Besov Spaces, Sobolev Spaces, and Cauchy Integrals

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1. Introduction and Statement of Results

Let B_n denote the unit ball in C^n with boundary S, the $(2n-1)$-dimensional sphere. If $d\sigma$ is normalized rotation invariant measure on S and $f \in L^1(d\sigma)$ then for $z \in B_n$ we define the Cauchy integral

$$Cf(z) = \int_S f(\zeta) \frac{d\sigma(\zeta)}{(1-\langle z, \zeta\rangle)^n}.$$

In this paper we obtain conditions on f sufficient to imply that Cf belongs to either the Besov space B_β^p or the Hardy–Sobolev space H_β^p, where $\beta > 0$ and $1 < p < \infty$. Recall that a holomorphic function F defined on B_n belongs to H_β^p if

$$\|F\|_{H_\beta^p}^p = \|R^\beta F\|_p^p = \sup_{0<r<1} \int_S |R^\beta F(r\zeta)|^p \, d\sigma(\zeta) < \infty,$$

where R^β denotes the radial fractional derivative operator defined on the class of harmonic functions on B_n by

$$R^\beta u(z) = \sum (1+k)^\beta P_k(z),$$

where $u = \sum P_k(z)$ is the expansion of u in homogeneous harmonic polynomials. Thus, if $z = r\zeta$ where $0 \le r < 1$ and $\zeta \in S$,

$$\begin{aligned} R^1 u(z) &= u(z) + r\frac{\partial u}{\partial r}(r\zeta) \\ &= u(z) + \sum_{j=1}^n \left(z_j \frac{\partial u}{\partial z_j} + \bar{z}_j \frac{\partial u}{\partial \bar{z}_j} \right). \end{aligned}$$

A holomorphic function F defined on B_n belongs to B_β^p if

$$\|F\|_{B_\beta^p}^p = \|R^{1+\beta} F\|_{p,p-1}^p = \int_{B_n} |R^{1+\beta} F(z)|^p (1-|z|)^{p-1} \, d\nu(z) < \infty,$$

where $d\nu$ denotes $2n$-dimensional Lebesgue measure defined on C^n.

The sufficient conditions we establish are of two types. The first we describe as "transverse" and the second we call "tangential". The transverse

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results, whose proofs are relatively simple, are motivated by earlier work of Ahern and Schneider [2; 3] and Phong and Stein [8]; see also [6] and [7]. The tangential theorems, whose proofs are more involved, are motivated by more recent results of Ahern and Bruna [1]. We proceed to discuss the transverse theorems.

In [3], Ahern and Schneider showed that if f is a bounded function on S satisfying a uniform Lipschitz condition of order $0<\alpha<1$ on almost every slice, that is, if there exists a constant C such that for almost all $\zeta\in S$

$$|f(e^{i(t+\theta)}\zeta)-f(e^{i\theta}\zeta)|\le Ct^{\alpha}, \tag{1.1}$$

then Cf belongs to the Lipschitz space $\Lambda_\alpha(B_n)$ of holomorphic functions F on B_n satisfying the condition

$$|F(z)-F(w)|\le C|z-w|^{\alpha}.$$

In this paper we prove a similar result which may be interpreted as giving a condition on the behavior of f on slices that is sufficient to imply that Cf belongs to the Besov space B^p_β. If f is a function defined on S, for $\zeta\in S$ and $t>0$ define

$$\Delta_t f(\zeta)=f(e^{it}\zeta)-f(\zeta).$$

In addition, for each $\zeta\in S$ define the slice function

$$f_\zeta(e^{i\theta})=f(e^{i\theta}\zeta).$$

If $\zeta\in S$ then an analogue of (1.1) appropriate for dealing with Besov spaces would involve some condition on the means

$$\int_0^{2\pi}|f(e^{i(\theta+t)}\zeta)-f(e^{i\theta}\zeta)|^p\,d\theta.$$

Now, if F is holomorphic on B_n and in B^p_β then the proof of Theorem A will show that an equivalent norm on B^p_β is given by

$$\|F\|_p+\left(\int_S\int_0^1\int_0^{2\pi}|R^mF(e^{i(\theta+t)}\zeta)-R^mF(e^{i\theta}\zeta)|^p\,d\theta\,\frac{dt}{t^{p(\beta-m)+1}}\,d\sigma(\zeta)\right)^{1/p}$$

if $m<\beta<m+1$. Interchanging the order of the integrations on ζ and θ and making the substitution $e^{i\theta}\zeta$ for ζ allows us to rewrite the second term as

$$\left(\int_0^1\int_S|\Delta_tR^mF(\zeta)|^p\,d\sigma(\zeta)\frac{dt}{t^{1+p(\beta-m)}}\right)^{1/p}. \tag{1.2}$$

It is therefore reasonable to wonder whether or not the boundedness of some norm similar to (1.2) on a not necessarily holomorphic function f is sufficient to imply that $Cf\in B^p_\beta$. The following result shows that this is indeed the case. To state it we define, for $f\in L(d\sigma)$, the transverse field

$$N_\tau f(\zeta)=i\sum_{j=1}^n\left(\zeta_j\frac{\partial f}{\partial\zeta_j}-\bar\zeta_j\frac{\partial f}{\partial\bar\zeta_j}\right).$$

Here and in what follows, all derivatives are to be interpreted in the sense of distributions.

THEOREM A. *Let* $1 < p < \infty$ *and suppose that* m *is a nonnegative integer such that* $m < \beta < m+1$. *Then a sufficient condition that* $Cf \in B^p_\beta$ *is that there be functions* $g_0, g_1, \ldots, g_m$ *in* $L^1(d\sigma)$ *satisfying*

$$(N_\tau)^j f(\zeta) = g_j(\zeta)$$

for $j = 0, \ldots, m$ *such that*

$$\int_0^1 \int_S |\Delta_t g_j(\zeta)|^p \, d\sigma(\zeta) \frac{dt}{t^{1+p(\beta-m)}} < \infty$$

for $j = 0, \ldots, m$.

REMARK. Notice that if f is sufficiently smooth then

$$N_\tau f(e^{i\theta}\zeta) = \frac{d}{d\theta} f_\zeta(e^{i\theta}).$$

In order to obtain a result that includes all real values of β it is necessary to consider second-order differences. Define

$$\Delta_t^2 f(\zeta) = f(e^{it}\zeta) + f(e^{-it}\zeta) - 2f(\zeta).$$

We prove the following result.

THEOREM A′. *Let* $1 < p < \infty$ *and suppose that* m *is a nonnegative integer such that* $m < \beta < m+2$. *Then a sufficient condition that* $Cf \in B^p_\beta$ *is that there be functions* $g_0, g_1, \ldots, g_m$ *in* $L^1(d\sigma)$ *satisfying*

$$(N_\tau)^j f(\zeta) = g_j(\zeta)$$

for $j = 0, \ldots, m$ *such that*

$$\int_0^1 \int_S |\Delta_t^2 g_j(\zeta)|^p \, d\sigma(\zeta) \frac{dt}{t^{1+p(\beta-m)}} < \infty$$

for $j = 0, \ldots, m$.

We remark that, as has already been indicated, our arguments will actually show that if f is the boundary function of an H^1 function then the sufficient conditions (stated in Theorem A and Theorem A′) that $f \in B^p_\beta$ are also necessary ones.

We also prove a version of Theorem A that is valid for Sobolev spaces.

THEOREM B. *Let* $1 < p < \infty$ *and suppose* m *is a positive integer. Then a sufficient condition that* $Cf \in H^p_m$ *is that there be functions* $g_0, \ldots, g_m$ *each in* $L^p(d\sigma)$ *such that*

$$(N_\tau)^j f(\zeta) = g_j(\zeta)$$

for $j = 0, \ldots, m$.

