On the Lebesgue Test for the Convergence of Vilenkin–Fourier Series

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1. Introduction

In the 1970's Onneweer and Waterman obtained analogues, in the context of bounded Vilenkin groups, for the Salem test [1] and the Lebesgue test [2] for the convergence of Fourier series. Recently, by localizing the Salem test, Waterman found a new criterion for the pointwise convergence of Fourier series [3]. Here we adapt Waterman's localization of the Salem test to obtain an extension of the Onneweer–Waterman Lebesgue test for convergence of Fourier series of functions defined on a bounded Vilenkin group.

2. Notation and Terminology

By a *Vilenkin group* G we mean a compact, 0-dimensional, metrizable abelian group. G contains a fundamental system $\{G_k\}$ of neighborhoods of 0 such that

(i) $G=G_0\supset G_1\supset\cdots\supset G_k\supset G_{k+1}\supset\cdots\supset\{0\}$;
(ii) the quotient G_{k-1}/G_k is of prime order p_k;
(iii) $\{0\}=\bigcap_{k=0}^{\infty} G_k$.

If the sequence of primes $\{p_k\}$ is a bounded sequence, we say that G is a *bounded Vilenkin group*; otherwise we say that G is *unbounded.* If $p_k=2$ for every k, then we will call G the *Walsh group* and denote G by 2^{ω}.

Set $m_0=1$ and $m_i=\prod_{\nu=1}^{i} p_\nu$. Each $n\in\mathbf{Z}^+$ has a unique representation $n=\sum_{i=0}^{s} a_i m_i$ with $0\le a_i<p_{i+1}$ for $0\le i\le s$. Denoting the dual of G by X, we may enumerate the elements χ_n of X in such a way that

$$\chi_n=\chi_{m_0}^{a_0}\cdot\chi_{m_1}^{a_1}\cdot\cdots\cdot\chi_{m_s}^{a_s}.$$

For each $k\in\mathbf{Z}^+$ there is an $x_k\in G_k\backslash G_{k+1}$ such that $\chi_{m_k}(x_k)=e^{2\pi i/p_{k+1}}=\zeta_k$. Each $x\in G$ has a unique representation $x=\sum_{i=0}^{\infty} b_i x_i$ with $0\le b_i<p_{i+1}$. We can enumerate the cosets of G_k in G by means of the lexicographic ordering of the coset representatives of the form $z=\sum_{i=0}^{k-1} b_i x_i$ $(0\le b_i<p_{i+1})$. In particular, we shall let $z_\alpha^{(k)}$ be that coset representative z for which $\alpha=\sum_{i=0}^{k-1} b_i(m_k/m_{i+1})$.

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As a compact abelian group, G has a normalized Haar measure, which is indicated by dx or dt in integrals. For $f \in L^1(G)$, the *Fourier series of* f is the series $S(f;x) = \sum_{i=0}^{\infty} c_i \chi_i(x)$, where the coefficients c_i are determined by the formula $c_i = \int_G f(t)\bar{\chi}_i(t)\,dt$. The partial sums of the Fourier series we denote by $S_n(f;x)$. Now

$$S_n(f;x) = \sum_{i=0}^{n-1} c_i \chi_i(x) = \int_G f(x-t) D_n(t)\,dt,$$

where the *Dirichlet kernel of* nth *order* is defined by $D_n(t) = \sum_{i=0}^{n-1} \chi_i(t)$. Standard properties of the Dirichlet kernel are listed in [1] and [2]. Additionally we present the following modification of a result in [1].

2.1. LEMMA. *Let G be a Vilenkin group, bounded or unbounded. If $n = a_k m_k + n'$, where a_k and n' are integers such that $1 \le a_k < p_{k+1}$ and $0 \le n' < m_k$, and if $z_\alpha^{(k)} \in G_\lambda \backslash G_{\lambda+1}$, then*

