A Formula for the Local Dirichlet Integral

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Dedicated to the memory of Allen L. Shields

1. Introduction

Let $H^2(\mathbf{D})$ denote the Hardy space of the open unit disc $\mathbf{D}$. The isometric isomorphism of $H^2(\mathbf{D})$ onto the closed subspace H^2 of L^2 of the unit circle is a map that is well understood. In fact, much of our knowledge about H^2 functions has been derived by an exploitation of the properties of this map.

The Dirichlet space D is the space of analytic functions f in $\mathbf{D}$ with finite Dirichlet integral; that is,

$$D(f)=\iint_{\mathbf{D}}|f'(z)|^2\,dA(z)<\infty,$$

where $dA(re^{it})=(1/\pi)r\,dr\,dt$ denotes the normalized area measure on $\mathbf{D}$. It is well known (and easy to verify) that D is contained in $H^2(\mathbf{D})$. It thus follows that the above-mentioned isomorphism maps D into a subset of L^2, and for problems involving variations within the class of Dirichlet functions it is important to know how changes in the boundary values affect the Dirichlet integral.

In his investigations about minimal surfaces, Douglas [9] used the following formula for the Dirichlet integral of f:

$$D(f)=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\left|\frac{f(e^{it})-f(e^{is})}{e^{it}-e^{is}}\right|^2 dt\,ds.$$

This or similar formulas have also been used by other authors; for example, Beurling used one in his original proof [3] that the Fourier series of a Dirichlet function converges everywhere except perhaps on a set of logarithmic capacity zero. In 1960 Carleson proved a formula that expresses the Dirichlet integral of f as a sum of three nonnegative terms, involving respectively the Blaschke factor of f, the singular inner factor, and the outer factor (see [6] and also Corollary 3.6 below). As one application of this we mention a result of Brown and Shields, who used Carleson's formula to show that the "cut-off" operation, which maps a Dirichlet space function f to the outer function defined by $|g|=\min\{|f|,1\}$ on the unit circle $\mathbf{T}$, does not increase the Dirichlet integral (see [4, p. 284]).

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Let $f \in L^1$ $(=L^1(\mathbf{T}))$. We assume that $f(e^{it})$ equals the nontangential limit of its Poisson extension whenever the latter exists. For $\zeta \in \mathbf{T}$ we define the *local Dirichlet integral* of f at ζ by

$$D_\zeta(f) = \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{f(e^{it})-f(\zeta)}{e^{it}-\zeta}\right|^2 dt.$$

If $f(\zeta)$ does not exist, then we set $D_\zeta(f)=\infty$. Note that it follows from Douglas' formula that one can obtain the Dirichlet integral of f by integrating the local Dirichlet integral with respect to normalized Lebesgue measure on $\mathbf{T}$. The definition of the local Dirichlet integral was motivated, of course, by Douglas' formula and by the results about 2-isometric operators that were proved in [13].

An operator T on a separable complex Hilbert space $\mathcal{H}$ is called a *2-isometry* if

$$\|T^2x\|^2-\|Tx\|^2=\|Tx\|^2-\|x\|^2 \quad \text{for all } x\in\mathcal{H}.$$

This definition is due to Agler [1]. It follows from the definition that any 2-isometry T satisfies $\|Tx\|\ge\|x\|$ for every $x\in\mathcal{H}$, and that the spectrum of T is contained in the closed unit disc; for details, see [12]. Thus, 2-isometries are examples of operators that do not belong to any of the classes that have been studied extensively. For example, 2-isometries in general are not contractions or hyponormal operators, yet their spectral theory relates them to the unit disc.