The second type of result presented here deals with complex tangential directions. Let $T_{j,k}$ and $\bar{T}_{j,k}$ denote the tangential differential operators

$$T_{j,k}F(z)=\bar{z}_j\frac{\partial F}{\partial z_k}-\bar{z}_k\frac{\partial F}{\partial z_j}$$

and

$$\bar{T}_{j,k}F(z)=z_j\frac{\partial F}{\partial \bar{z}_k}-z_k\frac{\partial F}{\partial \bar{z}_j}.$$

It is a well-known principle that at a given point w on the unit ball, a holomorphic function can be expected to behave "twice as nicely" in the direction given by a vector v in C^n which is "complex tangential" in the sense that $\langle w, v\rangle=0$, as opposed to the direction determined by a vector v for which $\langle w, v\rangle\neq 0$. This phenomenon is reflected in the fact that in many situations one may formally replace the radial derivative operator R^k by a generic operator L^{2k} obtained by composing $2k$ of the operators $T_{j,k}$ or $\bar{T}_{j,k}$. For example, it is easy to show that a holomorphic function F belongs to H_k^p if and only if the admissible maximal function of $L^{2k}F$ belongs to L^p for all operators represented by L^{2k} as discussed above. This is because R^1 actually appears among the operators represented by L^2. On the other hand, if F is holomorphic then $\bar{T}_{j,k}F=0$ for all $\bar{T}_{j,k}$, and it is natural to ask for characterizations of H_β^p and B_β^p that do not involve the operators $\bar{T}_{j,k}$. Such characterizations have been recently given by Ahern and Bruna in [1]. They proved, among other things, that if T^k denotes a generic operator obtained by composing k of the operators $T_{i,j}$ then a holomorphic function F on the unit ball belongs to $H_{k/2}^p$ if and only if T^kF has admissible (or even radial) maximal function in L^p; see [1, Thm. 4.2]. Concerning Besov spaces, their results show that F belongs to B_β^p if and only if

$$\int_{B_n}|T^kF(z)|^p(1-|z|)^{p(k/2-\beta)-1}\,d\nu(z)<\infty,$$

where $k>2\beta$.

In this paper we give conditions on a function $f\in L^1(d\sigma)$ which are in terms of (the distributional derivative) T^kf and its behavior on "complex tangential balls" embedded in S which are sufficient to imply that Cf belongs to B_β^p. We will use the following notation. For $\eta\in S$, $S'(\eta)$ will denote the set of points λ on the unit sphere in C^n orthogonal to η; that is,

$$S'(\eta)=\{\lambda\in S:\langle\lambda,\eta\rangle=0\}. \tag{1.3}$$

The symbol $d\sigma'(\lambda)$ will denote normalized $(2n-3)$-dimensional Lebesgue measure on $S'(\eta)$. Similarly, $B'(\eta)$ will denote the set of points w on the unit ball in C^n orthogonal to η; that is,

$$B'(\eta)=\{w\in B_n:\langle w,\eta\rangle=0\}. \tag{1.4}$$

The symbol $d\nu'(w)$ will denote normalized $(2n-2)$-dimensional Lebesgue measure on $B'(\eta)$. In polar coordinates we therefore have that

$$w=\rho\lambda, \quad 0\le\rho\le 1,$$

and

$$dv'(w)=(2n-2)\rho^{2n-3}\,d\rho\,d\sigma'(\lambda).$$

If $\eta\in S$ and $w\in B'(\eta)$ let

$$\Phi(\eta,w)=\sqrt{1-|w|^2}\,\eta+w \tag{1.5}$$

be the mapping obtained by projecting the point $\eta+w$ (which lies in the complex tangent space to S at η) in the direction orthogonal to that complex tangent space until it hits the sphere. For a fixed $\eta\in S$ the points ζ on the sphere may be parametrized by

$$\zeta=\zeta(t,w)=e^{it}\Phi(\eta,w), \tag{1.6}$$

where $0\le t\le 2\pi$ and $w\in B'(\eta)$. In terms of this parametrization $d\sigma(\zeta)$ may be written as (see [9, p. 15])

$$d\sigma(\zeta)=\frac{1}{2\pi}\,dt\,dv'(w). \tag{1.7}$$

For a function g defined on S and for $\eta\in S$ with $w\in B_n$ and $\langle\eta,w\rangle=0$, define

$$\Delta g(\eta,w)=g(\Phi(\eta,w))-g(\eta). \tag{1.8}$$

Suppose that $1\le p<\infty$ and $\gamma>0$. Let $g\in\mathcal{C}^\infty(S)$. Let $\|g\|_p$ be the norm of g in $L^p(d\sigma)$:

$$\|g\|_p=\left(\int|g(\zeta)|^p\,d\sigma(\zeta)\right)^{1/p}.$$

Then

$$\|g\|_{\Delta_{p,\gamma}}=\|g\|_1+\left(\int_0^1\int_S\int_{S'(\eta)}|\Delta g(\eta,\rho\lambda)|^p\,d\sigma'(\lambda)\,d\sigma(\eta)\frac{d\rho}{\rho^{1+p\gamma}}\right)^{1/p} \tag{1.9}$$

defines a norm on $\mathcal{C}^\infty(S)$ which is stronger than the L^1 norm. Let $\Delta_{p,\gamma}$ denote the Banach space obtained by completing $\mathcal{C}^\infty(S)$ in the norm $\|\ \|_{\Delta_{p,\gamma}}$. It is easy to see that $\Delta_{p,\gamma}$ is contained in $L^1(d\sigma)$. Furthermore, from the definitions given by (1.8) and (1.9), it is reasonable to assert that a function $f\in\Delta_{p,\gamma}$ possesses a degree of "smoothness" in the complex tangential directions.

We prove the following result.

THEOREM C. *Let $1<p<\infty$, suppose k is a nonnegative integer, and let $k/2<\beta<(k+1)/2$. Then for $f\in L^1(d\sigma)$ a condition sufficient to imply that Cf belongs to B^p_β is that T^kf belongs to $\Delta_{p,2\beta-k}$ whenever T^k is an operator obtained by composing k of the operators $T_{i,j}$.*

We also prove a version of Theorem C involving second-order differences. With g, η, and w as above, let

$$\Delta^2 g(\eta,w)=g(\sqrt{1-|w|^2}\,\eta+w)+g(\sqrt{1-|w|^2}\,\eta-w)-2g(\eta).$$

Then, for $g \in \mathcal{C}^\infty(S)$,

$$\|g\|_{\Delta^2_{p,\gamma}} = \|g\|_1 + \left(\int_0^1 \int_S \int_{S'(\eta)} |\Delta^2 g(\eta, \rho\lambda)|^p \, d\sigma'(\lambda) \, d\sigma(\eta) \frac{d\rho}{\rho^{1+p\gamma}} \right)^{1/p}$$

also defines a norm on $\mathcal{C}^\infty(S)$. Let $\Delta^2_{p,\gamma}$ denote the Banach space obtained by the usual process of completion. As before, $\Delta^2_{p,\gamma}$ is a subset of $L^1(d\sigma)$.

THEOREM C′. *Let $1<p<\infty$, suppose k is a nonnegative integer, and let $k/2<\beta<(k+2)/2$. Then for $f \in L^1(d\sigma)$ a condition sufficient to imply that Cf belongs to B^p_β is that $T^k f$ belongs to $\Delta^2_{p,2\beta-k}$ whenever T^k is an operator obtained by composing k of the operators $T_{i,j}$.*

Concerning the necessity of the conditions appearing in Theorems C and C′, we obtain the following result.

THEOREM D. *Let $F \in B^p_\beta$ where $1<p<\infty$.*

(a) *If $k/2<\beta<(k+1)/2$ and T^k is obtained by composing k of the operators $T_{i,j}$, then there is a constant C such that*

$$\|T^k F\|_{\Delta_{p,2\beta-k}} \le C \|R^{1+\beta} F\|_{p,p-1}.$$

(b) *If $k/2<\beta<(k+2)/2$ and T^k is obtained by composing k of the operators $T_{i,j}$, then there is a constant C such that*

$$\|T^k F\|_{\Delta^2_{p,2\beta-k}} \le C \|R^{1+\beta} F\|_{p,p-1}.$$

Finally, we consider sufficient conditions in tangential directions that imply that $Cf \in H^p_{k/2}$ for k a positive integer.

THEOREM E. *Sufficient that $Cf \in H^p{}_{k/2}$ for an L^p function f where $1<p<\infty$ is that $T^k f \in L^p$ for all operators T^k obtained as the composition of k of the operators $T_{i,j}$.*

In addition to the notation already introduced, we also adopt the following two conventions. First, the letter C will stand for various numbers that remain independent of the parameters with which they appear in context. Second, if F is a function on the ball B_n and $\zeta \in S$, then (for $0<r<1$) by F_r we will mean the function on the sphere S given by $F_r(\zeta) = F(r\zeta)$.

