and $z \in C_k$ chosen so that $\rho(C_h, C_k) = \rho(u, z)$, $v = B^-_h(u) \cap E^+_k$, and $w = B^+_k(v) \cap C(k)$. See Figure 1. Noting that $\alpha(hk)$ can be written as $\mathbf{B}(hv - k^{-1}v, x^0_{hk})$, the third term of (4) is seen to be equal to

(7) $$|\mathbf{B}((hv - v) + (v + k^{-1}v) - d_h - d_k, x^0_{hk})|.$$

v can be regarded as $u + r$ or $w + t$, where r is a multiple of x^-_h with norm $\rho(u, v)$ and t is a multiple of x^+_k with norm $\rho(v, w)$. (7) is bounded above by $(\rho(u, v) + \rho(v, w))\|x^0_{hk}\|$. By the law of sines applied to the triangle with vertices u, v, and z, $\rho(u, v) \leq \rho(u, z)/\sin(\angle(u - v, z - v)) \leq 2\rho(u, z)/\epsilon^3$. For the triangle with vertices u, v, and w, $\rho(v, w) \leq \rho(v, z)/\sin(\angle(v - u, z - u)) \leq \rho(v, z)/\epsilon$, which can be bounded above by $2\rho(u, z)/\epsilon^3$ by use of the triangle inequality for the triangle with vertices u, v, and z. Thus we have established

(8) $$|\alpha(hk) - \mathbf{B}(d_h + d_k, x^0_{hk})| < 4\rho(C_h, C_k)\|x^0_{hk}\|/\epsilon^3.$$

The application of (6) and (8) to (4) results in (2). □

The following proposition will be used in the proof of Lemma 2.

PROPOSITION. *For each p in $\mathcal{E}$ and ϵ-hyperbolic h in $\mathbf{H}$,*

(9) $$\rho(p, E^+_h) \leq \frac{\lambda(h)}{1 - \lambda(h)} \rho(h^{-1}p, p).$$

Proof. For each p in $\mathcal{E}$ let s in E_h^+ be chosen so that $p-s$ is parallel to x_h^-. There exists a constant k, $0<k<1$, such that $\rho(p,E_h^+)=k\cdot\rho(p,s)$ for all p in $\mathcal{E}$, and $\rho(h^{-1}p,E_h^+)=\lambda(h)^{-1}\rho(p,E_h^+)$. By the triangle inequality, $\rho(h^{-1}p,E_h^+)\leq\rho(h^{-1}p,p)+\rho(p,E_h^+)$ and

$$\lambda(h)^{-1}\rho(p,E_h^+)\leq\rho(h^{-1}p,p)+\rho(p,E_h^+). \tag{10}$$

(10) can then rewritten in the form of (9). □

LEMMA 2. *For ϵ-hyperbolic h in* **H** *and for each q in $\mathcal{E}$,*

$$\rho(q,C_h)<\frac{2}{\epsilon}\frac{\lambda(h)}{1-\lambda(h)}\min\{\rho(q,h(q)),\rho(q,h^{-1}(q))\}. \tag{11}$$

Proof. First, we will estimate $\rho(q,C(h))$ by examining the plane Q which contains q and is perpendicular to $C_h=E_h^+\cap E_h^-$. Let $\{w\}=Q\cap C_h$, y in $Q\cap E_h^+$, and z in $Q\cap E_h^-$ be chosen so that $\rho(q,C_h)=\rho(q,w)$, $\rho(q,E_h^+)=\rho(q,y)$, and $\rho(q,E_h^-)=\rho(q,z)$. See Figure 2.

