Lewy Unsolvability and Several Complex Variables

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I. Introduction

The history of surjectivity of linear partial differential operators L may be thought of as beginning with the Cauchy-Kowalewski theorem, which shows that if L has analytic coefficients then L maps analytic functions locally onto analytic functions. Thus, in the local analytic realm, the equation

$$Lf = g \tag{1-1}$$

always has a solution.

Many "special" surjectivity $\mathcal{C}^\infty$ theorems were proved (e.g., for elliptic or hyperbolic L), but it took the development of the theory of distributions by L. Schwartz to yield a general surjectivity theorem independent of type. It was proven by Malgrange [15] and Ehrenpreis [4] in 1953 that surjectivity in (1-1) holds in the space of $\mathcal{C}^\infty$ functions, locally or globally (at least on convex sets), for operators with constant coefficients (see also [2]).

It was very surprising, therefore, that surjectivity on the $\mathcal{C}^\infty$ level (even locally) fails for an operator L that is first order and has linear coefficients. This was discovered by Lewy [13] in 1956. Lewy's operator is

$$L = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} - 2i(x+iy)\frac{\partial}{\partial t} \tag{1-2}$$

in the three variables x, y, t.

Somewhat later, Mizohata gave the example in two variables:

$$M = \frac{\partial}{\partial x} + ix\frac{\partial}{\partial t}. \tag{1-3}$$

Starting with Lewy's original paper, many proofs have been given for the unsolvability of (1-1). Some of these proofs, such as Hörmander's [10], have led to vast generalizations.

In this paper we shall present two new proofs of the unsolvability of L. We shall see that each proof puts L in a new setting and leads to an interesting theory.

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(a) The first proof depends on the theory of holomorphic functions of several complex variables. The unsolvability of (1-1) results from the existence of peak points in the topological algebra that is the kernel of L. The proof yields a removable singularities theorem for L.

(b) The second proof also depends on several complex variables. It is now the extension property of Hartogs type which accounts for the nonsolvability.

Both proofs (a) and (b) have ramifications in the area of topological algebra. They use the fact that L is first order so its kernel is an algebra.

Although our proofs are formally different and depend on different aspects of several complex variables (peak points vs. Hartogs' extension), there is an important principle underlying both proofs: Suppose we try to solve (1-1) for a g that satisfies an operator equation of the form

$$Ag=0, \tag{1-4}$$

where A is an operator that commutes with L. (In our applications A is a multiplication.) Then (1-1) gives

$$LAf=0. \tag{1-5}$$

That is, Af is in the kernel of L.

Suppose now that A has a right inverse R. Denote by Q the commutator of L with R. We might wonder whether $RAf=f$. Note that

$$A(RAf-f)=0 \tag{1-6}$$

because AR is the identity. Moreover,

$$L(RAf-f)=RLAf+QAf-g=QAf-g \tag{1-7}$$

because $LAf=Ag=0$.

Call $\alpha=RAf-f$. Thus α is in the kernel of A and (1-7) computes $L\alpha$. In our applications we can compute the kernel of A explicitly because A is multiplication by a simple function. Also, Af has a certain nice behavior because it is in the kernel of L. What is crucial is that the commutator Q is "nice". Thus $L\alpha$ is nice and this is impossible for something in the kernel of A unless $\alpha\equiv 0$.

This yields $g=QAf$. Thus g must be in the range of Q on the kernel of L. If this kernel has special properties such as Hartogs' extension, this limits the possibilities for g. This is proof (b).

For proof (a) we use duality; that is, we study the equation $L'f=g$. Now it is clear that if $Ag=0$ then for suitable h the value of $g\cdot Rh$ should be formally ∞. On the other hand,

$$L'f\cdot Rh=f\cdot LRh=f\cdot RLh+f\cdot Qh. \tag{1-8}$$

If $Lh=0$ and Q is nice then the left side of (1-8) is nice, contradicting the "non-niceness" of $g\cdot Rh$. This suggests that we find an appropriate approximation R_ϵ to R and then apply (1-1) to $R_\epsilon h$ to obtain a contradiction. This is proof (a).

In our paper [3] we study two other proofs of the nonsurjectivity of L. Our first proof is vaguely modeled after Lewy's original proof. In that paper he shows that if $g=g(t)$ then g must be real analytic to be in the range of L. Lewy's idea also uses commuting operators. In fact, call $s=x^2+y^2$ and θ the angle in the x, y plane, so that $z=x+iy=s^{1/2}e^{i\theta}$.

Lewy introduces the transformation

$$Tu = i\int zu\,d\theta, \tag{1-9}$$

so Tu is a function of s and t. We call $w=t+is$. It is clear that $T(g/z)=g$ if g depends only on t. Moreover,

$$\frac{\partial}{\partial\bar{w}}Tu = T\frac{1}{z}Lu \tag{1-10}$$

and

$$Tu=0 \quad \text{on } s=0. \tag{1-11}$$

Lewy's equation $Lf=g$ becomes, for $g=g(t)$,

$$\frac{\partial}{\partial\bar{w}}Tf=g \quad \text{with } Tf=0 \text{ on } s=0. \tag{1-12}$$

We call A the operator $\partial/\partial s$, so $Ag=0$ and A commutes with $\partial/\partial\bar{w}$. Thus ATf is holomorphic in w and this implies, by our above ideas (proof (b)), that g is real analytic.

Thus Lewy's original proof fits into our ideas concerning commuting operators. In terms of the original operator L, the commuting operator is given formally by

$$\frac{\partial}{\partial\bar{w}}T = T\frac{1}{z}L, \tag{1-13}$$

so

$$\frac{\partial}{\partial\bar{w}} = T\frac{1}{z}LT^{-1},$$

and

$$A\frac{\partial}{\partial\bar{w}} = \frac{\partial}{\partial\bar{w}}A$$

becomes

$$AT\frac{1}{z}LT^{-1} = T\frac{1}{z}LT^{-1}A$$

or

$$\frac{1}{z}L(T^{-1}A^{-1}T) = (T^{-1}A^{-1}T)\left(\frac{1}{z}L\right). \tag{1-14}$$

The second proof given in [3] depends on the discrete series for semi-simple Lie groups. The operator A should be something like projection on the discrete series. (This commutes with the enveloping algebra.) But I cannot yet fit this precisely into the above framework.

Actually, in the case of semi-simple Lie groups, the enveloping algebra should map locally surjectively on the "projection on the most continuous series" for groups with a Cartan subgroup that is a vector space, and the nonsurjectivity should be determined by "projection on discrete-like series". But I have no idea how to make this idea precise.

The interpretation of Lewy's equation as given by Hörmander [10] and many authors following is in terms of the commutator of L and $\bar{L}$ rather than of L and R. Of course $\bar{L}$ and A have a simple commutation relation and perhaps this accounts for the interchangeability of $\bar{L}$ and R.

II. Use of Peak Points

Let Ω be a piece of smooth (real) hypersurface in C^2 (with coordinates z, w). Let $p \in \Omega$ and suppose that the Levi form is definite near p (we allow it to degenerate at p). It is standard that there is a unique (up to left multiplication) smooth operator

$$L = A(z, w)\frac{\partial}{\partial \bar{z}} + B(z, w)\frac{\partial}{\partial \bar{w}} \tag{2-1}$$

that is tangent to Ω near p. Another way of saying this is: If Ω is defined by $\psi(z, w) = 0$ then $L\psi = 0$ near Ω. Thus we can choose

$$L = \psi_{\bar{w}}\frac{\partial}{\partial \bar{z}} - \psi_{\bar{z}}\frac{\partial}{\partial \bar{w}}.$$

We call L the *Lewy operator* on Ω.