$$|D_{n'}(z_\alpha^{(k)})| \le p_{\lambda+1} \cdot \frac{m_k}{\alpha}.$$

In particular, if G is a bounded Vilenkin group with $p_i \le p$ for all i, then

$$|D_{n'}(z_\alpha^{(k)})| \le p \cdot \frac{m_k}{\alpha}.$$

2.2. *Proof.* If $z_\alpha^{(k)} \in G_\lambda \backslash G_{\lambda+1}$, then $z_\alpha^{(k)} = b_\lambda x_\lambda + \cdots + b_{k-1} x_{k-1}$ $(b_\lambda \ne 0)$. Then $\alpha = \sum_{i=\lambda}^{k-1} b_i (m_k/m_{i+1})$ and $m_k/m_{\lambda+1} \le \alpha < m_k/m_\lambda$. The Dirichlet kernel $D_{n'}(t)$ may be written

$$D_{n'}(t) = \sum_{i=0}^{k-1} \frac{\chi_{n'}(t)}{\chi_{m_i}^{a_i}(t)} D_{m_i}(t) \frac{1-\chi_{m_i}^{a_i}(t)}{1-\chi_{m_i}(t)}.$$

Now if $z_\alpha^{(k)} \in G_\lambda \backslash G_{\lambda+1}$, we have

$$D_{m_i}(z_\alpha^{(k)}) = \begin{cases} m_i & i < \lambda+1, \\ 0 & i \ge \lambda+1. \end{cases}$$

Consequently,

$$\begin{aligned} D_{n'}(z_\alpha^{(k)}) &= \sum_{i=0}^{\lambda} \frac{\chi_{n'}(z_\alpha^{(k)})}{\chi_{m_i}^{a_i}(z_\alpha^{(k)})} D_{m_i}(z_\alpha^{(k)}) \frac{1-\chi_{m_i}^{a_i}(z_\alpha^{(k)})}{1-\chi_{m_i}(z_\alpha^{(k)})} \\ &= \sum_{i=0}^{\lambda} \frac{\chi_{n'}(z_\alpha^{(k)})}{\chi_{m_i}^{a_i}(z_\alpha^{(k)})} m_i \frac{1-\chi_{m_i}^{a_i}(z_\alpha^{(k)})}{1-\chi_{m_i}(z_\alpha^{(k)})}. \end{aligned}$$

Moreover, if $z_\alpha^{(k)} \in G_{i+1}$ (i.e., if $i < \lambda$) then $\chi_{m_i}(z_\alpha^{(k)}) = 1$. We may write

$$D_{n'}(z_\alpha^{(k)}) = \sum_{i=0}^{\lambda-1} \chi_{n'}(z_\alpha^{(k)}) \cdot a_i m_i + \frac{\chi_{n'}(z_\alpha^{(k)})}{\chi_{m_\lambda}^{a_\lambda}(z_\alpha^{(k)})} m_\lambda \frac{1-\chi_{m_\lambda}^{a_\lambda}(z_\alpha^{(k)})}{1-\chi_{m_\lambda}(z_\alpha^{(k)})}.$$

Hence

$$|D_{n'}(z_\alpha^{(k)})| < \sum_{i=0}^{\lambda} a_i m_i < p_{\lambda+1} m_\lambda.$$

From the inequalities $m_k/m_{\lambda+1} \le \alpha < m_k/m_\lambda$, it follows that $m_\lambda < m_k/\alpha$, so that

$$|D_{n'}(z_\alpha^{(k)})| < p_{\lambda+1} \cdot \frac{m_k}{\alpha}. \qquad \square$$

A point $x \in G$ is said to be a *Lebesgue point of* f provided that

$$\frac{1}{m(G_k)} \int_{G_k} |f(x+t) - f(x)|\, dt = o(1) \quad \text{as } k \to \infty.$$

If $f \in L^1(G)$ then almost every $x \in G$ is a Lebesgue point of f.

Next we define a quantity $f^*(x)$ whose existence is analogous to the existence of $\lim_{h\to 0} 1/h \int_0^h f(x+t)\, dt$ for functions on the real line.

2.3. DEFINITION. Let $f^*(x) = \lim_{k\to\infty} 1/m(G_{k+1}) \int_{jx_k + G_{k+1}} f(x-t)\, dt$, provided that the limit exists uniformly in $j \in \{0, 1, 2, \ldots, p_{k+1} - 1\}$.

For the class of all Vilenkin groups, there is no simple relationship between the existence of $f^*(x)$ and x being a Lebesgue point.