2. Proofs of Transverse Results

Proof of Theorem A. Let g denote any one of the functions g_j, where $j=0,\dots,m$. Since Cf is holomorphic,

$$R^1 Cf(z) = Cf(z) + NCf(z),$$

where N is the formal field

$$N = \sum_{j=1}^n z_j \frac{\partial}{\partial z_j}.$$

Calculate that

$$\begin{aligned} NCf(z) &= \int_S f(\zeta) N(1-\langle z,\zeta\rangle)^{-n}\,d\sigma(\zeta) \\ &= \int_S f(\zeta)\frac{n\langle z,\zeta\rangle}{(1-\langle z,\zeta\rangle)^{n+1}}\,d\sigma(\zeta) \\ &= -i\int_S f(\zeta)(N_\tau)_\zeta(1-\langle z,\zeta\rangle)^{-n}\,d\sigma(\zeta), \end{aligned}$$

where in the last equation the symbol $(N_\tau)_\zeta$ indicates that the differentiations are with respect to ζ and not z. The argument given in [9, 18.2.2] allows us to integrate by parts and arrive at

$$\begin{aligned} NCf(z) &= i\int_S N_\tau f(\zeta)\frac{1}{(1-\langle z,\zeta\rangle)^n}\,d\sigma(\zeta) \\ &= i\int_S g(\zeta)\frac{1}{(1-\langle z,\zeta\rangle)^n}\,d\sigma(\zeta). \end{aligned}$$

If we apply this reasoning to $R^mCf=(I+N)^mCf$, it follows that

$$\begin{aligned} R^mCf(z) &= \int_S G(\zeta)\frac{1}{(1-\langle z,\zeta\rangle)^n}\,d\sigma(\zeta) \\ &= CG(z), \end{aligned}$$

where G denotes a linear combination of the functions represented by the symbol g. Let $F(z)=Cf(z)$, so that

$$R^mF=CG. \tag{2.1}$$

If $\lambda\in B_1$ it is easy to check that

$$R^kF(\lambda\zeta)=\left(1+\lambda\frac{\partial}{\partial\lambda}\right)^k F_\zeta(\lambda).$$

It is shown in [4] that with $m+1>\beta$, $F\in B_\beta^p$ if and only if

$$\int_0^1 \|R^{m+1}F_r\|_p^p(1-r)^{(m+1-\beta)p-1}\,dr<\infty.$$

Use [9, p. 15, Prop. 1.4.7] to calculate that

$$\int_0^1 \|R^{m+1}F_r\|_p^p(1-r)^{(m+1-\beta)p-1}\,dr$$

equals

$$\frac{1}{2\pi}\int_0^1\int_0^{2\pi}\int_S |R^{m+1}F(re^{i\theta}\zeta)|^p(1-r)^{(m+1-\beta)p-1}\,d\sigma\,d\theta\,dr,$$

which, with $\lambda=re^{i\theta}$, equals

$$\frac{1}{2\pi}\int_S\int_0^1\int_0^{2\pi}\left|\left(1+\lambda\frac{\partial}{\partial\lambda}\right)^{m+1}F_\zeta(\lambda)\right|^p(1-r)^{(m+1-\beta)p-1}\,d\theta\,dr\,d\sigma.$$

Since F_ζ is holomorphic on the unit disk B_1, the Besov norm

$$\|F_\zeta\|_p + \left(\int_0^1\int_0^{2\pi}\left|\left(1+\lambda\frac{\partial}{\partial\lambda}\right)^{m+1}F_\zeta(\lambda)\right|^p(1-r)^{(m+1-\beta)p-1}\,d\theta\,dr\right)^{1/p}$$

is equivalent to the norm (see [10, Chap. V])

$$\|F_\zeta\|_p + \left(\int_0^1\int_0^{2\pi}\left|\Delta_t\left(1+\lambda\frac{\partial}{\partial\lambda}\right)^{m}F_\zeta(e^{i\theta})\right|^p d\theta\frac{dt}{t^{1+(\beta-m)p}}\right)^{1/p}, \tag{2.2}$$

where

$$\Delta_t h(\lambda) = h(e^{it}\lambda) - h(\lambda).$$

We may rewrite the integral in (2.2) as

$$\int_0^1\int_0^{2\pi}|\Delta_t R^m F(e^{i\theta}\zeta)|^p\,d\theta\frac{dt}{t^{1+(\beta-m)p}}.$$

It follows that

$$\int_0^1\|R^{m+1}F_r\|_p^p(1-r)^{(m+1-\beta)p-1}\,dr$$

is finite if and only if

$$\frac{1}{2\pi}\int_S\int_0^1\int_0^{2\pi}|\Delta_t R^m F(e^{i\theta}\zeta)|^p\,d\theta\frac{dt}{t^{1+(\beta-m)p}}\,d\sigma(\zeta)$$
$$= \int_S\int_0^1|\Delta_t R^m F(\zeta)|^p\frac{dt}{t^{1+(\beta-m)p}}\,d\sigma(\zeta) < \infty.$$

It is easy to check that the Cauchy projection C commutes with the difference operator Δ_t. Recalling (2.1) and using the fact that $1 < p < \infty$ to apply the theorem of Korànyi and Vagi [9, 6.3.1], it follows that

$$\begin{aligned}\int_S|\Delta_t R^m F(\zeta)|^p\,d\sigma(\zeta) &= \int_S|\Delta_t CG(\zeta)|^p\,d\sigma(\zeta)\\ &= \int_S|C\Delta_t G(\zeta)|^p\,d\sigma(\zeta)\\ &\le C\int_S|\Delta_t G(\zeta)|^p\,d\sigma(\zeta).\end{aligned}$$

Therefore

$$\int_S\int_0^1|\Delta_t R^m F(\zeta)|^p\frac{dt}{t^{1+(\beta-m)p}}\,d\sigma \le C\int_S\int_0^1|\Delta_t G(\zeta)|^p\frac{dt}{t^{1+(\beta-m)p}}\,d\sigma. \tag{2.3}$$

But the right-hand side of (2.3) is finite by the hypothesis of the theorem. This completes the proof. □

REMARK 1. The proof of Theorem B follows easily from the formula

$$R^m Cf(z) = \int_S G(\zeta)\frac{1}{(1-\langle z,\zeta\rangle)^n}\,d\sigma(\zeta),$$

where G is a linear combination of the functions $g_0, \ldots, g_m$ which was obtained in the proof of Theorem A.

REMARK 2. In case f is holomorphic, in the proof of Theorem A we will have $CG = G$, and the argument by slice integration shows that

$$\int_S \int_0^1 |\Delta_t G(\zeta)|^p \frac{dt}{t^{1+(\beta-m)p}} d\sigma(\zeta) < \infty$$

is a necessary condition that $f \in B^p_\beta$.

Theorem A′ may be proved in a manner entirely similar to Theorem A; we omit the details.

3. Proofs of Tangential Results

The following lemma will be needed for the proofs of Theorems C, C′, and E.

LEMMA 1. *Let T^m be an operator obtained by composing m of the operators $T_{j,k}$. Then*

$$T^m Cf(z) = c_m \int_S T^m f(\zeta) \frac{(1-\langle \zeta, z\rangle)^m}{(1-\langle z, \zeta\rangle)^{n+m}} d\sigma(\zeta),$$

where $c_m = (-1)^m(\Gamma(n+m)/\Gamma(n)\Gamma(m+1))$.

Proof. The proof will be by induction; notice that the case when $m=0$ is just the definition of Cf. Assume then that the lemma holds for m. Then

$$T^m Cf(z) = c_m \int_S T^m f(\zeta) \frac{(1-\langle \zeta, z\rangle)^m}{(1-\langle z, \zeta\rangle)^{n+m}} d\sigma(\zeta),$$

and therefore

$$\begin{aligned} T^{m+1} Cf(z) &= c_m T_{j,k} \int_S T^m f(\zeta) \frac{(1-\langle \zeta, z\rangle)^m}{(1-\langle z, \zeta\rangle)^{n+m}} d\sigma(\zeta) \\ &= c_m \int_S T^m f(\zeta) T_{j,k} \frac{(1-\langle \zeta, z\rangle)^m}{(1-\langle z, \zeta\rangle)^{n+m}} d\sigma(\zeta) \\ &= c_m \int_S T^m f(\zeta) \frac{(n+m)\langle z_{j,k}, \zeta\rangle(1-\langle \zeta, z\rangle)^m}{(1-\langle z, \zeta\rangle)^{n+m+1}} d\sigma(\zeta), \end{aligned}$$

where if $w = (w_1, \ldots, w_n) \in C^n$ then $w_{j,k}$ denotes the point in C_n whose kth coordinate is $\bar{w}_j$, whose jth coordinate is $-\bar{w}_k$, and whose other coordinates are 0. Observe that

$$\begin{aligned} (m+1)\langle z_{j,k}, \zeta\rangle(1-\langle \zeta, z\rangle)^m &= -(m+1)\langle \zeta_{j,k}, z\rangle(1-\langle \zeta, z\rangle)^m \\ &= T_{j,k}(1-\langle \zeta, z\rangle)^{m+1}, \end{aligned}$$

where now the operator $T_{j,k}$ denotes differentiation with respect to ζ instead of z. It follows that

$$T^{m+1}Cf(z)=c_m\frac{n+m}{m+1}\int_S T^m f(\zeta)\frac{T_{j,k}(1-\langle\zeta,z\rangle)^{m+1}}{(1-\langle z,\zeta\rangle)^{n+m+1}}\,d\sigma(\zeta)$$
$$=c_{m+1}\int_S T_{j,k}T^m f(\zeta)\frac{(1-\langle\zeta,z\rangle)^{m+1}}{(1-\langle z,\zeta\rangle)^{n+m+1}}\,d\sigma(\zeta),$$

where to obtain the second equality we have integrated by parts according to [9, p. 396, eq. (4)], and have also used the fact that $T_{j,k}$ annihilates $(1-\langle z,\zeta\rangle)$. This completes the proof. □