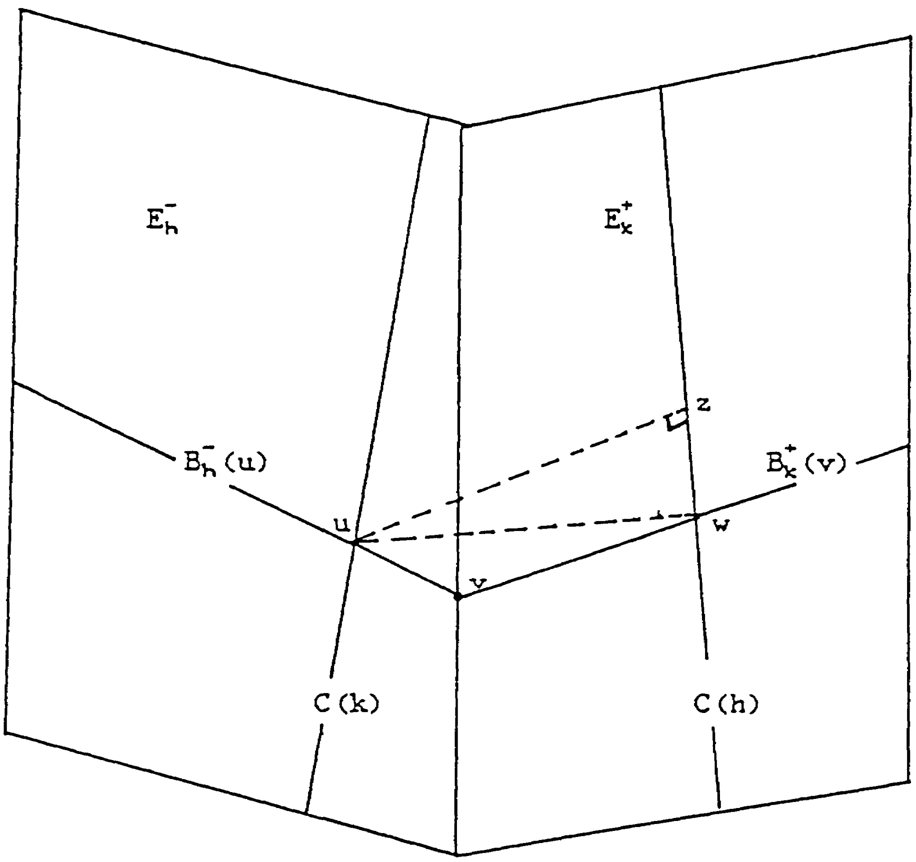

Figure 2

Consider the right triangles having vertices (q,y,w) and (q,z,w). Because h is assumed to be ϵ-hyperbolic, $\max(\angle(z-w,q-w),\angle(q-w,y-w))>\epsilon/2$ since $\angle(z-w,y-w)>\epsilon$. $\rho(q,w)$, the length of the hypotenuse of both triangles, is equal to $\rho(q,y)/\sin(\angle(q-w,y-w))$ and $\rho(q,z)/\sin(\angle(q-w,z-w))$, with an upper bound of either $\pi\rho(q,y)/\epsilon$ or $\pi\rho(q,z)/\epsilon$ since $\sin\theta\geq 2\theta/\pi$ for $0\leq\theta\leq\pi/2$;

$$\begin{aligned}\rho(q, C_h) &= \rho(q, w)\\ &< 2(\rho(q,y)+\rho(q,z))/\epsilon \qquad (12)\\ &= 2(\rho(q, E_h^+)+\rho(q, E_h^-))/\epsilon.\end{aligned}$$

Apply the Proposition to (12), and (11) is proven since $\lambda(h)=\lambda(h^{-1})$. □

Proof of Theorem 1. It suffices to construct a compact set K so that there are infinitely many elements $\mathfrak{h}_i$ in Γ of the form $h_1^n h_2^m$ with $\mathfrak{h}_i(K)\cap K \neq \emptyset$. Suppose that $\alpha(h_1)>0$ and $\alpha(h_2)<0$. By replacing h_1 and h_2 with sufficiently large powers of h_1 and h_2, respectively, one may assume that the corresponding conical subsets $A_h^\pm$ and $A_k^\pm$ of W, as described in Section 1, are separated by ϵ. By Lemma 2.2 of [DG], the $\mathfrak{h}_i$'s are ϵ-hyperbolic and $\|x^0(\mathfrak{h}_i)\|<2/\epsilon$.

To construct the set K, first find upper bounds for $\alpha(\mathfrak{h}_i)$ and $\rho(q, C_{\mathfrak{h}_i})$ for a chosen q in $\mathcal{E}$. For x in $C_{\mathfrak{h}_i}$, $\rho(\mathfrak{h}_i x, x)<2\alpha(\mathfrak{h}_i)/\epsilon$. If $L=\sup_i(\alpha(\mathfrak{h}_i))$ and $M=\sup_i(\rho(q, C_{\mathfrak{h}_i}))$, then $K=\{x\in\mathcal{E} \mid \rho(q,x)\le M+2L/\epsilon\}$ is constructed to be a ball with center q so that $\mathfrak{h}_i(q)$ is in K for all i. K is compact if both L and M are finite.