2.4. THEOREM. *Let G be a Vilenkin group and suppose that $f \in L^1(G)$. If G is bounded and if x is a Lebesgue point of f, then $f^*(x)$ exists and is equal to $f(x)$, but the existence of $f^*(x)$ does not imply that x is a Lebesgue point of f. Moreover, if G is unbounded, then $f^*(x)$ need not exist even at Lebesgue points.*

2.5. *Proof.* First let G be bounded, and suppose that x is a Lebesgue point of f. To estimate the magnitude of the difference

$$\frac{1}{m(G_{k+1})} \int_{jx_k + G_{k+1}} f(x-t)\, dt - f(x),$$

observe that this difference is dominated by

$$\frac{1}{m(G_{k+1})} \int_{jx_k + G_{k+1}} |f(x-t) - f(x)|\, dt,$$

which in turn is bounded above by

$$\frac{1}{m(G_{k+1})} \int_{G_k} |f(x-t) - f(x)|\, dt = \frac{p_{k+1}}{m(G_k)} \int_{G_k} |f(x-t) - f(x)|\, dt$$
$$= p_{k+1} \cdot o(1).$$

That $1/m(G_{k+1}) \int_{jx_k + G_{k+1}} f(x-t)\, dt$ tends to $f(x)$ follows from the boundedness of the Vilenkin group G.

To show that $f^*(x)$ may exist at points that are not Lebesgue points of f, let G be the Walsh group 2^ω, define f by the formula

$$f(t)=\begin{cases} 1 & t\in x_k+G_{k+2}, \quad k=0,1,\dots, \\ -1 & t\in x_k+x_{k+1}+G_{k+2}, \quad k=0,1,\dots, \\ 0 & t=0, \end{cases}$$

and examine the behavior of f at 0. We observe that

$$\int_{G_{k+1}} f(0-t)\,dt=\int_{x_k+G_{k+1}} f(0-t)\,dt=0,$$

so that $f^*(0)=0$ also. But 0 cannot be a Lebesgue point of f, since

$$\frac{1}{m(G_k)}\int_{G_k} |f(0-t)-f(0)|\,dt=\frac{1}{m(G_k)}\int_{G_k} 1\,dt=1.$$

Finally, let G be an unbounded Vilenkin group having an increasing sequence $\{p_k\}$ of primes. Define the function

$$f(t)=\begin{cases} 1 & t\in x_k+G_{k+1}, \quad k=0,1,\dots, \\ 0 & \text{otherwise.} \end{cases}$$

Then $f(0)=0$ while $f^*(0)$ does not exist. But

$$\begin{aligned}\frac{1}{m(G_k)}\int_{G_k} |f(t)-f(0)|\,dt&=m_k\cdot\left[\frac{1}{m_{k+1}}+\frac{1}{m_{k+2}}+\frac{1}{m_{k+3}}+\cdots\right]\\ &\le\frac{1}{p_{k+1}}\cdot\left[1+\frac{1}{p_{k+2}}+\frac{1}{p_{k+2}p_{k+3}}+\cdots\right]\\ &\le\frac{1}{p_{k+1}}\cdot\frac{1}{1-(1/p_{k+2})},\end{aligned}$$

so that x is a Lebesgue point of f. □

3. Convergence of Fourier Series on Bounded Vilenkin Groups

A straightforward connection between the existence of $f^*(x)$ and the convergence of Fourier series is given by the following result.

3.1. THEOREM. *Let G be a bounded Vilenkin group, and suppose that $f\in L^1(G)$. Additionally, let $n=a_km_k+n'$, where a_k and n' are integers such that $1\le a_k<p_{k+1}$ and $0\le n'<m_k$. At an $x\in G$ for which $f^*(x)$ exists, we have*

$$S_n(f;x)-f^*(x)=o(1)+\sum_{\alpha=1}^{m_k-1}\chi_{m_k}^{a_k}(z_\alpha^{(k)})D_{n'}(z_\alpha^{(k)})\int_{G_k} f(x-z_\alpha^{(k)}-t)\chi_{m_k}^{a_k}(t)\,dt$$

as $n\to\infty$. Thus the necessary and sufficient condition that the Fourier series of f converges at such an x is that

$$(*)\qquad \sum_{\alpha=1}^{m_k-1}\chi_{m_k}^{a_k}(z_\alpha^{(k)})D_{n'}(z_\alpha^{(k)})\int_{G_k} f(x-z_\alpha^{(k)}-t)\chi_{m_k}^{a_k}(t)\,dt=o(1)$$

uniformly in a_k and n' as $k\to\infty$.