We recall the following notation. For $\eta\in S$, $S'(\eta)$ will denote the set of points λ on the unit sphere in C^n orthogonal to η, that is, $\langle\lambda,\eta\rangle=0$. The symbol $d\sigma'(\lambda)$ will denote normalized $(2n-3)$-dimensional Lebesgue measure on $S'(\eta)$.

Proof of Theorem C. Let T^k denote an operator obtained by composing k of the operators $T_{i,j}$. Then T^{k+1} has the form T^1T^k. By the results [1] referred to earlier, it is sufficient to show that there is a constant C such that the inequality

$$\int_{B_n}|T^{k+1}Cf(z)|^p(1-|z|)^{p((k+1)/2-\beta)-1}\,d\nu(z)\le C\|T^kf\|^p_{\Delta_{p,2\beta-k}}$$

holds for all $f\in\mathcal{C}^\infty(S)$.

Let $z=r\eta$, where $\eta\in S$ and $0<r<1$. By Lemma 1,

$$T^{k+1}Cf(r\eta)=c_kT^1\int_S T^kf(\zeta)\frac{(1-\langle\zeta,r\eta\rangle)^k}{(1-\langle r\eta,\zeta\rangle)^{n+k}}\,d\sigma(\zeta),$$

where T^1 is one of the operators $T_{i,j}$. It follows that there is a point w_1 on the ball B_n satisfying $\langle w_1,\eta\rangle=0$ for which

$$T^{k+1}Cf(r\eta)=c_k\int_S T^kf(\zeta)\frac{(1-\langle\zeta,r\eta\rangle)^k}{(1-\langle r\eta,\zeta\rangle)^{n+k+1}}\langle rw_1,\zeta\rangle\,d\sigma(\zeta).$$

We again use the parametrization of S given by

$$\zeta=e^{it}(\sqrt{1-|w|^2}\,\eta-w),$$

where w ranges over the $(2n-2)$-dimensional subset of the unit ball, $B'(\eta)$, and $0\le t\le 2\pi$. Recalling that

$$d\sigma(\zeta)=\frac{1}{2\pi}\,dt\,d\nu'(w),$$

where $d\nu'$ is normalized $(2n-2)$-dimensional Lebesgue measure on $B'(\eta)$, it follows that $(1/c_k2\pi)T^{k+1}Cf(r\eta)$ may be written as

$$\text{(3.1)}\qquad \int_0^{2\pi}\int_{B'(\eta)}T^kf(\zeta)\frac{(1-r\sqrt{1-|w|^2}\,e^{it})^kre^{-it}}{(1-r\sqrt{1-|w|^2}\,e^{-it})^{n+k+1}}\langle w_1,w\rangle\,d\nu'(w)\,dt.$$

Since

$$\int_{B'(\eta)}\frac{(1-r\sqrt{1-|w|^2}\,e^{it})^kre^{-it}}{(1-r\sqrt{1-|w|^2}\,e^{-it})^{n+k+1}}\langle w_1,w\rangle\,d\nu'(w)=0,$$

for each t, (3.1) remains unchanged if we replace $T^k f(\zeta)$ by

$$T^k f(\zeta) - T^k f(e^{it}\eta).$$

Taking absolute value signs inside the integrals therefore gives the estimate

$$\begin{aligned}
&|T^{k+1}Cf(r\eta)| \\
&\leq C\int_0^{2\pi}\int_{B'(\eta)} \frac{|T^k f(\zeta) - T^k f(e^{it}\eta)||w|}{|1 - r\sqrt{1-|w|^2}\, e^{it}|^{n+1}}\, d\nu\, dt \\
&= C\int_0^{2\pi}\int_0^1\int_{S'(\eta)} \frac{|T^k f(\Phi(e^{it}\eta, \rho e^{it}\lambda)) - T^k f(e^{it}\eta)|\rho^{2n-2}}{|1 - r\sqrt{1-\rho^2}\, e^{it}|^{n+1}}\, d\sigma'\, d\rho\, dt,
\end{aligned}$$

where $\Phi(\eta, w)$ is given by equation (1.5). Note that

$$\Delta g(\eta, w) = g(\Phi(\eta, w)) - g(\eta).$$

Minkowski's integral inequality implies that

$$\left(\int_S |T^{k+1}Cf(r\eta)|^p\, d\sigma(\eta)\right)^{1/p}$$

is dominated by a constant times

$$\int_0^{2\pi}\int_0^1 \left(\int_S\left(\int_{S'(\eta)} |\Delta T^k f(e^{it}\eta, \rho e^{it}\lambda)|\, d\sigma'(\lambda)\right)^p d\sigma(\eta)\right)^{1/p} \frac{\rho^{2n-2}\, d\rho\, dt}{|1 - r\sqrt{1-\rho^2}\, e^{it}|^{n+1}}.$$

Notice that

$$\int_S\left(\int_{S'(\eta)} |\Delta T^k f(e^{it}\eta, \rho e^{it}\lambda)|\, d\sigma'(\lambda)\right)^p d\sigma(\eta)$$

equals

$$\int_S\left(\int_{S'(e^{-it}\eta)} |\Delta T^k f(\eta, \rho e^{it}\lambda)|\, d\sigma'(\lambda)\right)^p d\sigma(\eta)$$

which, since $S'(e^{it}\eta) = S'(\eta)$, is equal to

$$\int_S\left(\int_{S'(\eta)} |\Delta T^k f(\eta, \rho\lambda)|\, d\sigma'(\lambda)\right)^p d\sigma(\eta),$$

which depends on ρ but not t.

Define then

$$G(\rho) = \left(\int_S\left(\int_{S'(\eta)} |\Delta T^k f(\eta, \rho\lambda)|\, d\sigma'(\lambda)\right)^p d\sigma(\eta)\right)^{1/p}.$$

It follows therefore that

$$\left(\int_S |T^{k+1}Cf(r\eta)|^p\, d\sigma(\eta)\right)^{1/p} \leq C\int_0^{2\pi}\int_0^1 \frac{G(\rho)\rho^{2n-2}}{|1 - r\sqrt{1-\rho^2}\, e^{it}|^{n+1}}\, d\rho\, dt.$$

If we interchange the order of integration and integrate out the variable t, then elementary considerations yield the estimate

$$\left(\int_S |T^{k+1}Cf(r\eta)|^p\, d\sigma(\eta)\right)^{1/p} \leq C\int_0^1 \frac{G(\rho)\rho^{2n-2}}{(1-r)^n + \rho^{2n}}\, d\rho. \tag{3.2}$$

Majorize the right-hand side of (3.2) by a constant times the sum

$$\int_0^{\sqrt{1-r}} \frac{G(\rho)}{1-r}\,d\rho + \int_{\sqrt{1-r}}^1 \frac{G(\rho)}{\rho^2}\,d\rho. \tag{3.3}$$

Combine (3.2) and (3.3) to obtain the estimate

$$\int_0^1 \int_S |T^{k+1}Cf(r\eta)|^p\,d\sigma(\eta)(1-r)^{p((k+1)/2-\beta)-1}\,dr \leq C(I_1+I_2). \tag{3.4}$$

where

$$I_1 = \int_0^1 \left(\int_0^{\sqrt{1-r}} \frac{G(\rho)}{1-r}\,d\rho\right)^p (1-r)^{p((k+1)/2-\beta)-1}\,dr$$

and

$$I_2 = \int_0^1 \left(\int_{\sqrt{1-r}}^1 \frac{G(\rho)}{\rho^2}\,d\rho\right)^p (1-r)^{p((k+1)/2-\beta)-1}\,dr.$$