In order to construct a sequence of $\mathfrak{h}_i$'s for which L is bounded, examine

$$|\alpha(\mathfrak{h}_i)-\alpha(h_1^n)-\alpha(h_2^m)|. \tag{13}$$

$\alpha(h_1^n)+\alpha(h_2^m)=n\alpha(h_1)+m\alpha(h_2)$, and if (13) is bounded above for all $\mathfrak{h}_i = h_1^{n_i}h_2^{m_i}$ where $n_i, m_i>0$ and $0<n_i\alpha(h_1)+m_i\alpha(h_2)<k$, then the set of $\alpha(\mathfrak{h}_i)$'s is bounded. This sequence is infinite since h_1 and h_2 have different signs.

(13) is the left side of inequality (2). The first term on the right side of inequality (2), $\rho(C_h, C_k)$, is independent of i since $C_{h^n}=C_h$.

The second and third terms on the right side of inequality (2) are similar and only the second will be examined. For $\mathfrak{h}=h^n k^m$, this term is

$$\|d_{h^n}\|\|x^+_{h^nk^m}-x^+_{h^n}\| = n\|d_h\|\|x^+_{\mathfrak{h}}-x^+_h\|. \tag{14}$$

For a particular v in W, $\rho(x_h^+, h^n v/\|h^n v\|)$ provides an upper bound for $\|x_{\mathfrak{h}}^+-x_h^+\|$. $\rho(x_h^+, h^n v/\|h^n v\|)\le C\lambda(h)^n$ for some $C>0$. Thus, (14) is bounded for any sequence of $\mathfrak{h}_i$'s and the same can be said for $\alpha(\mathfrak{h}_i)$, for each term in the sum is bounded from above.

For the sequence of $\mathfrak{h}_i$'s described above, we will show that $\rho(q, C_{\mathfrak{h}_i})$ is bounded for all i. Examining (11),

$$\min\{\rho(q,h(q)), \rho(q,h^{-1}(q))\}\le[\rho(q,h(q))+\rho(q,h^{-1}(q))]/2$$

and $1/(1-\lambda(\mathfrak{h}_i))\le 1/(1-\lambda(h))$ so that the terms in (11) are bounded above by a constant times $\lambda(h^nk^m)(\rho(q,h^nk^m(q))+\rho(q,(h^nk^m)^{-1}(q)))$. For convenience, choose q in $E_h^+\cap E_k^-$ and let $p_1=k^m(q)$ and $p_2=h^{-n}(q)$. $\rho(q, C_{\mathfrak{h}_i})$ is bounded above by a constant times

$$\lambda(h^nk^m)[\rho(q,p_1)+\rho(p_1,h^n(p_1))+\rho(q,p_2)+\rho(p_2,k^{-m}(p_2))]. \tag{15}$$

In the following discussion about (15), the C_i's denote positive constants depending only on ϵ. That is, they depend upon h and k, but do not depend upon n and m.

q was chosen to be in E_k^- so that $\rho(q, p_1) \leq mC_1$ for some C_1, and q was chosen to be in E_h^+ so that $\rho(q, p_2) \leq nC_2$ for some C_2. Further, $\rho(p_1, h^n(p_1)) \leq C_3 m\lambda(h)^{-n}$ for some C_3 and $\rho(p_2, k^{-m}(p_2)) \leq C_4 n\lambda(k)^{-m}$ for some C_4. But

$$\|h^n k^m(x_k^+)\| = \lambda(k)^m \|h^n(x_k^+)\| \leq C_5 \lambda(h)^n \lambda(k)^m$$

for some C_5, and $\lambda(h^n k^m) \leq C_5 \lambda(h)^n \lambda(k)^m$. Thus, the terms in (11) are bounded above for all $\mathfrak{h}_i$. □

3. Generators of the Same Sign

It is interesting to note that the converse of Theorem 1 is not true.

THEOREM 2. *There exist $\Gamma = \langle h_1, h_2 \rangle$, h_1 and h_2 in $\mathbf{H}$ with the same sign, and $l(\Gamma)$ acting on W as a Schottky group such that Γ does not act properly discontinuously on $\mathcal{E}$.*

Proof. It suffices to construct h_1 and h_2 with positive sign such that $h_1 h_2$ has negative sign. Then by Margulis's sign condition, Γ would not act properly discontinuously on $\mathcal{E}$.

Choose g_1 and g_2 so that $\langle g_1, g_2 \rangle$ acts as a Schottky group on W, the line $X(g_1, g_2) = \langle x_{g_1}^+, x_{g_1}^- \rangle \cap \langle x_{g_2}^+, x_{g_2}^- \rangle$ lies inside of C, and $\mathbf{B}(x_{g_1}^+, x_{g_2}^0) > 0$. Choose v_1 so that $\mathbf{B}(v_1, x_{g_1}^0) > 0$ and $\|v_1\| = 1$. Let $h_1(x) = g_1 x + v_1$ and then $\alpha(h_1) > 0$.