3.2. *Proof.* Consider the integral representation of the difference

$$S_n(f;x)-f^*(x).$$

We have

$$\begin{aligned} S_n(f;x)-f^*(x) &= \int_G [f(x-t)-f^*(x)]D_n(t)\,dt \\ &= \int_{G_k} [f(x-t)-f^*(x)]D_n(t)\,dt+\int_{G\setminus G_k} [f(x-t)-f^*(x)]D_n(t)\,dt \\ &= A+B. \end{aligned}$$

First we observe that

$$\begin{aligned} B &= \int_{G\setminus G_k} [f(x-t)-f^*(x)]\chi_{m_k}^{a_k}(t)D_{n'}(t)\,dt \\ &= \sum_{\alpha=1}^{m_k-1} \chi_{m_k}^{a_k}(z_\alpha^{(k)})D_{n'}(z_\alpha^{(k)}) \int_{G_k} [f(x-z_\alpha^{(k)}-t)-f^*(x)]\chi_{m_k}^{a_k}(t)\,dt \\ &= \sum_{\alpha=1}^{m_k-1} \chi_{m_k}^{a_k}(z_\alpha^{(k)})D_{n'}(z_\alpha^{(k)}) \int_{G_k} f(x-z_\alpha^{(k)}-t)\chi_{m_k}^{a_k}(t)\,dt, \end{aligned}$$

so that B is equal to the expression in (*).

It remains to analyze A. We proceed by separating A into two parts:

$$\begin{aligned} A &= \int_{G_k} [f(x-t)-f^*(x)]\left\{m_k\cdot\frac{1-\chi_{m_k}^{a_k}(t)}{1-\chi_{m_k}(t)}+n'\cdot\chi_{m_k}^{a_k}(t)\right\}dt \\ &= \frac{1}{m(G_k)}\int_{G_k} (f(x-t)-f^*(x)]\,dt \\ &\quad +\frac{1}{m(G_k)}\int_{G_k} [f(x-t)-f^*(x)]\left\{\chi_{m_k}(t)+\cdots+\chi_{m_k}^{a_k-1}(t)+\frac{n'}{m_k}\chi_{m_k}^{a_k}(t)\right\}dt \\ &= A_1+A_2. \end{aligned}$$

Now

$$A_1=\frac{1}{m(G_k)}\int_{G_k} f(x-t)\,dt-f^*(x)=o(1)$$

by the definition of $f^*(x)$. To estimate A_2, fix $\lambda\in\{1,2,\ldots,p_{k+1}-1\}$ and consider

$$\frac{1}{m(G_k)}\int_{G_k} [f(x-t)-f^*(x)]\chi_{m_k}^{\lambda}(t)\,dt,$$

which we may rewrite as

$$\begin{aligned} &\frac{1}{m(G_k)}\sum_{j=0}^{p_{k+1}-1}\int_{jx_k+G_{k+1}} [f(x-t)-f^*(x)]\chi_{m_k}^{\lambda}(t)\,dt \\ &=\frac{1}{p_{k+1}}\sum_{j=0}^{p_{k+1}-1}\frac{1}{m(G_{k+1})}\int_{G_{k+1}} [f(x-jx_k-t)-f^*(x)]\chi_{m_k}^{\lambda}(t+jx_k)\,dt= \end{aligned}$$

$$=\frac{1}{p_{k+1}}\sum_{j=0}^{p_{k+1}-1}\frac{1}{m(G_{k+1})}\int_{G_{k+1}}[f(x-jx_k-t)-f^*(x)]\chi_{m_k}^{\lambda}(jx_k)\,dt$$

$$=\frac{1}{p_{k+1}}\sum_{j=0}^{p_{k+1}-1}\frac{1}{m(G_{k+1})}\int_{G_{k+1}}[f(x-jx_k-t)-f^*(x)]\zeta_k^{j\lambda}\,dt$$

$$=\frac{1}{p_{k+1}}\sum_{j=0}^{p_{k+1}-1}\zeta_k^{j\lambda}\left\{\frac{1}{m(G_{k+1})}\int_{jx_k+G_{k+1}}f(x-t)\,dt-f^*(x)\right\}.$$