Use Hardy's inequality to conclude that

$$\begin{aligned} I_1 &= \int_0^1 \left(\int_0^{\sqrt{1-r}} G(\rho)\,d\rho\right)^p (1-r)^{p((k-1)/2-\beta)-1}\,dr \\ &= 2\int_0^1 \left(s^{(k-1-2\beta)} \int_0^s G(\rho)\,d\rho\right)^p \frac{ds}{s} \\ &\leq C\int_0^1 s^{p(k-2\beta)-1} G(s)^p\,ds, \end{aligned}$$

where we have made the substitution $s=\sqrt{1-r}$. Argue in a similar fashion to deduce that

$$\begin{aligned} I_2 &= \int_0^1 \left(\int_{\sqrt{1-r}}^1 \frac{G(\rho)}{\rho^2}\,d\rho\right)^p (1-r)^{p((k+1)/2-\beta)-1}\,dr \\ &= 2\int_0^1 \left(s^{(k+1-2\beta)} \int_s^1 \frac{G(\rho)}{\rho^2}\,d\rho\right)^p \frac{ds}{s} \\ &\leq C\int_0^1 s^{p(k-2\beta)+2p-1}(s^{-2}G(s))^p\,ds \\ &= C\int_0^1 s^{p(k-2\beta)-1} G(s)^p\,ds. \end{aligned}$$

From the definition of G and the fact that $1<p<\infty$, it follows that

$$G(s)^p \leq \int_S \int_{S'(\eta)} |\Delta T^k f(\eta, s\lambda)|^p\,d\sigma'(\lambda)\,d\sigma(\eta).$$

Therefore

$$I_1+I_2 \leq C\int_0^1 s^{p(k-2\beta)-1} \int_S \int_{S'(\eta)} |\Delta T^k f(\eta, s\lambda)|^p\,d\sigma'(\lambda)\,d\sigma(\eta)\,ds.$$

If we insert this last estimate into (3.4), we arrive at the conclusion that

$$\int_{B_n} |T^{k+1}Cf(r\eta)|^p (1-r)^{p((k+1)/2-\beta)-1}\, d\nu(r\eta)$$

is dominated by an absolute constant times

$$\int_0^1 s^{p(k-2\beta)-1} \int_S \int_{S'(\eta)} |\Delta T^k f(\eta, s\lambda)|^p \, d\sigma'(\lambda)\, d\sigma(\eta)\, ds \le \|T^k f\|^p_{\Delta_{p,2\beta-k}}.$$

This completes the proof. □

The proof of Theorem C′ proceeds in a similar fashion.

Proof of Theorem C′. Recall that for η and w in S with $\langle \eta, w\rangle = 0$ we have

$$\Delta^2 g(\eta, w) = g(\Phi(\eta, w)) + g(\Phi(\eta, -w)) - 2g(\eta),$$

where Φ is defined by equation (3). Again, let T^k denote an operator obtained by composing k of the operators $T_{i,j}$. Then T^{k+2} has the form T^2T^k. By the results [1] referred to earlier, it is sufficient to show that there is a constant C such that the inequality

$$\int_{B_n} |T^{k+2}Cf(z)|^p (1-|z|)^{p((k+2)/2-\beta)-1}\, d\nu(z) \le C\|T^k f\|^{p2}_{\Delta_{p,2\beta-k}}$$

holds for all $f \in \mathcal{C}^\infty(S)$.

By Lemma 1,

$$T^{k+2}Cf(r\eta) = c_k T_1 T_2 \int_S T^k f(\zeta) \frac{(1-\langle \zeta, r\eta\rangle)^k}{(1-\langle r\eta, \zeta\rangle)^{n+k}}\, d\sigma(\zeta),$$

where T_1 and T_2 are chosen among the $T_{i,j}$. It follows that there are points w_1 and w_2 on the sphere S satisfying $\langle w_i, \eta\rangle = 0$, $i = 1, 2$, for which

$$T^{k+2}Cf(r\eta) = c_k \int_S T^k f(\zeta) \frac{(1-\langle \zeta, r\eta\rangle)^k}{(1-\langle r\eta, \zeta\rangle)^{n+k+2}} \prod_{i=1}^{2} \langle rw_i, \zeta\rangle \, d\sigma(\zeta).$$

We will use parametrizations of S given by

$$\zeta = e^{it}(\sqrt{1-|w|^2}\,\eta + w) \quad \text{and} \quad \zeta' = e^{it}(\sqrt{1-|w|^2}\,\eta - w),$$

where once again w ranges over the $(2n-2)$-dimensional subset of the unit ball in C^n,

$$B'(\eta) = \{w \in B_n : \langle w, \eta\rangle = 0\},$$

and $0 \le t \le 2\pi$. Then we may write

$$d\sigma(\zeta) = \frac{1}{2\pi}\, dt\, d\nu'(w) \quad \text{and} \quad d\sigma(\zeta') = \frac{1}{2\pi}\, dt\, d\nu'(w),$$

where $d\nu'$ is normalized $(2n-2)$-dimensional Lebesgue measure on $B'(\eta)$. It follows that $(1/c_k 2\pi)T^{k+2}Cf(r\eta)$ may be written as either

$$(3.5) \quad \int_0^{2\pi} \int_{B'(\eta)} T^k f(\zeta) \frac{(1-r\sqrt{1-|w|^2}\, e^{it})^k r^2 e^{-2it}}{(1-r\sqrt{1-|w|^2}\, e^{-it})^{n+k+2}} \prod_{i=1}^{2} \langle w_i, w\rangle \, d\nu'(w)\, dt$$

or

$$(3.6)\quad \int_0^{2\pi}\int_{B'(\eta)} T^k f(\zeta')\frac{(1-r\sqrt{1-|w|^2}\,e^{it})^k r^2 e^{-2it}}{(1-r\sqrt{1-|w|^2}\,e^{-it})^{n+k+2}}\prod_{i=1}^{2}\langle w_i, w\rangle\, dv'(w)\, dt,$$

because

$$\prod_{i=1}^{2}\langle w_i, w\rangle = \prod_{i=1}^{2}\langle w_i, -w\rangle.$$

Since

$$\int_{B'(\eta)} \frac{(1-r\sqrt{1-|w|^2}\,e^{it})^k r^2 e^{-2it}}{(1-r\sqrt{1-|w|^2}\,e^{-it})^{n+k+2}}\prod_{i=1}^{2}\langle w_i, w\rangle\, dv'(w) = 0,$$

for each t, (3.5) and (3.6) remain unchanged if we replace $T^k f(\zeta)$ or $T^k f(\zeta')$ by

$$T^k f(\zeta) - T^k f(e^{it}\eta) \quad \text{or} \quad T^k f(\zeta') - T^k f(e^{it}\eta).$$

Adding the modified versions of (3.5) and (3.6) and taking absolute value signs inside the integrals therefore gives the estimate

$$|T^{k+2}Cf(r\eta)| \le C\int_0^{2\pi}\int_{B'(\eta)} \frac{|\Delta^2 T^k f(e^{it}\zeta, e^{it}w)|\,|w|^2}{|1-r\sqrt{1-|w|^2}\,e^{it}|^{n+2}}\, dv\, dt.$$

The remainder of the proof follows the path taken in the proof of Theorem C from the point where Minkowski's integral inequality was applied; we omit the details. □

We now establish Theorem D, which may be regarded as a converse to Theorems C and C′, at least for functions that are holomorphic on B_n. We will need to know that certain mappings are uniformly local diffeomorphisms. To prepare for this we state the following result.

LEMMA 2. *Let $U: R^n \to R^n$ be given by*

$$U(x) = x + tW(x),$$

where $x = (x_1, \ldots, x_n)$ and $W(x) = (W_1(x), \ldots, W_n(x))$, and suppose that there exists a constant M such that

$$\|W_j\|_\infty < M, \quad \left\|\frac{\partial W_j}{\partial x_i}\right\|_\infty < M, \quad \text{and} \quad \left\|\frac{\partial^2 W_j}{\partial x_k \partial x_i}\right\|_\infty < M$$

for all j, i, and k between 1 and n. Then there exists a constant ϵ independent of W such that, for $M|t| < \epsilon$ and $y \in R^n$, U is 1:1 on the ball centered at y of radius 1/4 and furthermore

$$|\text{Det}\, U(x)| > 1/2$$

for all $x \in R^n$.