Choose $w_2 \in (W - \{A_1^+ \cup A_1^- \cup A_2^+ \cup A_2^-\}) \cap S^2$ so that $\mathbf{B}(w_2, x_{g_2}^0) > 0$ and $\{w_2, x_{g_2}^-, x_{g_1}^+\}$ is a left-handed basis for $\mathcal{E}$. See Figure 3.

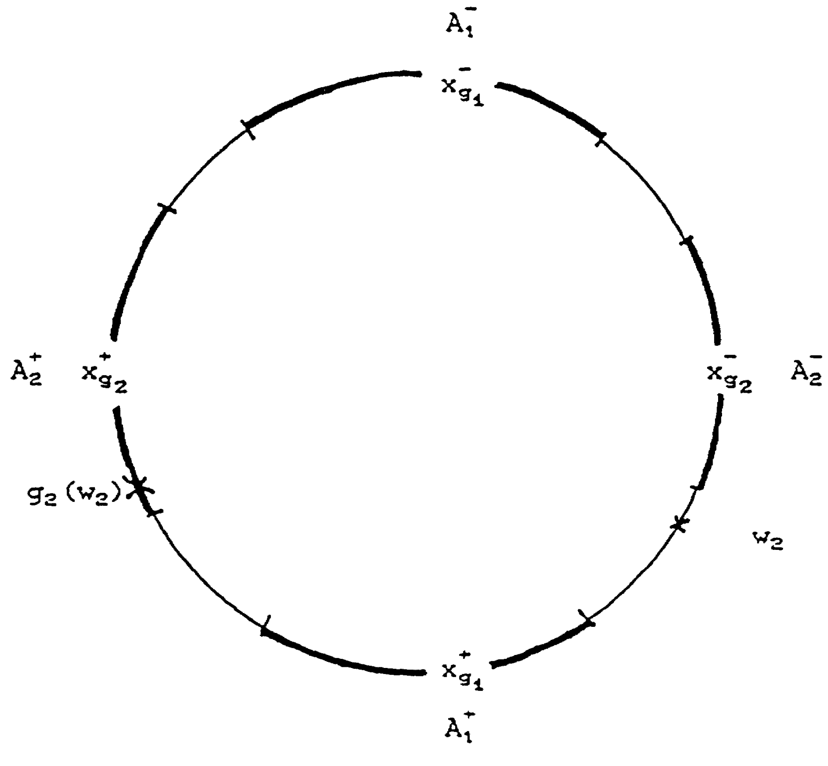

Figure 3

$$\mathbf{B}(w_2, x^0_{g_2}) = \mathbf{B}(g_2 w_2, g_2 x^0_{g_2}) = \mathbf{B}(g_2 w_2, x^0_{g_2}).$$

For any $k>0$, if $v_2 = kg_2 w_2$ and $h_2(x) = g_2 x + v_2$ then $\alpha(h_2) > 0$.

Note that

$$(h_1 h_2)(x) = (g_1 g_2)x + (g_1 v_2 + v_1)$$

and

$$\alpha(h_1 h_2) = \mathbf{B}(g_1 v_2 + v_1, x^0_{g_1 g_2}) = \mathbf{B}(g_1 v_2, x^0_{g_1 g_2}) + \mathbf{B}(v_1, x^0_{g_1 g_2}).$$

$\|v_1\| = 1$ and $\|x^0_{g_1 g_2}\|$ is bounded since $\langle g_1, g_2\rangle$ acts as a Schottky group on W. Thus $|\mathbf{B}(v_1, x^0_{g_1 g_2})|$ is bounded. $(g_1 g_2)^{-1}(A_2^-) \subset A_2^-$ and $(g_1 g_2)(A_1^+) \subset A_1^+$ so that $x^-_{g_1 g_2} \in A_2^-$ and $x^+_{g_1 g_2} \in (g_1 g_2)(A_1^+)$. The choice of w_2 ensures that $\mathbf{B}(g_1 g_2(w_2), x^0_{g_1 g_2}) < 0$. k can be chosen so that $k|\mathbf{B}(g_1 g_2(w_2), x^0_{g_1 g_2})| > |\mathbf{B}(v_1, x^0_{g_1 g_2})|$ and $\alpha(h_1 h_2) < 0$. □

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