Hence we have that

$$\left|\frac{1}{m(G_k)}\int_{G_k}[f(x-t)-f^*(x)]\chi_{m_k}^{\lambda}(t)\,dt\right|$$

is bounded above by

$$\sup_j\left|\frac{1}{m(G_{k+1})}\int_{jx_k+G_{k+1}}f(x-t)\,dt-f^*(x)\right|,$$

and it follows that

$$|A_2|\le p_{k+1}\cdot\sup_j\left|\frac{1}{m(G_{k+1})}\int_{jx_k+G_{k+1}}f(x-t)\,dt-f^*(x)\right|=o(1)\quad\text{as } k\to\infty.$$

Consequently, $A=o(1)$, and we have shown that if $f^*(x)$ exists then

$$S_n(f;x)-f^*(x)=o(1)+\sum_{\alpha=1}^{m_k-1}\chi_{m_k}^{a_k}(z_\alpha^{(k)})D_{n'}(z_\alpha^{(k)})\int_{G_k}f(x-z_\alpha^{(k)}-t)\chi_{m_k}^{a_k}(t)\,dt,$$

which in turn implies that $S_n(f;x)\to f^*(x)$ if and only if

$$\sum_{\alpha=1}^{m_k-1}\chi_{m_k}^{a_k}(z_\alpha^{(k)})D_{n'}(z_\alpha^{(k)})\int_{G_k}f(x-z_\alpha^{(k)}-t)\chi_{m_k}^{a_k}(t)\,dt=o(1). \qquad \square$$

The following result localizes the Onneweer–Waterman Salem test for the convergence of Fourier series on bounded Vilenkin groups.

3.3. THEOREM. *Let G be a bounded Vilenkin group, and suppose that $f\in L^1(G)$. Then the Fourier series of f converges to $f^*(x)$ at every point $x\in G$ at which the following conditions hold:*

(S1) *$f^*(x)$ exists, and*

(S2) $$\lim_{k\to\infty}\frac{1}{m(G_{k+1})}\int_{G_{k+1}}\sum_{\alpha=1}^{m_k-1}\frac{1}{\alpha}\left|\sum_{j=0}^{p_{k+1}-1}f(x-z_\alpha^{(k)}-jx_k-t)\zeta_k^{ja_k}\right|dt=0$$

uniformly in $a_k\in\{1,2,\dots,p_{k+1}-1\}$.

The convergence is uniform in x on any closed set of points where f is continuous and (S2) *holds uniformly.*

3.4. *Proof.* Let $n=a_km_k+n'$, where a_k and n' are integers such that $1\le a_k<p_{k+1}$ and $0\le n'<m_k$. Then (S1) implies

$$S_n(f;x)-f^*(x)=o(1)+\sum_{\alpha=1}^{m_k-1}\chi_{m_k}^{a_k}(z_\alpha^{(k)})D_{n'}(z_\alpha^{(k)})\int_{G_k}f(x-z_\alpha^{(k)}-t)\chi_{m_k}^{a_k}(t)\,dt.$$

Thus we need only show that (S2) implies

$$(*)\qquad \sum_{\alpha=1}^{m_k-1}\chi_{m_k}^{a_k}(z_\alpha^{(k)})D_{n'}(z_\alpha^{(k)})\int_{G_k}f(x-z_\alpha^{(k)}-t)\chi_{m_k}^{a_k}(t)\,dt=o(1).$$