Proof. The proof of the inverse function theorem given in [5] can easily be modified to yield the lemma. □

Proof of Theorem D(a). Suppose that $F \in B^p_\beta$ and $k/2 \leq \beta$. Since $p > 1$ and since $B^p_\beta \subseteq H^p_\beta$ for $p \leq 2$, it follows that $F \in H^1_\beta$ and therefore $R^{k/2}F \in H^1$. By the results in [1] it follows that, if T^k is an operator obtained by composing k of the $T_{i,j}$, then T^kF has admissible limits a.e. $d\sigma$ and the boundary function of T^kF is in $L^1(d\sigma)$. Since F_r converges to F in B^p_β, it is therefore enough to prove the inequality

$$\int_0^1 \int_S \int_{S'(\eta)} |\Delta F(\eta, \rho\lambda)|^p \, d\sigma'(\lambda) \, d\sigma(\eta) \, \frac{d\rho}{\rho^{1+p(2\beta-k)}} \leq C \|R^{1+\beta}F\|^p_{p,p-1}$$

for all functions F which are holomorphic on a neighborhood of B_n.

Assuming then that F is holomorphic on a neighborhood of the closure of B_n, write

$$T^kF(\sqrt{1-\rho^2}\,\eta + \rho\lambda) - T^kF(\eta) = I_1(\eta, \rho\lambda) + I_2(\eta, \rho\lambda) + I_3(\eta, \rho\lambda), \tag{3.7}$$

where

$$I_1(\eta, \rho\lambda) = \int_0^\rho \frac{\partial}{\partial\phi} T^kF(r(\sqrt{1-\phi^2}\,\eta + \phi\lambda)) \, d\phi, \tag{3.8}$$

$$I_2(\eta, \rho\lambda) = \int_r^1 \frac{\partial}{\partial s} T^kF(s(\sqrt{1-\rho^2}\,\eta + \rho\lambda)) \, ds, \tag{3.9}$$

$$I_3(\eta, \rho\lambda) = -\int_r^1 \frac{\partial}{\partial s} T^kF(s\eta) \, ds, \tag{3.10}$$

and r is a function of ρ that we will choose later. If we set

$$\Phi(\eta, w) = \sqrt{1-|w|^2}\,\eta + w$$

as before, it is not hard to show from (3.8), (3.9), and (3.10) that

$$|I_1(\eta, \rho\lambda)| \leq C \int_0^\rho |XF(r(\Phi(\eta, \phi\lambda)))| \, d\phi,$$

$$|I_2(\eta, \rho\lambda)| \leq C \int_r^1 |YF(s(\Phi(\eta, \phi\lambda)))| \, ds,$$

and

$$|I_3(\eta, \rho\lambda)| \leq C \int_r^1 |ZF(s\eta)| \, ds,$$

where, in the terminology of [1], X is a differential operator of "weight" $(k+1)/2$ and Y and Z are differential operators of weight $(k+2)/2$.

For each fixed value of η_0 on S we may find a neighborhood N of η_0 for which it is possible to choose $n-1$ smooth functions $\omega_j(\eta)$, $j = 1, \ldots, n-1$ such that for each $\eta \in N$ the set $\{\omega_j(\eta) : j = 1, \ldots, n-1\}$ is an orthonormal basis for $C^n \ominus \eta C$. If η belongs to N then it follows that $S'(\eta)$ may be parametrized by

$$\lambda(\xi) = \lambda_n(\xi) = \sum_{j=1}^{n-1} \xi_j \omega_j(\eta),$$

where

$$\xi = (\xi_1, \ldots, \xi_{n-1}) \in S_{2n-3}.$$

We may therefore write

$$V(\eta, \phi\xi) = \sqrt{1-\phi^2}\,\eta + \phi \sum_{j=1}^{n-1} \xi_j \omega_j(\eta),$$

and assert that with $w = \phi\lambda$

$$\Phi(\eta, w) = V(\eta, \tau), \quad \tau = \phi\xi.$$

Fixing τ, we may apply the result of Lemma 2 to the map V by letting $\phi/\sqrt{1-\phi^2}$ play the role of t and $\sum_{j=1}^{n-1} \xi_j \omega_j(\eta)$ play the role of W and conclude that there are constants δ_1, δ_2, and C such that for any η_0 on S it is possible to choose N so that N contains all points ζ on the sphere satisfying $|\zeta - \eta| < \delta_1$, and for this N and all $|\tau| < \delta_2$ the measure $V_*(d\sigma)$ defined on $V(N)$ by

$$V_*(d\sigma)(E) = \sigma(V^{-1}(E))$$

is given by the equation

$$V_*(d\sigma)(\zeta) = H(\zeta)\, d\sigma(\zeta),$$

where $\|H\|_\infty \le C$.

Cover the sphere S with finitely many neighborhoods $N_1, \ldots, N_m$, where each N_l is as the neighborhood N described above. Then for $j = 1, \ldots, 3$,

$$\int_S \int_{S'(\eta)} |I_j(\eta, \rho\lambda)|^p \, d\sigma'(\lambda)\, d\sigma(\eta)$$

is less than a finite sum of the integrals

$$\begin{aligned} &\int_{N_l} \int_{S'(\eta)} |I_j(\eta, \rho\lambda)|^p \, d\sigma'(\lambda)\, d\sigma(\eta) \\ (3.11) \qquad &= \int_{S_{2n-3}} \int_{N_l} |I_j(\eta, \rho\lambda(\xi))|^p \, d\sigma(\eta)\, d\sigma'(\xi), \end{aligned}$$

where $\lambda(\xi)$ (which depends on l) is the function defined above. If $j = 1$, the $(1/p)$th power of (3.11) is bounded by a constant times

$$\int_0^\rho \left(\int_{S_{2n-3}} \int_{N_l} |XF(r(\Phi(\eta, \phi\lambda(\xi)))|^p \, d\sigma(\eta)\, d\sigma'(\xi) \right)^{1/p} d\phi.$$

Changing variables with the help of the remarks based on Lemma 2 allows us to conclude that there is a constant C such that for $\phi \le \delta_2$,

$$\begin{aligned} \int_{N_l} |XF(r(\Phi(\eta, \phi\lambda(\xi)))|^p \, d\sigma(\eta) &= \int_{N_l} |XF(r(V(\eta, \tau))|^p \, d\sigma(\eta) \\ &= \int_{V(N_l)} |XF(r\eta)|^p \, V_*(d\sigma)(\eta) \\ &\le C \int_S |XF(r\eta)|^p \, d\sigma(\eta). \end{aligned}$$

Thus, for $\rho \le \delta_2$, the $(1/p)$th power of (3.11) is bounded by a constant times

$$\int_0^\rho \left(\int_{S_{2n-3}} \int_S |XF(r\eta)|^p \, d\sigma(\eta)\, d\sigma'(\xi) \right)^{1/p} d\phi = \int_0^\rho \left(\int_{S_{2n-3}} \|XF_r\|_p^p \, d\sigma'(\xi) \right)^{1/p} d\phi$$
$$= \rho \|XF_r\|_p.$$

If $j=2$ a similar argument yields a bound on the $(1/p)$th power of (3.11) of

$$C \int_r^1 \|YF_s\|_p \, ds.$$

For $j=3$ the conclusion that the $(1/p)$th power of (3.11) is bounded by

$$C \int_r^1 \|ZF_s\|_p \, ds$$

is immediate.

Recall that

$$\Delta T^k F(\eta, \rho\lambda) = T^k F(\sqrt{1-\rho^2}\,\eta + \rho\lambda) - T^k F(\eta).$$

Then from (3.7), the estimates just discussed, and the fact that it is behavior near $\rho = 0$ which is important, it follows that

$$\int_0^1 \int_S \int_{S'(\eta)} |\Delta T^k F(\eta, \rho\lambda)|^p \, d\sigma'(\lambda)\, d\sigma(\eta) \frac{d\rho}{\rho^{1+p(2\beta-k)}}$$

is bounded by a constant times the sum

$$\int_0^1 \rho^{p(k+1-2\beta)-1} \|XF_r\|_p^p \, d\rho + \int_0^1 \rho^{p(k-2\beta)-1} \left(\int_r^1 \|YF_s\|_p + \|ZF_s\|_p \, ds \right)^p d\rho.$$

Recall that r was an arbitrary function of ρ. We choose $r = 1-\rho^2$ and write the last two integrals in terms of r and get $1/2$ times the sum

$$\int_0^1 (1-r)^{p((k+1)/2-\beta)-1} \|XF_r\|_p^p \, dr + \int_0^1 \left((1-r)^{(k/2-\beta)} \int_r^1 \|YF_s\|_p + \|ZF_s\|_p \, ds \right)^p \frac{dr}{1-r}.$$