In the particular case

$$\psi(z, w) \equiv \frac{w - \bar{w}}{2i} - z\bar{z} \equiv \operatorname{Im} w - z\bar{z}, \tag{2-2}$$

the operator L is exactly Lewy's (1-2) if we use the coordinates $t = \operatorname{Re} w$, $x = \operatorname{Re} z$, and $y = \operatorname{Im} z$ on the surface $\psi = 0$.

We shall make some remarks about the case of C^n for $n > 2$ at the end of this section.

We choose some smooth measure $d\omega$ on Ω and we call L' the *formal adjoint* of L with respect to this measure. Our results do not depend on ω, which means that they hold for $L + \lambda$ where λ is a more or less arbitrary function.

Since the Levi form is definite, we can consider the "inside" Δ of Ω as being the (local) pseudoconvex part of C^2 bounded by Ω, that is, Δ is the envelope of holomorphy of Ω.

As mentioned in the introduction, the proof given here depends on the fact that the points in Ω are peak points for the algebra defined by the kernel of L, which is the same as the algebra of restrictions of functions holomorphic on Δ to Ω. Another way of putting this is that the whole of Ω is the Shilov boundary of this algebra.

We claim there is no distribution f on Ω such that

$$L'f = \delta_p. \tag{2-3}$$

We shall explain in Section III how to replace δ_p by a $\mathcal{C}^\infty$ function.

Let η be a small number and let χ be a cut-off function on Ω, that is, $\chi(x)=1$ for $|x-p|\le\eta$ and $\chi(x)=0$ for $|x-p|\ge 2\eta$. It is assumed that the defining function ψ for Ω is defined at least in the ball $|(z,w)-p|<3\eta$ so that this makes sense.

Since Ω is strictly pseudoconvex near p we can construct peak functions h_j, that is, h_j are restrictions to Ω of holomorphic functions, and for each N

$$|h_j(p)|\ge j, \tag{2-4}$$

with

$$|h_j^{(l)}(x)|\le 1 \quad \text{for } \eta\le|x-p|\le 2\eta \text{ and } |l|\le N+1.$$

By adjusting η to be small enough it is easy to construct h_j explicitly. For example, if p is a strictly pseudoconvex point then there is a quadratic polynomial P on C^2 whose zero set intersects $\bar{\Omega}$ exactly at p. Then set

$$h_j = \frac{\epsilon_j}{P-c_j} \tag{2-5}$$

for suitable ϵ_j, c_j.

To put this construction in the format of Section I, A is multiplication by the function P so $R=1/P$. In our case Q is formally identical to 0. The h in (1-8) can be chosen $\equiv 1$. To make things work precisely we had to use h_j and even χh_j in place of $1/P$. If the Levi form degenerates at p, then we choose h_j of the form $\epsilon_j(P_{p_j}-c_j)^{-1}$, where $p_j\in\Omega$, $p_j\to p$, and P_{p_j} is the quadratic polynomial associated to p_j.

We claim that there is no solution of (2-3) for f a distribution of order N. Since N is arbitrary, this proves our result.

Suppose that f did exist. Then we deduce from (2-3) that

$$L'f\cdot\chi h_j = \delta_p\cdot\chi h_j. \tag{2-6}$$

The right side of (2-6) is $h_j(p)$ and $\to\infty$. The left side of (9) is $f\cdot L\chi h_j$. Now, by Leibniz's formula,

$$L\chi h_j = \begin{cases} 0 & \text{if } |x-p|\le\eta, \\ \text{bounded and so are its} & \\ \text{derivatives of order} \le N & \text{if } \eta\le|x-p|\le 2\eta, \\ 0 & \text{if } |x-p|\ge 2\eta. \end{cases} \tag{2-7}$$

In particular, the set $\{L\chi h_j\}$ is bounded in the space of test functions of order $\le N$. Thus the left side of (2-6) is bounded, which is a contradiction. Hence the unsolvability of (2-3) is established.

One can push the theory still further. Suppose Ω is real analytic. It is easy to estimate the size of the derivatives of $(P-c_j)^{-1}$ or $(P_{p_j}-c_j)^{-1}$ on $\eta \leq |x-p| \leq 2\eta$, thereby establishing the following lemma.

LEMMA 2.1. *Suppose the Levi form is nondegenerate at p. There are constants A, B such that, for all sufficiently small c, we have*

$$\left|\frac{\partial^l}{\partial x^l}(P-c)^{-1}\right| \leq Bl!\, A^{|l|} \tag{2-8}$$

on $\eta \leq |x-p| \leq 2\eta$. (*A and B depend on* η.)

A similar property holds for real-analytic Ω if the Levi form degenerates at p, but we have to suitably normalize the P_{p_j} so we shall omit this. (Or else we can use polynomials of degree greater than 2 and proceed as before.)

Let $\{M_l\}$ be a sequence of positive numbers such that $M_l \geq (bl)!\, a^{|l|}$ for some b, a with $b > 1$. We form the space $\mathfrak{D}(\{M_l\})$ of functions Y with support near p so that

$$|Y^{(l)}(x)| \leq B_\epsilon M_l \epsilon^{|l|} \tag{2-9}$$

for all $\epsilon > 0$ (see [2]). The topology of $\mathfrak{D}(\{M_l\})$ is defined in the natural manner. We assume that $\{M_l\}$ is nonquasi-analytic, that is, there are plenty of functions in $\mathfrak{D}(\{M_l\})$. We assume also that $\mathfrak{D}(\{M_l\})$ is a ring.

From Lemma 2.1 we deduce exactly as before the following theorem.

THEOREM 2.2. *Let Ω be strictly pseudoconvex at p. Then equation* (2-3) *has no local solution $f \in \mathfrak{D}'(\{M_l\})$.*

REMARK 1. We could replace the right side of (2-3) by $\delta_p^{(l)}$ for any l. Thus the cokernel of L' has infinite dimension.

REMARK 2. A variation of the above argument shows that there is no local hyperfunction solution f.

We have shown that there is no distribution solution f of $L'f = \delta_p$. Suppose we know that f is a distribution defined in the neighborhood of p such that $L'f = c\delta_p$. Then it follows that $c = 0$ and f is actually a solution of $L'f = 0$ in a full neighborhood of p.

The strongest form of this result is given in the following theorem.

THEOREM 2.3. *Suppose Ω is real analytic and strictly pseudoconvex at p. Suppose f is a distribution in the neighborhood of p such that $L'f = 0$ except at p, that is, the support of $L'f = \{p\}$. Then we can modify f by adding a distribution supported at p to obtain a solution in a full neighborhood of p.*

Our result can be formulated as follows: Suppose $L'f = g$ has support at p. Then $g = L'\tilde{f}$, where support $\tilde{f} = \{p\}$.

REMARK. Theorem 2.3 can be extended to the case where f belongs to $\mathfrak{D}'(\{M_l\})$ for $\{M_l\}$ nonquasi-analytic. As such, we can think of our result as being in the same spirit as that of Riemann's removable singularities theorem, since being a distribution can be thought of as imposing a type of growth condition at p.

PROBLEM 2.1. Is Theorem 2.3 true without any growth condition at p; that is, if f is a distribution defined (near p) on the complement of p and $L'f=0$ there, can we redefine the value of f at p to make it a solution in a full neighborhood of p?

Problem 2.1 involves several difficulties; for example, what is meant by redefining f at p? A positive solution to Problem 2.1 would constitute a sort of Hartogs extension theorem for the kernel of L.

Proof of Theorem 2.3. In order to prove Theorem 2.3, we need to know when a distribution g supported at p is of the form $L'f$. For simplicity we assume $p=0$.

As the case of general Ω involves some technical complications, we shall give the complete proof in the case of Lewy's original example (2-2). The complex 3-dimensional tangent space to Ω at the origin is easily seen to be spanned by

$$L=\frac{\partial}{\partial \bar{z}}-2iz\frac{\partial}{\partial \bar{w}},$$

(2-10) $$\bar{L}=\frac{\partial}{\partial z}+2i\bar{z}\frac{\partial}{\partial w},\quad \text{and}$$

$$M=\frac{1}{2i}[L,\bar{L}]=\frac{\partial}{\partial w}+\frac{\partial}{\partial \bar{w}}.$$

Note that $M, L, \bar{L}$ form a Heisenberg Lie algebra H.