We begin by writing the integrals that appear in (∗) as sums of integrals over the cosets of G_{k+1} lying inside G_k, and observe that

$$\begin{aligned}\int_{G_k}f(x-z_\alpha^{(k)}-t)\chi_{m_k}^{a_k}(t)\,dt&=\sum_{j=0}^{p_{k+1}-1}\int_{jx_k+G_{k+1}}f(x-z_\alpha^{(k)}-t)\chi_{m_k}^{a_k}(t)\,dt\\&=\sum_{j=0}^{p_{k+1}-1}\int_{G_{k+1}}f(x-z_\alpha^{(k)}-jx_k-t)\chi_{m_k}^{a_k}(jx_k)\,dt\\&=\int_{G_{k+1}}\sum_{j=0}^{p_{k+1}-1}f(x-z_\alpha^{(k)}-jx_k-t)\,\zeta_k^{ja_k}\,dt,\end{aligned}$$

so that

$$\begin{aligned}&\sum_{\alpha=1}^{m_k-1}\chi_{m_k}^{a_k}(z_\alpha^{(k)})D_{n'}(z_\alpha^{(k)})\int_{G_k}f(x-z_\alpha^{(k)}-t)\chi_{m_k}^{a_k}(t)\,dt\\&\quad=\int_{G_{k+1}}\sum_{\alpha=1}^{m_k-1}\chi_{m_k}^{a_k}(z_\alpha^{(k)})D_{n'}(z_\alpha^{(k)})\sum_{j=0}^{p_{k+1}-1}f(x-z_\alpha^{(k)}-jx_k-t)\,\zeta_k^{ja_k}\,dt.\end{aligned}$$

On bounded Vilenkin groups $|D_{n'}(z_\alpha^{(k)})|\le p\cdot(m_k/\alpha)$, so that

$$\begin{aligned}&\left|\sum_{\alpha=1}^{m_k-1}\chi_{m_k}^{a_k}(z_\alpha^{(k)})D_{n'}(z_\alpha^{(k)})\int_{G_k}f(x-z_\alpha^{(k)}-t)\chi_{m_k}^{a_k}(t)\,dt\right|\\&\qquad\le\int_{G_{k+1}}\sum_{\alpha=1}^{m_k-1}p\cdot\frac{m_k}{\alpha}\left|\sum_{j=0}^{p_{k+1}-1}f(x-z_\alpha^{(k)}-jx_k-t)\,\zeta_k^{ja_k}\right|dt\\&\qquad=\left(\frac{p}{p_{k+1}}\right)\frac{1}{m(G_{k+1})}\int_{G_{k+1}}\sum_{\alpha=1}^{m_k-1}\frac{1}{\alpha}\left|\sum_{j=0}^{p_{k+1}-1}f(x-z_\alpha^{(k)}-jx_k-t)\,\zeta_k^{ja_k}\right|dt\\&\qquad=o(1)\end{aligned}$$

uniformly in a_k by (S2). Thus (S1) and (S2) together imply (∗), and $S_n(f;x)\to f^*(x)$. For x in a closed set of points of continuity, $f^*(x)=f(x)$ and

$$\frac{1}{m(G_k)}\int_{G_k}[f(x-t)-f(x)]\,dt=o(1)$$

uniformly in x as $k\to\infty$. Hence the convergence of the Fourier series is uniform in x so long as (S2) holds uniformly. □

3.5. EXAMPLES. Here we mention briefly two examples which illustrate the scope of the preceding theorem.

3.5.1 *Conditions* (S1) *and* (S2) *together do not imply the continuity of f at* x. On the Walsh group 2^ω, let f be defined by

$$f(t)=\begin{cases} 1 & t\in x_{k-1}+G_{k^2},\ k=1,2,\ldots, \\ 0 & \text{otherwise.}\end{cases}$$

Then f is continuous except at 0 but $f^*(0)$ exists and equals 0.

To show that (S2) holds, evaluate

$$\frac{1}{m(G_{k+1})}\int_{G_{k+1}}\sum_{\alpha=1}^{m_k-1}\frac{1}{\alpha}\left|\sum_{j=0}^{p_{k+1}-1} f(0-z_\alpha^{(k)}-jx_k-t)\zeta_k^{ja_k}\right|dt,$$

which reduces to

$$(1)\qquad 2^{k+1}\cdot\sum_{\alpha=1}^{2^k-1}\frac{1}{\alpha}\int_{G_{k+1}}|f(z_\alpha^{(k)}+t)-f(z_\alpha^{(k)}+x_k+t)|\,dt$$

on the Walsh group.