Apply Hardy's inequality to the second term and deduce that the last sum is bounded by a constant times

$$\int_0^1 (1-r)^{p((k+1)/2-\beta)-1} \|XF_r\|_p^p \, dr + \int_0^1 (1-r)^{p((k+2)/2-\beta)-1} \|YF_r\|_p^p + \|ZF_r\|_p^p \, dr.$$

Since X has weight $(k+1)/2$ and both Y and Z have weight $(k+2)/2$ and since $\beta < (k+1)/2$, it follows from [1] that the last sum is dominated by a constant multiple of $\|R^{1+\beta}F\|_{p,p-1}^p$. This concludes the proof. □

Proof of Theorem D(b). We note again that it is enough to prove the inequality

$$\int_0^1 \int_S \int_{S'(\eta)} |\Delta^2 F(\eta, \rho\lambda)|^p \, d\sigma'(\lambda)\, d\sigma(\eta) \frac{d\rho}{\rho^{1+p(2\beta-k)}} \le C \|R^{1+\beta}F\|_{p,p-1}^p$$

for all functions F which are holomorphic on a neighborhood of B_n. Write

$$\Delta^2 T^k F(\eta, \rho\lambda) = I_1(\eta, \rho\lambda) + I_2(\eta, \rho\lambda), \tag{3.12}$$

where

$$I_1(\eta, \rho\lambda) = \Delta^2 T^k F_r(\eta, \rho\lambda)$$

and

$$I_2(\eta, \rho\lambda) = \int_r^1 \frac{\partial}{\partial s}(T^k F(s(\Phi(\eta, \rho\lambda))) + T^k F(s(\Phi(\eta, -\rho\lambda))) - 2T^k F(s(\eta)))\, ds,$$

and r is a function of ρ that we will choose later. Then

$$I_1(\eta, \rho\lambda) = \int_0^\rho \int_{-\phi}^{\phi} \frac{d^2}{d\theta^2} T^k F_r(\Phi(\eta, \theta\lambda))\, d\theta\, d\phi,$$

and it can be seen that

$$|I_1(\eta, \rho\lambda)| \le C \int_0^\rho \int_{-\phi}^{\phi} |XF_r(\Phi(\eta, \theta\lambda))|\, d\theta\, d\phi,$$

where X is a differential operator of weight $(k+2)/2$. It also can be seen that $|I_2(\eta, \rho\lambda)|$ is dominated by a sum of the terms

$$\int_r^1 |YF(s(\Phi(\eta, \phi\lambda)))|\, ds,$$

$$\int_r^1 |ZF(s(\Phi(\eta, -\phi\lambda)))|\, ds,$$

and

$$\int_r^1 |WF(s\eta)|\, ds,$$

where Y, Z, and W are also differential operators of weight $(k+2)/2$.

Once again cover the sphere S with finitely many neighborhoods $N_1, \ldots, N_m$, where each N_l is as the neighborhood N described in the proof of Theorem D(a). Then for $j = 1, \ldots, 4$,

$$\int_S \int_{S'(\eta)} |I_j(\eta, \rho\lambda)|^p\, d\sigma'(\lambda)\, d\sigma(\eta)$$

is less than a finite sum of the integrals

$$\begin{aligned} &\int_{N_l} \int_{S'(\eta)} |I_j(\eta, \rho\lambda)|^p\, d\sigma'(\lambda)\, d\sigma(\eta) \\ &\quad = \int_{S_{2n-3}} \int_{N_l} |I_j(\eta, \rho\lambda(\xi))|^p\, d\sigma(\eta)\, d\sigma'(\xi). \end{aligned} \tag{3.13}$$

If $j = 1$, the $(1/p)$th power of (3.13) is bounded by a constant times

$$\int_0^\rho \int_{-\phi}^{\phi} \left(\int_{S_{2n-3}} \int_{N_l} |XF(r(\Phi(\eta, \theta\lambda(\xi))))|^p\, d\sigma(\eta)\, d\sigma'(\xi) \right)^{1/p} d\theta\, d\phi.$$

It follows as before that the $(1/p)$th power of (3.13) is bounded by a constant times

$$\int_0^\rho \int_{-\phi}^{\phi} \left(\int_{S_{2n-3}} \int_S |XF(r\eta)|^p \, d\sigma(\eta) \, d\sigma'(\xi) \right)^{1/p} d\theta \, d\phi$$
$$= \int_0^\rho \int_{-\phi}^{\phi} \left(\int_{S_{2n-3}} \|XF_r\|_p^p \, d\sigma'(\xi) \right)^{1/p} d\theta \, d\phi$$
$$= \int_0^\rho \int_{-\phi}^{\phi} \|XF_r\|_p \, d\theta \, d\phi$$
$$= C\rho^2 \|XF_r\|_p.$$

For $j = 2, \ldots, 4$, similar arguments yield successive bounds on the $(1/p)$th power of (3.13) by

$$C \int_r^1 \|YF_s\|_p \, ds,$$

$$C \int_r^1 \|ZF_s\|_p \, ds$$

and

$$C \int_r^1 \|WF_s\|_p \, ds.$$

Therefore, from (3.12) and the estimates just discussed it follows that

$$\int_0^1 \int_S \int_{S'(\eta)} |\Delta^2 T^k F(\eta, \rho\lambda)|^p \, d\sigma'(\lambda) \, d\sigma(\eta) \frac{d\rho}{\rho^{1+p(2\beta-k)}}$$

is bounded by a constant times the sum

$$\int_0^1 \rho^{p(k+2-2\beta)-1} \|XF_r\|_p^p \, d\rho + \int_0^1 \rho^{p(k-2\beta)-1} \left(\int_r^1 G(s) \, ds \right)^p d\rho,$$

where $G(s) = \|YF_s\|_p + \|ZF_s\|_p + \|WF_s\|_p$. We now choose $r = 1 - \rho^2$ and write the last two integrals in terms of r and get $1/2$ times the sum

$$\int_0^1 (1-r)^{p((k+2)/2-\beta)-1} \|XF_r\|_p^p \, dr + \int_0^1 \left((1-r)^{(k/2-\beta)} \int_r^1 G(s) \, ds \right)^p \frac{dr}{1-r}.$$

Since $k/2 < \beta$, we may apply Hardy's inequality to the second term and deduce that the last sum is bounded by a constant times the sum

$$\int_0^1 (1-r)^{p((k+2)/2-\beta)-1} \|XF_r\|_p^p \, dr$$
$$+ \int_0^1 (1-r)^{p((k+2)/2-\beta)-1} (\|YF_r\|_p^p + \|ZF_r\|_p^p + \|WF_r\|_p^p) \, dr.$$

Since X, Y, Z, and W have weight $(k+2)/2$ and since $\beta < (k+2)/2$, it follows from [1] that the last sum is dominated by a constant multiple of $\|R^{1+\beta}F\|_{p,p-1}^p$. This concludes the proof. □

The remainder of the paper is concerned with the proof of Theorem E. We require some preliminary groundwork.

LEMMA 3 (Krein). *Let H be a Hilbert space with inner product $\langle\ ,\ \rangle$. Suppose that Y is a dense subspace of H and that there is a norm $\|\ \|_Y$ defined on Y which makes Y a Banach space and for which the inclusion map $i: Y \to H$ is bounded. Let T and T^* be operators defined and bounded on $(Y, \|\ \|_Y)$ satisfying the relation $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x, y \in Y$. Then T and T^* extend to be bounded operators on H.*

Proof. Suppose that $x \in Y$. Then

$$\begin{aligned}\|Tx\|_H^2 &= \langle Tx, Tx\rangle \\ &= \langle x, T^*Tx\rangle \\ &\le \|x\|_H \|T^*Tx\|_H.\end{aligned}$$

If we square both sides of the last inequality, we get that

$$\begin{aligned}\|Tx\|_H^{2^2} &\le \|x\|_H^2 \|T^*Tx\|_H^2 \\ &= \|x\|_H^2 \langle x, (T^*T)^2 x\rangle \\ &\le \|x\|_H^{2^2-1} \|(T^*T)^2 x\|_H.\end{aligned}$$

Squaring once more gives

$$\begin{aligned}\|Tx\|_H^{2^3} &\le \|x\|_H^{2^3-2} \|(T^*T)^2 x\|_H^2 \\ &= \|x\|_H^{2^3-2} \langle (T^*T)^2 x, (T^*T)^2 x\rangle \\ &= \|x\|_H^{2^3-2} \langle x, (T^*T)^2 (T^*T)^2 x\rangle \\ &\le \|x\|_H^{2^3-1} \|(T^*T)^{2^2} x\|_H.\end{aligned}$$