Now suppose that $L'f=g$ has its support at the origin. Then we can write

(2-11) $$g=[P(M,L,\bar{L})]\delta,$$

where P is a polynomial. By rearranging terms using the bracket relations in H, we can write P in the form

(2-12) $$P=\Sigma\alpha_{jkl}M^j\bar{L}^kL^l.$$

CLAIM. *Consider the map sending $P\to P^A$, where P^A is the "analytic part" of P, obtained from P by replacing $\partial/\partial\bar{z}$ and $\partial/\partial\bar{w}$ by zero. Then $P^A\equiv 0$ if and only if $P=P_0L$ for some polynomial P_0 in the enveloping algebra of H.*

Proof of Claim. Note that, by (2-12),

(2-13) $$P^A=\Sigma\alpha_{jk0}\frac{\partial^j}{\partial w^j}\bar{L}^k.$$

This vanishes if and only if all terms in P contain a factor of L on the right, whence our claim is established.

Thus, to complete the proof of the theorem, it suffices to prove that if $L'f=P'\delta$ then $P^A\equiv 0$. For, if we knew this, then $P'=L'P'_0$ so $P'\delta=L'P'_0\delta$ and $L'(f-P'_0\delta)=0$.

To show that $L'f=P'\delta$ implies $P^A\equiv 0$, we proceed just as in our proof that we cannot solve $L'f=\delta$. Namely, if $P^A\not\equiv 0$ then we claim that we can construct "peak" holomorphic functions h_j so that, instead of (2-4), we have

$$|(P^A)h_j(0)|\geq j, \tag{2-4*}$$

with

$$|h_j^{(l)}(x)|\leq 1 \quad \text{for } \eta\leq|x|\leq 2\eta,\ |l|\leq N+1.$$

The proof is essentially the same as in case $P^A\equiv 1$ because P^A is a differential operator in $\partial/\partial z$ and $\partial/\partial w$.

The proof of the impossibility of solving (2-3) now shows that we cannot solve

$$L'f=P'\delta \tag{2-3*}$$

because

$$\begin{aligned} P'\delta\cdot\chi h_j &= [P(\chi h_j)](0) \\ &= [(P^A)(\chi h_j)](0)\to\infty, \end{aligned} \tag{2-14}$$

since $P=P^A$ on boundary values of holomorphic functions and χh_j is holomorphic near the origin.

Thus $L'f=P'\delta$ implies $P^A\equiv 0$ and, as we have noted, this completes the proof of Theorem 2.3 for the case of Lewy's original example (2-2). For the general case the Heisenberg algebra must be replaced by a more complicated Lie algebra that is still nilpotent. We shall develop these ideas elsewhere. □

REMARK. The proof shows some relation between nonsolvability of inhomogeneous equations like $L'f=g$ and removable singularities, that is, extendability of solutions of $L'f=0$. For, when L is not surjective then there are few g in the range of L'. Thus this set of g should be "controllable" and this is the nature of our proof of Theorem 2.3 (namely, the g supported at the origin can be determined).

One might be puzzled by the following example. Let k be a holomorphic function whose zero set meets Ω exactly at p and which has no zeros in Ω. Then for any l we have $Lk^{-l}=0$ on Ω except at p. By choosing a suitable measure for defining the adjoint (e.g., in Lewy's original example use $dx\,dy\,dt$), we find also that $L'k^{-l}=0$ off p since L' is essentially the same as L. This means that $L'k^{-l}$ is some distribution μ_l supported at p. By Theorem 2.3 we expect that $\mu_l=0$. In fact, it is easily seen that $\mu_l=0$, since k^{-l} is the limit in the sense of distributions of the holomorphic functions k_ϵ^{-l} obtained by "sliding" the zero set of k off Ω. This computation uses one definition of k^{-l}. It would be interesting to compute μ_l using other definitions of k^{-l}, for example, the principal value definition.

As we have noted, the proof of the nonsurjectivity of L depends on the existence of "peak" points for the kernel of L, that is, the whole of Ω constitutes the Shilov boundary for this algebra. (Of course, we are speaking somewhat loosely since the algebra is the algebra of $\mathcal{C}^\infty$ functions in the kernel of L. This is a topological, not a Banach, algebra, so the notion of Shilov boundary may not be precisely defined.)

We can formulate things precisely as follows.

THEOREM 2.4. *Let M be a subset of the algbra $\mathcal{E}(U)$ of $\mathcal{C}^\infty$ functions on the smooth manifold U. Suppose $p \in U$ is a peak point for M, meaning that there are functions $f \in M$ with $f(p) \to \infty$ but $f \to 0$ in the $\mathcal{C}^\infty$ topology of the open set $U_0 \subset U$, where one component of $U - U_0$ is a compact neighborhood of p. Let A be a* local *continuous linear map of $\mathcal{E}(U)$ into $\mathcal{E}(U)$ that annihilates M. Then there is no solution of*

$$A'T = \delta \tag{2-15}$$

for $T \in \mathfrak{D}'(U)$.

The proof of Theorem 2.4 is contained in our above ideas.

We can formulate a similar result in case A is a semilocal operator. Then we need an estimate on the size of U_0.

The fact that Ω is the Shilov boundary would seem to indicate that the kernel of L is large since for each $p \in \Omega$ it contains a peak function at p. A similar property is true of the kernel of L'. Now, L' is a map of

$$L': \frac{\mathfrak{D}'}{\text{kernel } L'} \to \mathfrak{D}. \tag{2-16}$$

The nonsurjectivity of L' would seem to be a consequence of the fact that kernel L' is large so that $\mathfrak{D}'/\text{kernel } L'$ is small. However, in a *linear* sense $\mathfrak{D}'/\text{kernel } L'$ is not smaller than $\mathfrak{D}'$, as is shown in the next theorem.

THEOREM 2.5. *When Δ is the unit ball, there exists a linear isomorphism*

$$\psi: \frac{\mathfrak{D}'}{\text{kernel } L'} \approx \mathfrak{D}'. \tag{2-17}$$

Proof. A basis for $\mathfrak{D}'(\Omega)$ can be found using the fact that the unitary group $G = U(2)$ acts on Δ and on Ω. In fact, it is well known that Ω can be identified with the special unitary group $G^0 = SU(2)$ under the identification

$$\begin{pmatrix} z \\ w \end{pmatrix} \leftrightarrow \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{2-18}$$

for

$$g = \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix}$$

an arbitrary element of G^0. Thus we can decompose the representation of G^0 on the functions on Ω to obtain a basis for $\mathfrak{D}'(\Omega)$.

The decomposition of the representation of G^0 on the holomorphic functions is easily obtained. The homogeneous polynomials of degree l constitute an irreducible representation space of dimension $2l+1$. These representations are all the irreducible representations of G^0. On the other hand, the representation of G^0 on $\mathfrak{D}'(\Omega)$ is the regular representation, so each irreducible representation of degree $2l+1$ occurs $2l+1$ times.

If we use a natural basis of representation functions $\{\alpha_{jk}^{il}\}$, where $i=1,2,\ldots,2l+1$ indexes the representations of degree $2l+1$ and j,k represent the corresponding matrix coefficients, then we can arrange things so that a distribution T in $\mathfrak{D}'$ has a Fourier series

$$T=\Sigma a_{il}^{jk}\alpha_{jk}^{il}, \tag{2-19}$$

where the a_{il}^{jk} are slowly increasing in the sense that

$$|a_{il}^{jk}|\leq C(1+l)^C. \tag{2-20}$$

The kernel of L' contains those T which involve only the holomorphic representations, say the ones corresponding to $i=1$. Thus the quotient can be thought of as the space of all slowly increasing sequences with $i\neq 1$. It is clear that this quotient is linearly isomorphic to the space of all slowly increasing sequences. This proves the result. □

The analog of Theorem 2.5 holds under great generality. The general proof is of a similar (albeit less explicit) nature. We use the theory of kernels of Schwarz or, rather, the proof of the Schwarz kernel theorem (see e.g. [8]) to obtain a basis for $\mathfrak{D}'(\Omega)$ that contains the kernel of L' as a suitable sub-basis. The completion of the proof is as before.