In analyzing this expression, it will be convenient to consider the representations

$$z_\alpha^{(k)}+t=b_0x_0+\cdots+b_{k-1}x_{k-1}+0x_k+b_{k+1}x_{k+1}+\cdots$$

and

$$z_\alpha^{(k)}+x_k+t=b_0x_0+\cdots+b_{k-1}x_{k-1}+1x_k+b_{k+1}x_{k+1}+\cdots.$$

By the definition of f, $f(x)=1$ only if there is a sufficient number of zero coefficients between the first two nonzero coefficients in the representation of x. Hence it is easy to see that if $f(z_\alpha^{(k)}+x_k+t)=1$ then $f(z_\alpha^{(k)}+t)=1$ also. Thus, in evaluating (1) we need consider only those points for which $f(z_\alpha^{(k)}+t)=1$ and $f(z_\alpha^{(k)}+x_k+t)=0$. Further, note that if $f(z_\alpha^{(k)}+t)=1$, then either $z_\alpha^{(k)}=\sum_{i=0}^{k-1}b_ix_i$ has two nonzero b_i's (which implies that the same is true of $z_\alpha^{(k)}+x_k+t$, so that $f(z_\alpha^{(k)}+x_k+t)=1$ also) or $z_\alpha^{(k)}=\sum_{i=0}^{k-1}b_ix_i$ has just one nonzero b_i, which implies that $\alpha=2^j$ and $z_\alpha^{(k)}=x_{k-j-1}$. In this case, however, if $f(z_\alpha^{(k)}+t)=1$ and $f(z_\alpha^{(k)}+x_k+t)=0$, it must follow that $x_k\notin G_{(k-j)^2}$. This in turn implies that $(k-j)^2>k$, which we may rewrite as $j<k-\sqrt{k}$. It follows, then, that the nonzero terms in (1) correspond to

$$2^{k+1}\cdot\frac{1}{2^j}\int_{G_{(k-j)^2}}1\,dt=2^{1+k-j-(k-j)^2}.$$

There is one term for each integer j satisfying $0\le j<k-\sqrt{k}$. Hence

$$2^{k+1}\cdot\int_{G_{k+1}}\sum_{\alpha=1}^{2^k-1}\frac{1}{\alpha}|f(z_\alpha^{(k)}+t)-f(z_\alpha^{(k)}+x_k+t)|\,dt=2^{1+k-k^2}\cdot\sum_j 2^{(2k-1)j-j^2},$$

where the sum extends over those integers j such that $0\le j<k-\sqrt{k}$. The last term in this sum is the largest, so it follows that

$$\begin{aligned}2^{k+1}\cdot&\int_{G_{k+1}}\sum_{\alpha=1}^{2^k-1}\frac{1}{\alpha}|f(z_\alpha^{(k)}+t)-f(z_\alpha^{(k)}+x_k+t)|\,dt\\ &\le 2^{1+k-k^2}\cdot(k-\sqrt{k})\cdot 2^{k^2-2k+\sqrt{k}}\\ &\le 2^{1-k+\sqrt{k}}\cdot(k-\sqrt{k})=o(1)\quad\text{as } k\to\infty.\end{aligned}$$

3.5.2. *In the convergence Theorem 3.3, condition* (S1), *the assumption that* $f^*(x)$ *exists, cannot be weakened to the requirement that*

$$\lim_{k\to\infty}\frac{1}{m(G_k)}\int_{G_k} f(x-t)\,dt$$

exists.

Define the group 3^ω to be the product of countably many copies of $\mathbf{Z}_3$. Let

$$f(t)=\begin{cases} 1 & t\in x_{k-1}+G_k, \quad k=1,2,\ldots,\\ -1 & t\in 2x_{k-1}+G_k,\ k=1,2,\ldots,\\ 0 & t=0.\end{cases}$$

Then f is continuous except at 0, and $\int_{G_k} f(0-t)\,dt=0$ for each k, so that

$$f(0)=\lim_{k\to\infty}\frac{1}{m(G_k)}\int_{G_k} f(0-t)\,dt.$$

Nonetheless $f^*(0)$ does not exist because

$$\frac{1}{m(G_{k+1})}\int_{jx_k+G_{k+1}} f(0-t)\,dt=\begin{cases} 0 & j=0,\\ -1 & j=1,\\ 1 & j=2.\end{cases}$$