In general the following relation holds:

$$\|Tx\|_H^{2^n} \le \|x\|_H^{2^n-1} \|(T^*T)^{2^{n-1}} x\|_H.$$

Since $i: Y \to H$ is continuous, there is a constant C such that $\|x\|_H \le C\|x\|_Y$ for all $x \in Y$. Therefore

$$\|Tx\|_H^{2^n} \le C \|x\|_H^{2^n-1} \|(T^*T)^{2^{n-1}} x\|_Y.$$

Let $\|T^*T\|$ denote the $\|\ \|_Y$ norm of the operator T^*T. It follows that

$$\|Tx\|_H^{2^n} \le C \|(T^*T)\|^{2^{n-1}} \|x\|_H^{2^n-1} \|x\|_Y.$$

Taking (2^n)th roots of both sides of the last inequality and letting n go to ∞ gives

$$\|Tx\|_H \le \|(T^*T)\|^{1/2} \|x\|_H.$$

This completes the proof. □

Notice that the proof also gives the relation

$$\|T\|_H \le \|(T^*T)\|_Y^{1/2}.$$

For $m=0,1,\ldots$ and $f\in L^2(d\sigma)$ define

$$Hf(z)=H_m f(z)=\int_S f(\zeta)\frac{(1-\langle\zeta,z\rangle)^m}{(1-\langle z,\zeta\rangle)^{n+m}}\,d\sigma(\zeta).$$

LEMMA 4. *There is a constant $C=C(m)$ such that*

$$\sup_{0<r<1}\int_S |Hf(r\eta)|^2\,d\sigma(\eta)\le C\|f\|_2^2.$$

Proof. For $0<r<1$ and a fixed m let H_r $(=(H_m)_r)$ be the operator H_r: $L^2\to L^2$ defined by

$$H_r f(\eta)=\int_S f(\zeta)\frac{(1-\langle\zeta,r\eta\rangle)^m}{(1-\langle r\eta,\zeta\rangle)^{n+m}}\,d\sigma(\zeta).$$

We must find a constant $C(m)$ such that $\|H_r\|\le C(m)$. Notice that $H_r=H_r^*$. By the inequality given at the end of the proof of Lemma 3, it is therefore sufficient to prove that there is a constant C such that

$$\|H_r f\|_Y\le C\|f\|_Y$$

for all $f\in Y$, where Y is some Bánach space continuously contained in $L^2(d\sigma)$ which is also dense in $L^2(d\sigma)$. We choose Y to be a Lipschitz space. Fix an α between 0 and 1 and let Λ denote the space of functions continuous on S satisfying

$$|f(\zeta)-f(\eta)|\le C|\zeta-\eta|^\alpha. \tag{3.14}$$

Norm Λ by

$$\|f\|_\Lambda=\|f\|_\infty+C_f,$$

where C_f is the infimum of all C satisfying (3.14). Let G denote any first-order derivative ($\partial/\partial z_j$ or $\partial/\partial\bar{z}_j$) of $Hf(z)$. It is enough to show that there is a constant $C(m)$ such that

$$|G(z)|\le C(m)\|f\|_\Lambda(1-|z|)^{\alpha-1}.$$

Write $z=r\eta$ where $|\eta|=1$. Then either

$$G(z)=\int_S f(\zeta)\frac{(1-\langle\zeta,z\rangle)^{m-1}\zeta_j}{(1-\langle z,\zeta\rangle)^{n+m}}\,d\sigma(\zeta)$$

or

$$G(z)=\int_S f(\zeta)\frac{(1-\langle\zeta,z\rangle)^{m}\bar{\zeta}_j}{(1-\langle z,\zeta\rangle)^{n+m+1}}\,d\sigma(\zeta),$$

for some j, where we have ignored harmless factors. Both expressions can be handled in the same way; we consider only the second one. As in the proof of Theorem C, use the parametrization

$$\zeta=e^{it}(\sqrt{1-|w|^2}\,\eta+w)$$

to write

$$G(z)=\int_{B'(\eta)}\frac{1}{2\pi}\int_0^{2\pi}\frac{f(\zeta)(1-r\sqrt{1-|w|^2}e^{it})^m\bar{\zeta}_j}{(1-r\sqrt{1-|w|^2}e^{-it})^{n+m+1}}\,dt\,dv'. \tag{3.15}$$

It can be verified that

$$\left|\int_0^{2\pi}\frac{(1-r\sqrt{1-|w|^2}e^{it})^m e^{-it}}{(1-r\sqrt{1-|w|^2}e^{-it})^{n+m+1}}\,dt\right|\le C(m),$$

where $C(m)$ is independent of η, r or w. Proceed as in [3] and replace $f(\zeta)$ in (3.15) by the difference $f(\zeta)-f(e^{-it}\zeta)$. Since $f(e^{-it}\zeta)$ depends only on w and not t, this introduces a bounded error term and yields the estimate

$$|G(z)|\le C(m)\|f\|_\infty + C\int_{B'(\eta)}\int_0^{2\pi}\frac{|f(\zeta)-f(e^{-it}\zeta)|}{|1-r\sqrt{1-|w|^2}e^{it}|^{n+1}}\,dt\,dv'. \tag{3.16}$$

Since $f\in\Lambda$, (3.14) and (3.16) imply that

$$|G(z)|\le C(m)\|f\|_\infty + C\|f\|_\Lambda\int_{B'(\eta)}\int_0^{2\pi}\frac{|t|^\alpha}{|1-r\sqrt{1-|w|^2}e^{it}|^{n+1}}\,dt\,dv'. \tag{3.17}$$

Working with the second term in (3.17), we calculate that

$$\begin{aligned}\int_{B'(\eta)}\int_0^{2\pi}\frac{|t|^\alpha}{|1-r\sqrt{1-|w|^2}e^{it}|^{n+1}}\,dt\,dv' &= C\int_0^{2\pi}\int_0^1\frac{\rho^{2n-3}|t|^\alpha}{|1-r\sqrt{1-\rho^2}e^{it}|^{n+1}}\,d\rho\,dt\\ &\le C\int_0^1\frac{t^\alpha}{(1-r)^2+t^2}\,dt\\ &\le C(1-r)^{\alpha-1}.\end{aligned}$$

Inserting this estimate into (3.13) gives the desired estimate and completes the proof. □

We will actually need to know that the operator H is bounded from L^p to L^p when $1<p<\infty$; that is, there is a constant $C=C(m,p)$ such that

$$\sup_{0<r<1}\int_S |Hf(r\eta)|^p\,d\sigma(\eta)\le C\|f\|_p^p.$$

To this end we state the following lemma.

LEMMA 5. *Suppose* $\zeta,\eta,\omega\in S$, $\alpha>1$, $|1-\langle\zeta,\eta\rangle|^{1/2}\le\delta$, $|1-\langle\omega,\zeta\rangle|^{1/2}\ge 2\delta$, *and* $|1-\langle z,\omega\rangle|<\alpha/2(1-|z|^2)$. *Let* $C_m(z,\zeta)$ *be the kernel*

$$C_m(z,\zeta)=\frac{(1-\langle\zeta,z\rangle)^m}{(1-\langle z,\zeta\rangle)^{n+m}}.$$

Then there is a constant $C(\alpha)$ *such that*

$$|C_m(z,\zeta)-C_m(z,\eta)|\le C(\alpha)\delta|1-\langle\omega,\zeta\rangle|^{-n-1/2}.$$

Proof. The proof given in [9] for the case where $m=0$ can be easily adapted to work for the general case where m is a nonnegative integer. □

LEMMA 6. *There is a constant $C(p,m)$ such that for $1<p<\infty$,*

$$\sup_{0<r<1}\int_S |Hf(r\eta)|^p\, d\sigma(\eta) \le C\|f\|_p^p.$$

Proof. The proof proceeds along the lines of the argument given in [9, Thm. 6.2.2]. The only difference is that instead of considering the maximal function $M_\alpha H[g]$ it is enough to work with $|H_r[g]|$ and apply the L^2 result given by Lemma 4. □

We can now prove Theorem E.

Proof of Theorem E. By the results of [1], it is enough to show that the radial maximal function of the tangential derivative T^kCf is in L^p. If we use Lemma 1 it follows that $T^kCf = Hg$ for an L^p function g. Apply Lemma 6 to deduce that

$$\sup_{0<r<1}\int_S |TCf(r\eta)|^p\, d\sigma(\eta) \le C\|g\|_p^p.$$

Now use the fact that T^kCf is a harmonic function (see [1]) to conclude that its radial maximal function is in L^p. This completes the proof. □

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