It should be noted that in complex dimension 1 there is an analog of the Lewy operator. This is the operator L that projects functions on Ω onto those which are antiholomorphic on Δ. Of course, L is not a local operator so there does not seem to be any analog of a removable singularities theorem; however, the other results of this section are easily established in this case.

Thus we see that it is the multiplicative structure in $\mathfrak{D}'$ and the kernel of L that prevent L' from being surjective. In fact, Theorem 2.4 shows that no differential operator ψ of any order whose kernel contains the kernel of L could define a surjective map of $\mathfrak{D}'/\text{kernel } L$ onto $\mathfrak{D}'$, because of the Shilov property of the kernel of L; namely, that for each p, N, η there exist functions h_j that satisfy the peak property (2-4).

Not only can ψ not be a differential operator, but ψ cannot even satisfy any of a series of "weak" behaviors with respect to multiplication in place of the Leibniz formula. The situation reminds one of the extension theory in topology where the multiplicative structure in the cohomology ring is often decisive.

PROBLEM 2.2. Put the nonsurjectivity in the framework of topological algebra. More precisely, let A be a function algebra and B a subalgebra whose Shilov boundary is the whole maximal ideal space of A. Can the nonsurjectivity of L' be formulated purely in terms of the topological algebraic structures of A and B?

All of our discussion up to now has concerned the case of $n=2$ complex variables. If we go to $n>2$ then the same argument (or Theorem 2.4) shows that no tangential Cauchy-Riemann operator is surjective on $\mathfrak{D}'(\Omega)$ if Ω is (locally) the boundary of a strictly pseudoconvex domain Δ.

Actually much more is true. We can study the tangential $\bar\partial$ or Lewy cohomology; that is, if $L_1,\dots,L_{n-1}$ form a basis for the tangential Cauchy-Riemann complex, then to what extent can we solve

$$L_j f = g_j \quad j=1,\dots,n-1 \tag{2-21}$$

given that the g_j satisfy the compatibility conditions

$$L_i g_j - L_j g_i = \Sigma a_{ij}^k g_k? \tag{2-22}$$

Here the a_{ij}^k are the structure constants for the Lie algebra formed by the vector fields L_j,

$$[L_i, L_j] = \Sigma a_{ij}^k L_k. \tag{2-23}$$

We define higher cohomology groups in a similar manner.

Our method shows simply that we cannot solve

$$\Sigma L_j' f_j = \delta_p \tag{2-24}$$

for p a peak point. But we do not know how to study the general (2-21) (with L_j replaced by L_j'). In fact, it is easily seen that there is no nontrivial cohomology class supported at one point so long as there are nontrivial compatibility conditions.

In particular, when Ω is defined by a single equation $\psi=0$ then we can choose, for L_j,

$$L_j = \psi_{\bar z_n}\frac{\partial}{\partial \bar z_j} - \psi_{\bar z_j}\frac{\partial}{\partial \bar z_n}. \tag{2-25}$$

III. Lewy's Operator and Hartogs' Extension

When I was first made aware of Lewy's example, I assumed that he had proceeded as follows: Let Δ_ϵ be a crescent in Δ, that is, let $\Delta_\epsilon \subset \Delta$ be bounded by Ω and some similar surface Ω_ϵ (see Figure 1). The shaded area is Δ_ϵ. Assume the Levi form is nondegenerate at all points of Ω and Ω_ϵ.

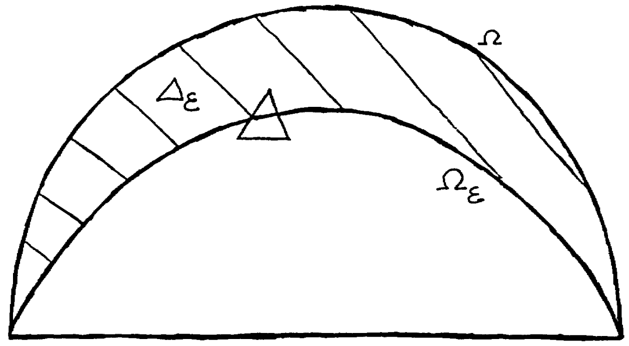

Figure 1

Now, Δ_ϵ is not a domain of holomorphy because we have taken out $\Delta-\Delta_\epsilon$ from the domain of holomorphy Δ. Thus, by standard theory, the cohomology group $H^1(\Delta_\epsilon,\Lambda)\neq 0$, where Λ is the sheaf of germs of holomorphic functions. The size of $H^1(\Delta_\epsilon,\Lambda)$ should be in "proportion" to the size of $\Delta-\Delta_\epsilon$ since this cohomology measures the departure from being a domain of holomorphy. Thus as $\Delta_\epsilon\to\Omega$ the cohomology group should grow. (This growth can be seen by a simple exact sequence.)

By Dolbeaut's theorem, the cohomology group of Δ_ϵ is measured by the lack of possibility of finding functions (or distributions) $\tilde{f}$ on Δ_ϵ satisfying

$$\frac{\partial\tilde{f}}{\partial\bar{z}}=\tilde{g}_1 \quad\text{and}\quad \frac{\partial\tilde{f}}{\partial\bar{w}}=\tilde{g}_2, \tag{3-1}$$

where

$$\frac{\partial\tilde{g}_1}{\partial\bar{w}}=\frac{\partial\tilde{g}_2}{\partial\bar{z}}. \tag{3-2}$$

We might hope that the groups $H^1(\Delta_\epsilon,\Lambda)$ have some limit as $\Delta_\epsilon\to\Omega$. The only meaningful equation one could derive from (3-1) and (3-2) on Ω is the Lewy equation (1-1). Thus the cokernel of L is our only hope for a reasonable description of this limit. Hence, in particular, L has a large cokernel. Theorem 3.4 and the remark following show a precise relation between this cokernel and a limit of $H^1(\Delta_\epsilon,\Lambda)$.

This viewpoint of the Lewy example is in conformity with my functional analysis solution of the Lewy problem for domains in C^n (see Chapter XI of [2]).

Professor Lewy has informed me that he never thought along these lines. This seems very strange to me because, as we shall see, the main ingredient of the proof is Lewy's theorem characterizing the kernel of L as the set of functions on Ω that have holomorphic extensions to Δ. Our proof uses distributions while Lewy's does not.

As a "warm-up" to showing the nontriviality of the cokernel of L, we construct the simplest cohomology classes in $H^1(\Delta_\epsilon,\Lambda)$. In situations such as that given by (2-2), we can arrange coordinates so that the sets $w=c$ for a certain range of constants c meet Δ_ϵ in an annulus A_ϵ, which we assume contains $z=0$ in its interior. We assume that the interior of A_ϵ is contained in Δ. Let $k(z)$ be holomorphic on A_ϵ but such that k cannot be extended to be holomorphic near $z=0$. We use k to define the cohomology class ($\bar\partial$ closed form)

$$\lambda=k(z)\delta_{w=c}\,d\bar{w}+0\,d\bar{z}. \tag{3-3}$$

Note that $\partial[k(z)\delta_{w=c}]/\partial\bar{z}=0$ on Δ_ϵ so λ is closed.