The function f does however satisfy (S2), since the integrand may be rewritten in the form

$$\sum_{\alpha=1}^{m_k-1}\frac{1}{\alpha}\left|\sum_{j=0}^{p_{k+1}-1} f(0-z_\alpha^{(k)}-jx_k-t)\,\zeta_k^{ja_k}\right|=\sum_{\alpha=1}^{m_k-1}\frac{1}{\alpha}\left|\sum_{j=0}^{p_{k+1}-1} f(-z_\alpha^{(k)})\,\zeta_k^{ja_k}\right|=0.$$

The following direct computation shows that $S_n(f;0)$ fails to converge as $n\to\infty$.

Observe that the integral expression for the partial sum of the Fourier series,

$$S_n(f;0)=\int_G f(-t)D_n(t)\,dt,$$

may be rewritten as the sum

$$\int_{G_k} f(-t)D_n(t)\,dt+\int_{G\setminus G_k} f(-t)D_n(t)\,dt=A+B.$$

Now $B=0$, since

$$\begin{aligned}&\int_{G\setminus G_k} f(-t)\chi_{m_k}^{a_k}(t)D_{n'}(t)\,dt\\ &=\sum_{\alpha=1}^{m_k-1}\chi_{m_k}^{a_k}(z_\alpha^{(k)})D_{n'}(z_\alpha^{(k)})\int_{G_k} f(-z_\alpha^{(k)}-t)\chi_{m_k}^{a_k}(t)\,dt\\ &=\sum_{\alpha=1}^{m_k-1}\chi_{m_k}^{a_k}(z_\alpha^{(k)})D_{n'}(z_\alpha^{(k)})\int_{G_k} f(-z_\alpha^{(k)})\chi_{m_k}^{a_k}(t)\,dt\\ &=\sum_{\alpha=1}^{m_k-1}\chi_{m_k}^{a_k}(z_\alpha^{(k)})D_{n'}(z_\alpha^{(k)})f(-z_\alpha^{(k)})\int_{G_k}\chi_{m_k}^{a_k}(t)\,dt,\end{aligned}$$

which is 0 because $\int_{G_k}\chi_{m_k}^{a_k}(t)\,dt=0$.

In estimating A, consider the two specific sequences of partial sums corresponding to $n=3^k$ and $n=2\cdot 3^k$. For $n=3^k$, $A=m_k\int_{G_k} f(-t)\,dt=0$, so that if $S_n(f;x)$ converges, it must converge to 0. On the other hand, for $n=2\cdot 3^k$,

$$
\begin{aligned}
A &= m_k \int_{G_k} f(-t)(1+\chi_{m_k}(t))\,dt = m_k \int_{G_k} f(-t)\chi_{m_k}(t)\,dt \\
&= m_k \int_{G_{k+1}} f(-t)\,dt + m_k \int_{G_{k+1}} (-1)\zeta_k\,dt + m_k \int_{G_{k+1}} (1)\zeta_k^2\,dt \\
&= 0+\frac{1}{3}(-1)\zeta_k+\frac{1}{3}(1)\zeta_k^2 = -\frac{i}{\sqrt{3}}.
\end{aligned}
$$

Since this value differs from 0, the sequence of partial sums cannot converge.

3.6. REMARKS. Theorem 3.3 may be regarded as an extension of the Onneweer-Waterman Lebesgue test for the convergence of Fourier series [2]. This theorem establishes the condition

$$
(**)\qquad \int_{G\setminus G_k} \left| \sum_{j=0}^{p_{k+1}-1} [f(x-t-jx_k)-f(x)]\zeta_k^{ja_k} \right| T_k^{-1}(t)\,dt = o(1) \quad \text{as } k\to\infty,
$$

where $T_k^{-1}=\alpha/m_k$ for $t\in z_\alpha^{(k)}+G_k$, as sufficient for the convergence of the Fourier series at Lebesgue points of f. This is equivalent to our condition (S2), as may be seen by expanding the integral in (**) into a sum of integrals over cosets of G_k and expanding these integrals in turn into sums of integrals over the cosets of G_{k+1}. Theorem 3.3, by comparison, replaces the requirement that x is a Lebesgue point of f by the requirement that $f^*(0)$ exists.

References

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