CLAIM 1. *λ is not cohomologous to zero in $H^1(\Delta_\epsilon,\Lambda)$.*

Proof of Claim 1. Suppose that $\lambda=\bar\partial h$ for some distribution h on Δ_ϵ. Then $\bar\partial h=0$ outside $\{w=c\}$ so h is holomorphic in Δ_ϵ off $\{w=c\}$. Consider the function $(w-c)h$. We claim that $(w-c)h$ is a holomorphic function on Δ_ϵ. For, by Leibniz's formula,

$$\frac{\partial}{\partial \bar{z}}[(w-c)h]=(w-c)\frac{\partial h}{\partial \bar{z}}=0 \tag{3-4}$$

while

$$\begin{aligned}\frac{\partial}{\partial \bar{w}}[(w-c)h]&=(w-c)\frac{\partial h}{\partial \bar{w}}\\ &=(w-c)k(z)\delta_{w=c}\\ &=0.\end{aligned} \tag{3-5}$$

Since $(w-c)h$ is holomorphic on Δ_ϵ, it follows from Hartogs' extension theorem (see e.g. [2]) that $(w-c)h$ extends to a holomorphic function $\tilde{h}$ on all of Δ. We want to use $\tilde{h}$ to extend k over the interior of A_ϵ.

CLAIM 2. $h=\tilde{h}/(w-c)$.

Proof of Claim 2. Since $1/(w-c)$ is the fundamental solution for $\partial/\partial\bar{w}$ with singularity at $w=c$, it follows from the holomorphicity of $\tilde{h}$ that

$$\frac{\partial}{\partial \bar{w}}\frac{\tilde{h}}{w-c}=\tilde{h}\delta_{w=c}. \tag{3-6}$$

(Here and in the following we use a suitable normalization as in [2], so that factors of π do not appear.)

We have shown that $h-\tilde{h}/(w-c)$ satisfies the equations

$$(w-c)\left[h-\frac{\tilde{h}}{w-c}\right]=0 \tag{3-7}$$

and

$$\frac{\partial}{\partial \bar{w}}\left[h-\frac{\tilde{h}}{w-c}\right]=[k(z)-\tilde{h}]\delta_{w=c}\quad \text{on } \Delta_\epsilon. \tag{3-8}$$

Equation (3-7) says that $\alpha=h-\tilde{h}/(w-c)$ is in the kernel of multiplication by $w-c$. This kernel is readily described; for simplicity we give the description for $c=0$. Then $w\alpha=0$ means that we can write α in the form

$$\alpha=\Sigma\frac{\partial^j}{\partial \bar{w}^j}\mu_j, \tag{3-9}$$

where μ_j are distributions on $w=0$ that are uniquely determined by α. (The definition of *distribution on* $w=0$ is clarified below.)

The proof of (3-9) is an easy consequence of the fact that support$(\alpha)\subset\{w=0\}$. Note that, for such an α, $\partial\alpha/\partial\bar{w}$ can never be a distribution on $w=0$ unless $\alpha\equiv 0$. This gives Claim 2. From Claim 2 and (3-6) we see that $\tilde{h}$ provides an extension of k over the interior of $w=c$. This completes the proof of Claim 1. □

Note how this construction fits into the framework set forth in Section I. A is multiplication by $w-c$, so $R=1/(w-c)$ and Q is multiplication by $\psi_{\bar{z}}\delta_{w=c}$.

Next we want to pass from $H^1(\Delta_\epsilon,\Lambda)$ to the cokernel of L. For this we must assume that $\{w=c\}$ meets Ω "nicely" (i.e., transversely). This enables us to define the product of $\delta_{w=c}$ with δ_Ω. Certainly we have a nice intersection in Lewy's original example (2-2).

In order to clarify things let me make some precise definitions. Let us recall that in one dimension we can write $\delta_{x=0}=\lim P_m(x)$. The polynomials $P_m(x)$ are thought of as distributions by multiplying by dx. We then define $\delta_\Omega=\lim P_m(\psi)$, where Ω is defined by $\psi=0$. Again we identify $P_m(\psi)$ with distributions by multiplying by the Euclidean measure. (We really care only about what happens locally near some point in Ω.) Thus δ_Ω is actually the Euclidean measure on Ω divided by $|\operatorname{grad}\psi|$. The product of distributions, when it exists, can be defined using the product of approximations. The existence of products can be defined in terms of wave front sets, but we shall not need such ideas here.

The important point about our definition is that it (generally) agrees with our geometric intuition. Even more important for us is the fact that it allows us to make constructions and calculations on the boundary Ω by means of Δ_ϵ.

The distributions we consider are in the space $\mathfrak{D}'(\bar{\Delta}_\epsilon)$, that is, they are in the dual of $\mathcal{C}^\infty$ functions on Δ_ϵ which have $\mathcal{C}^\infty$ extensions to the "upper boundary" that is Ω. (The other part of the boundary of Δ_ϵ is irrelevant because we care only about what happens near Ω.)

An $S\in\mathfrak{D}'(\bar{\Delta}_\epsilon)$ is called a *distribution* on Ω if $\psi S=0$. A smooth function f in $\bar{\Delta}_\epsilon$ is identified with the distribution $f\delta_\Omega$ on Ω. A function on Ω is identified with the distribution $\tilde{g}\delta_\Omega$ on Ω, where $\tilde{g}$ is any extension of g to a neighborhood of Ω. This distribution is $g/|\operatorname{grad}\psi|$ times the Euclidean measure on Ω.

A differentiable operator M in Euclidean space is said to be a *differentiable operator on* Ω (or *tangent to* Ω) if M annihilates the ideal $\mathcal{I}_\Omega$ of smooth functions that vanish on Ω. Thus M is defined on the quotient of $\mathcal{C}^\infty$ by $\mathcal{I}_\Omega$ which is $\mathcal{C}^\infty(\Omega)$. ($M$ is also defined on $\mathfrak{D}'(\Omega)$.) In particular, L is a differential operator on Ω.

As in the previous case of Δ_ϵ, we define

$$\lambda=\psi_{\bar{z}}k(z)\delta_{w=c}\delta_\Omega, \tag{3-10}$$

where $k(z)$ is holomorphic on $\{w=c\}$ near $\{w=c\}\cap\Omega$ but $k(z)$ does not have a holomorphic extension to the whole interior of $\{w=c\}\cap\Omega$. Thus λ can be thought of as a distribution on Ω. Here $\delta_{w=c}\delta_\Omega$ is the product of $\delta_{w=c}$ with δ_Ω, which exists by our assumption. We assume that $\psi_{\bar{z}}$ does not vanish identically on $\{w=c\}\cap\Omega$.

The main result of this section is the following theorem.

THEOREM 3.1. *λ is not in the range of L.*

To prove Theorem 3.1, we need some facts relating Ω to the ambient space.

LEMMA 3.2. *Let M be a smooth first-order operator on C^2 of the form $M=\Sigma a_j(\partial/\partial x_j)$ which is tangent to Ω; then, for any distribution S defined near Ω which restricts to Ω (i.e., $S\delta_\Omega$ is defined), we have*

$$M(S|_\Omega)=(MS)|_\Omega. \tag{3-11}$$

Proof. $S|_\Omega$ means $S\delta_\Omega$. Thus the left side of (3-11) is

$$M(S\delta_\Omega)=(MS)\delta_\Omega+SM\delta_\Omega=(MS)\delta_\Omega$$

because $M\delta_\Omega=\lim MP_m(\psi)=0$ on Ω, since M annihilates ψ (hence all powers of ψ) and constants because M has no constant term. □

It is readily verified that any distribution T on Ω is the restriction of some $\tilde{T}$ to Ω, that is, $T=\tilde{T}\delta_\Omega$. Thus MT is the restriction of $M\tilde{T}$ by our lemma. In particular, if $T=h$ is a function on Ω and $\tilde{h}$ is an extension of h, then we associate with h the distribution $h=\tilde{h}\delta_\Omega$. The function Mh defines the distribution $(M\tilde{h})\delta_\Omega=Mh$ by the lemma.

In [12] Lewy proved a sharp form of the Hartogs extension theorem in which he characterized functions on Ω that are boundary values of holomorphic functions on Δ. It is not difficult to extend his result to distributions by a regularization or other approximation argument. (Note that, by our above remarks, if f is a function on Ω then the vanishing of Lf as a function is the same as the vanishing of Lf as a distribution.)

We can state Lewy's result in the following form.

THEOREM 3.3. *The kernel of L on distributions on Ω is exactly the set of distributions that have holomorphic extensions to Δ.*

Let us return to the proof of Theorem 3.1. Suppose that $\lambda=Lh$ (see (3-10)). By Leibniz's formula and the fact that L is first order, we deduce that for $h_1=(w-c)h$

$$Lh_1=0, \tag{3-12}$$

since L annihilates holomorphic functions (or else use Lemma 3.2). By Theorem 3.3 this means that h_1 extends to a holomorphic function $\tilde{h}_1$ on Δ.

Let us denote by h_2 the meromorphic function $\tilde{h}_1/(w-c)$. We can use (3-6) to compute $\bar{\partial}h_2$. It is clear that $\partial h_2/\partial\bar{z}=0$; thus, calling $\tilde{L}$ the operator $\psi_{\bar{z}}(\partial/\partial\bar{w})-\psi_{\bar{w}}(\partial/\partial\bar{z})$ considered as an operator on functions on Δ_ϵ,

$$\tilde{L}h_2=\psi_{\bar{z}}\tilde{h}_1\delta_{w=c}. \tag{3-13}$$

By Lemma 3.2 this gives

$$\begin{aligned} L(h_2|_\Omega)&=(\tilde{L}h_2)\delta_\Omega\\ &=\psi_{\bar{z}}\tilde{h}_1\delta_{w=c}\delta_\Omega\\ &=\psi_{\bar{z}}h_1\delta_{w=c}. \end{aligned} \tag{3-14}$$

(Recall that h and h_1 are distributions on Ω.) Note that $h_1\delta_{w=c}$ makes sense because $\tilde{h}_1$ and hence h_2 have restrictions to Ω.

On the other hand,

$$\begin{aligned} (w-c)[h-h_2\delta_\Omega]&=h_1-\tilde{h}_1\delta_\Omega\\ &=0 \qquad \text{in } \mathfrak{D}'(\Omega), \end{aligned} \tag{3-15}$$

since $\tilde{h}_1\delta_\Omega$ is (by definition) the distribution that the function $\tilde{h}_1$ defines on Ω, and $\tilde{h}_1$ is the function whose boundary value on Ω is the distribution h_1.

As in the above case of Δ_ϵ, let $\alpha = h - h_2\delta_\Omega$ and think of α as a distribution on Ω. Then equation (3-15) says that $(w-c)\alpha = 0$, while (3-14) says that

$$L\alpha = [(k(z) - \psi_{\bar{z}} h_1)\delta_{w=c}]\delta_\Omega. \tag{3-16}$$

We analyze equations (3-15) and (3-16) as we did equations (3-7) and (3-8). Equation (3-15) says that α satisfies

$$(w-c)\alpha = 0 \quad \text{modulo } \mathcal{I}, \tag{3-17}$$

where $\mathcal{I}$ is the ideal of Ω. Because of the smoothness of Ω we can extend α to a distribution $\tilde{\alpha}$ in the kernel of $w-c$. Letting, for example, $c=0$ we obtain an expression like (3-9) for $\tilde{\alpha}$.

Next we apply L and use Lemma 3.2 to obtain a contradiction as in the proof of Claim 2, unless $\alpha \equiv 0$. Theorem 3.1 is thereby proven. □

Extensions of the Construction

Extension 1. We have used curves $w=c$. Naturally the same idea works for smooth holomorphic curves Γ that intersect Δ_ϵ in a similar manner. We describe Γ by $z = u(s)$, $w = v(s)$, with s in the unit disc. The image of $|s|=1$ lies outside Δ, the image of the origin is in $\Delta - \Delta_\epsilon$, and some annulus-like region A gets mapped into Δ_ϵ.

The analog of $\delta_{w=c}$ is the measure δ_Γ. By using functions $k(s)$ that are holomorphic near $\Gamma \cap \Delta_\epsilon$ but cannot be continued analytically near $s=0$, we construct nontrivial cohomology classes in $H^1(\Delta_\epsilon, \Lambda)$. Then, as in the case of $w=c$, we use $k(s)\delta_\Gamma\delta_\Omega$ to construct elements in the cokernel of L.

It should be noted that as $\epsilon \to 0$ the number of curves Γ and the number of functions k increase. In another work we shall show how the cohomology classes constructed here generate the cohomology group, so this shows again why $H^1(\Delta_\epsilon, \Lambda)$ increases as $\epsilon \to 0$. It also shows why the cokernel of L contains the limit of these cohomology groups.

Although our result on the generation of $H^1(\Delta_\epsilon, \Lambda)$ is difficult, let us show why the cokernel of Λ is larger than the $H^1(\Delta_\epsilon, \Lambda)$. The basic tool for this is the following theorem.

THEOREM 3.4. *Suppose ψ is real analytic. Use coordinates ψ, θ near Ω, where θ represents coordinates on Ω. Then for any given g on Ω there are formal power series*

$$\alpha = \Sigma\alpha_n(\theta)\psi^n \quad \textit{and} \quad \beta = \Sigma\beta_n(\theta)\psi^n \tag{3-18}$$

such that

$$\alpha_{\bar{w}} = \beta_{\bar{z}} \tag{3-19}$$

and

$$\psi_{\bar{w}}\alpha - \psi_{\bar{z}}\beta \equiv g \quad \textit{modulo multiples of } \psi. \tag{3-20}$$

Proof. The proof is a computation. We express α and β as in (3-18). We write $\psi_{\bar{w}}$, $\psi_{\bar{z}}$, $\theta_{\bar{w}}$, and $\theta_{\bar{z}}$ in a similar fashion, say

$$\psi_{\bar{w}} = \sum \psi_j^{\bar{w}}(\theta)\psi^j, \qquad \theta_{\bar{w}} = \sum \theta_j^{\bar{w}}(\theta)\psi^j,$$

and so forth. Then the coefficient of ψ^n in (3-19) is

$$\sum_{j=0}^{n}(\alpha_{n-j})_\theta\theta_j^{\bar w}+\sum_{j=0}^{n}(n-j+1)\alpha_{n-j+1}\psi_j^{\bar w} = \sum_{j=0}^{n}(\beta_{n-j})_\theta\theta_j^{\bar z}+\sum_{j=0}^{n}(n-j+1)\alpha_{n-j+1}\psi_j^{\bar z}. \tag{3-21}$$

The coefficient of ψ^n in the left side of (3-20) is

$$\sum_{j=0}^{n}(\psi_j^{\bar w}\alpha_{n-j}-\psi_j^{\bar z}\beta_{n-j}). \tag{3-22}$$

We use this expression for $n=0$ to determine α_0 and β_0 by means of

$$\psi_0^{\bar w}\alpha_0-\psi_0^{\bar z}\beta_0=g, \tag{3-23}$$

which can be solved since $\psi_{\bar z}$ and $\psi_{\bar w}$ have no common zero on Ω.

Note that the highest subscript of α in (3-21) is $n+1$ and the coefficients of α_{n+1} and β_{n+1} are respectively $\psi_0^{\bar w}$ and $\psi_0^{\bar z}$, so that we can determine α_{n+1} and β_{n+1} respectively. Theorem 3.4 is thereby proven. □

REMARK. Theorem 3.4 relates the cokernel of L to the formal cohomology that we denote by $H^1(\Delta_0,\Lambda)$. For it is clear that, for α,β,g as above, if $\alpha\,d\bar z+\beta\,d\bar w$ is formally $\bar\partial$ exact then g is in the range of L. Conversely, if $g=Lf$ then we could take some extension $\tilde f$ of f and let α,β be the respective formal power series of $\tilde f_{\bar z},\tilde f_{\bar w}$. Thus there is some formally exact $\alpha\,d\bar z+\beta\,d\bar w$ that satisfies (3-20).

All this shows that the map $(\alpha,\beta)\to g$ of (3-20) sends

$$H^1(\Delta_0,\Lambda)\to\frac{C^\infty(\Omega)}{LC^\infty(\Omega)}\to 0. \tag{3-24}$$

We have already remarked that $H^1(\Delta_\epsilon,\Lambda)$ grows as $\epsilon\to 0$. In fact, $H^1(\Delta_0,\Lambda)$ can be regarded as the limit of $H^1(\Delta_\epsilon,\Lambda)$.

PROBLEM 3.1. To what extent is the cokernel of L smaller than the limit cohomology group; that is, what is the kernel of the map in (3-24)?

Extension 2. We mentioned at the end of Section II how to study the case of dimension $n>2$. We have a basis of $n-1$ tangential $\bar\partial$ vector fields $L_1,\dots,L_{n-1}$ given, for example, by (2-25), and we want to study the cohomology problem defined by (2-21), (2-22), and (2-23). Our first task is to find $\{g_i\}$ satisfying these compatibility conditions.

PROPOSITION 3.5. *Let $\Sigma h_j\,d\bar z_j$ be a $\bar\partial$ closed 1-form on C^n (locally near Ω). Define*

$$g_j=\psi_{\bar z_n}h_j-\psi_{\bar z_j}h_n. \tag{3-25}$$

Then the g_j satisfy the compatibility conditions (2-22) *and the analogous higher commutation relations.*

Proposition 3.5 is a simple computation. It depends on the fact that (2-22) is a consequence of the formal (Leibniz) property of differentiation, and this is basically the origin of the $\bar{\partial}$ closure of $\Sigma h_j\, d\bar{z}_j$. (In fact, if we write the form $\Sigma h_j\, d\bar{z}_j$ locally as $\bar{\partial} u$ then the result is obvious.)

We can now use Proposition 3.5 to construct nontrivial cohomology classes for $\bar{\partial}_b$ in much the same way as we constructed elements of the cokernel of L in case $n=2$. For simplicity we give the construction for $n=3$, but there is no difficulty in passing to $n>3$.

Let us use coordinates (z,s,w). We want to construct a domain that plays the same role as the domains used for $n=2$. The crucial property we used for $n=2$ is that $\{w=0\}\cap\Delta_\epsilon$ is an annulus which, together with its interior, belongs to the envelope of holomorphy of Ω.

To obtain a similar situation in case $n=3$, we want $\{w=0)\cap\Delta_\epsilon$ to be a neighborhood of a torus (say, $|z|=a_\epsilon$, $|s|=b_\epsilon$) which, together with its interior, is contained in the envelope of holomorphy of the "outer boundary" Ω of Δ or Δ_ϵ. To get a precise example, start with Ω as a piece of the boundary of a convex set in C^3. By scaling we can imagine that Ω looks like the upper hemisphere of the unit sphere. Remove from Ω a small neighborhood N of the torus $\{w=\frac{1}{2}, |z|=\frac{1}{2}, |s|=\frac{1}{2}\}$. Call $\tilde{\Delta}=\Delta-N$, where Δ is the convex hull of Ω; see Figure 2. (We are using a single N here rather than a family of N_ϵ as in case $n=2$. There is no difficulty in using a family N_ϵ for which $\Delta-N_\epsilon\to\Omega$.)

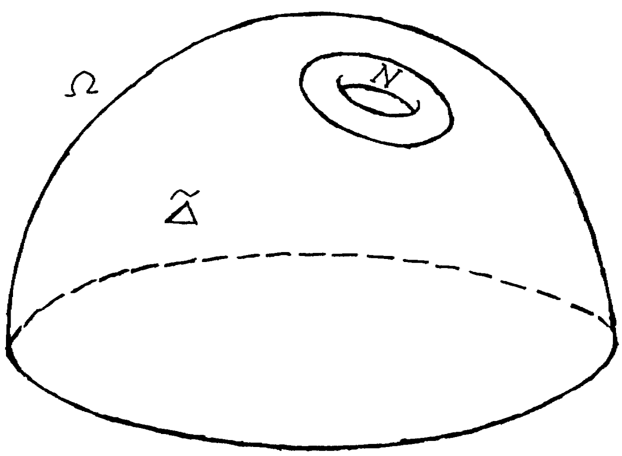

Figure 2

Consider a form like

$$\lambda=\frac{k(z,w)}{zw}\delta_{w=1/2}\,d\bar{s}+0\,d\bar{z}+0\,d\bar{w}, \tag{3-26}$$

where k is holomorphic on the torus.

It is clear that $\bar{\partial}\lambda=0$. However, we cannot write

$$\lambda=\bar{\partial}h \tag{3-27}$$

because (by Leibniz's formula) we deduce, for $\tilde{h}=(w-\frac{1}{2})h$, that $\bar{\partial}\tilde{h}=0$ on $\tilde{\Delta}$. Thus, by Hartogs' theorem, $\tilde{h}$ extends to be holomorphic on all of Δ and, as in case $n=2$, this is impossible. Using Proposition 3.5 and our method for $n=2$, this construction yields a nontrivial class in the first $\bar{\partial}_b$ cohomology group on Ω.

To study higher cohomology groups we must use the analog of Hartogs' theorem (and Lewy's form of it, which is Theorem 3.3) for the extension of closed l-forms, where $l>1$. (The first results of this nature appear in [7]; see also [2].)

There is no difficulty in extending Theorem 3.4 to the present case of $n>2$.

Results starting from Lewy [14] give analogs of the Lewy–Hartogs extension theorem from manifolds M of real codimension >1. These results can be used to prove nonvanishing results for the cohomology defined by tangential $\bar{\partial}$ operators on M.

For example, if M is a 4-dimensional subvariety of C^3, say M defined by $\psi=\phi=0$, then there is a single tangential $\bar{\partial}$ operator L, namely

$$L=A\frac{\partial}{\partial\bar{z}}+B\frac{\partial}{\partial\bar{w}}+C\frac{\partial}{\partial\bar{s}}, \tag{3-28}$$

where

$$A=\phi_{\bar{w}}\psi_{\bar{s}}-\phi_{\bar{s}}\psi_{\bar{w}},$$

$$B=\phi_{\bar{s}}\psi_{\bar{z}}-\phi_{\bar{z}}\psi_{\bar{s}},$$

and

$$C=\phi_{\bar{z}}\psi_{\bar{w}}-\phi_{\bar{w}}\psi_{\bar{z}}.$$

Extension 3. We should like to find the extension of the above to higher-order operators and systems. The result we have in mind is the following: Given an overdetermined system $\vec{D}f=0$ for a single unknown function f, there exists—at least if $\vec{D}$ has constant coefficients and has some unique continuation properties and if Ω has a certain type of convexity property—a boundary system $\boxed{D_b}$ that is a matrix of operators acting on $\vec{f}_b$ which consists of f and some "normal derivatives" on Ω so that $\boxed{D_b}\vec{f}_b=0$ if and only if $\vec{f}_b$ is the "Cauchy data" of a solution in Δ near Ω. This is the analog of the Hartogs phenomenon (see [2]).

In case $\boxed{D_b}$ is not first order, so there is no Leibniz formula, the relation of Hartogs phenomena to unsolvability is not clear.

Extension 4. All the above applies to solutions of (1-1) when g (hence f) are distributions. How about allowing g to be a $\mathcal{C}^\infty$ function (as Lewy allowed in his proof)?

The two proofs given above can be modified so as to produce such a g. For simplicity I shall treat the second proof (the one given in this section), as it is more instructive. Presumably one could also derive the $\mathcal{C}^\infty$ result from the distribution result by use of functional analysis. In order to avoid complicated notation, we shall use only curves Γ of the form $w=c$; this curve is denoted by Γ_c.

Let $k(z)$ be holomorphic on Γ_0 near $\Gamma_0 \cap \Delta$ (which we can think of as something like $|z| \leq 1$; in any case, for Lewy's original example we could actually make $\Gamma_0 \cap \Omega = \{|z|=1\}$). We consider restriction to Ω of the function

$$g = \psi_{\bar{z}} \frac{k(z)}{z} \alpha(w); \tag{3-29}$$

where α is a $\mathcal{C}^\infty$ function of support close to $w=0$. We claim that for suitable k and α there is no $\mathcal{C}^\infty$ (or even distribution) f satisfying $Lf = g$. We sometimes write f^α instead of f.

Suppose such an f existed. Then $Lf = 0$ except on the neighborhood $N = \bigcup \Gamma_w \cap \Omega$ for $w \in \operatorname{support}(\alpha)$. We need the following sharpening of Lewy's Theorem 3.4 which can be thought of as a *propagation of singularities*.

LEMMA 3.6. *f extends to be holomorphic on*

$$\Delta - \bigcup_{c \in \operatorname{supp}(\alpha)} \Gamma_c. \tag{3-30}$$

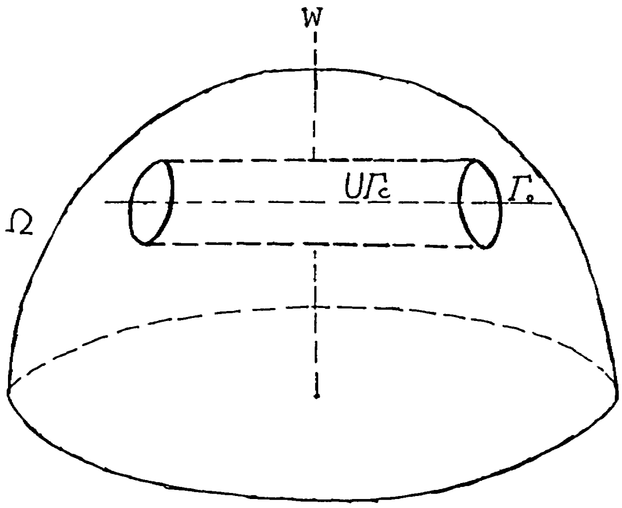

Figure 3

At the beginning of this section we used the form of this lemma when N is replaced by $\Gamma_0 \cap \Omega$. In that case we multiplied f by w and applied the usual Lewy theorem.

Before proving Lemma 3.6 we need a preliminary construction. We need to know something about the behavior of f in terms of α. This can be analyzed in various ways; the simplest for us is derivable from the following lemma.

LEMMA 3.7. *If the distribution f exists then it is the restriction to Ω of $f_1 + h/z$, where h is holomorphic on Δ and*

$$f_1(z, w) = \frac{k(z)}{z} \iint \frac{\alpha(w')\, dw' \wedge d\bar{w}'}{w - w'}. \tag{3-31}$$

More precisely, f_1+h/z is defined on a neighborhood of Ω in Δ and its restriction to Ω is f.

Proof. It is clear that

(3-32) $$\frac{\partial f_1}{\partial \bar{z}} = a\delta_{z=0}$$

and

$$\frac{\partial f_1}{\partial \bar{w}} = \frac{k(z)\alpha(w)}{z}$$

in Δ. Here $a=\int\int(\alpha(w')\,dw'\wedge d\bar{w}')/(w-w')$. Thus Lf_1 restricts to g on Ω off $z=0$. By Lemma 3.2 it follows that $L(f-f_1)=0$ on $\Omega-\{z=0\}$. Moreover, $Lf=0$ near $z=0$ on Ω because α has small support (see Figure 3) so that by (3-31) we have $Lz(f-f_1)=0$ on all of Ω. Hence $z(f-f_1)=\tilde{h}$ is holomorphic on Δ. This shows that

(3-33) $$f-f_1=\frac{\tilde{h}}{z}+\mu,$$

where μ is as in (3-9). Now, $Lf=0$ near $\{z=0\}\cap\Omega$, so f extends to be holomorphic near $\{z=0\}\cap\Omega$. By the explicit formula for f_1, it is of the form h_1/z near $\{z=0)\cap\Omega$, where h_1 is holomorphic. Hence $\mu=0$. This is Lemma 3.7. □

We can now complete the proof of Lemma 3.6. By Lemma 3.7 and the explicit formula for f_1, it follows that the singularities of f are only on $\bigcup\Gamma_c$ and $z=0$. But $Lf=0$ on Ω near $z=0$. Thus, by standard several complex variables arguments, f cannot have any singularity on $z=0$ off $\bigcup\Gamma_c$. This proves Lemma 3.6. □

Now we have good control over f_1, so we want to "factor out" the annoying holomorphic term h. We do this by applying the Cauchy integral in the w variable; that is, we form

(3-34) $$\int\int f(z,w)\,dw=\int f_1(z,w)\,dw.$$

The path of integration surrounds the support of α; for example, $|w|=\epsilon$. This makes sense for z when (z,w) is near $\Omega\cap\Gamma_0$. We now integrate (3-34) around a z path of the form $|z|=1-\epsilon'$. Since w in (3-34) is outside supp(α), the function f is regular in z inside the z contour. Hence, by changing orders of integration, we get zero. We have proven

(3-35) $$\int\int_{\text{torus}} f_1(z,w)\,dw\,dz=0$$

for all α.

We know from the explicit formula that $f_1=f_1^{\alpha}$ depends nicely on α. As $\alpha\to\delta_{w=0}=\alpha^0$, we still have (3-35). But for $\alpha^0=\delta_{w=0}$ we know from explicit calculation that $f_1^{\alpha^0}=k(z)/wz$. The integral in (3-35) is then

$$\int \frac{k(z)\,dz}{z}, \tag{3-36}$$

which can be assumed not to vanish if k is properly chosen. This proves the result on the unsolvability for $\mathcal{C}^\infty$ right sides.

REMARK 1. We needed to use f_1 since the behavior of f in terms of α is unclear.

REMARK 2. It might happen that f exists for some α. But the determination of those α for which f exists seems difficult.

REMARK 3. It is interesting to compare the above ideas for $\bar\partial$ with a corresponding construction for d. Of course the local d cohomology is trivial, so the closest we can come must involve a global construction. The d analog of holomorphic function is a constant. For the analog $\tilde\Delta$ of Δ we use the unit disc in R^2 and $\tilde\Delta_\epsilon$ is the annulus $1-\epsilon<|x|<1$. The analog of k is the function on $x_2=0$:

$$\tilde k(x_1,0)=\begin{cases} +1 & \text{for } \frac12<x_1\le 1,\\ -1 & \text{for } -\frac12>x_1\ge -1.\end{cases} \tag{3-37}$$

Then the restriction of $\tilde k\delta_{x_2=0}$ to the unit circle $\tilde\Omega$ is $\delta_{1,0}-\delta_{-1,0}$. Strangely this is in the range of

$$\frac{\partial}{\partial\theta}=x_2\frac{\partial}{\partial x_1}-x_1\frac{\partial}{\partial x_2},$$

whereas if we had replaced k by the actual constant $\tilde{\tilde k}\equiv 1$ on the x_1 axis then the restriction to $\tilde\Omega$ would be $\delta_{1,0}+\delta_{-1,0}$, which is not in the range of $\partial/\partial\theta$.

I do not understand why it is the locally constant $\tilde k$ rather than the actual constant $\tilde{\tilde k}$ whose restriction to $\tilde\Omega$ is in the range of $\partial/\partial\theta$, while the locally holomorphic k on $w=c$ discussed above has a restriction to Ω that is not in the range of L.

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