In [13] it was shown that any 2-isometric operator T, which also satisfies $\bigcap_{n>0}T^n\mathcal{H}=(0)$ and $\dim\ker T^*=1$, can be represented as multiplication by z on a Dirichlet-type space $D(\mu)$. Here μ denotes a finite nonnegative Borel measure on $\mathbf{T}$. We shall see in Section 2 that the space $D(\mu)$ as defined in [13] coincides with the space of all H^2 functions f such that $D_\zeta(f)$ is in $L^1(\mu)$; that is, $f\in H^2$ and $\int_{\mathbf{T}} D_\zeta(f)\,d\mu(\zeta)<\infty$.

It is easy to see that the Dirichlet shift (M_z, D), that is, multiplication by z on the Dirichlet space with the norm

$$\|f\|_D^2=\|f\|_{H^2}^2+\iint_{\mathbf{D}}|f'(z)|^2\,dA(z),$$

is a 2-isometry. Hence the restriction of (M_z, D) to any nonzero invariant subspace must be a 2-isometry as well. In fact, it was shown in [13] that every invariant subspace of the Dirichlet space is of the form $fD(m_f)$, where m_f is the absolutely continuous measure on $\mathbf{T}$ defined by $dm_f=(1/2\pi)|f|^2\,dt$. Thus, a systematic study of 2-isometric operators and the $D(\mu)$ spaces will in particular produce information about the invariant subspace structure of the Dirichlet shift.

A connection of the local Dirichlet integral to the Hilbert spaces of square summable power series considered by de Branges and Rovnyak [8] was pointed out to the authors by D. Sarason. It turns out that the space of all H^2 functions with finite local Dirichlet integral at $\zeta\in\mathbf{T}$ equals the space

$\mathcal{M}(\bar{\zeta}-S^*)=\{(\bar{\zeta}-S^*)g,\ g\in H^2\}$, where S^* denotes the backward shift on H^2 and the norm on $\mathcal{M}(\bar{\zeta}-S^*)$ is chosen so that $\bar{\zeta}-S^*$ is an isometry of H^2 onto $\mathcal{M}(\bar{\zeta}-S^*)$. These particular spaces have been considered by Sarason (see [16, §8]). We shall give a few more details in Section 2.

In the present paper we shall prove an analog of Carleson's formula for the local Dirichlet integral of an H^2 function (Theorem 3.1), and we shall indicate how our formula implies Carleson's. In particular, we shall see that the local Dirichlet integral of an inner function equals the absolute value of its angular derivative (see Section 3 for definitions).

Furthermore, as in the case of the classical Dirichlet integral, it will follow that cut-off operations do not increase the local Dirichlet integral. As a consequence we shall show that every function in $D(\mu)$ can be written as the quotient of two bounded functions in $D(\mu)$ (Corollary 3.8). This extends a result for the Dirichlet space D that was stated and proved in [14].

From our formula for the local Dirichlet integral, one can easily deduce that any inner function which defines a (bounded) multiplication operator on $D(\mu)$ must be a 2-isometric multiplication operator. In Section 4 we shall prove a result that implies that the converse to this statement is true as well: If a multiplication operator M_φ on $D(\mu)$ acts as a 2-isometry, then φ must be an inner function. This can be viewed as an extension of the well-known fact that the isometric multiplication operators on H^2 are given exactly by multiplications with inner functions.

Finally, in Section 5 we apply the local Dirichlet integral to some questions about cyclic vectors in the Dirichlet space. In fact, all of our results are true in the generality of all $D(\mu)$ spaces. Recall that a function f in $D(\mu)$ is called a *cyclic vector* if the polynomial multiples of f are dense in $D(\mu)$. Corollary 5.5 implies that, if g is cyclic in $D(\mu)$ and if $|f(z)|\geq|g(z)|$ for all $z\in\mathbf{D}$, then f must be cyclic. This answers a question of Brown and Shields in [4].

In Corollary 5.6 we answer another question of Brown and Shields by proving that two bounded functions f and g are cyclic in $D(\mu)$ if and only if their product fg is.

2. The $D(\mu)$ Spaces

Throughout this paper we shall only be interested in H^2 functions with finite local Dirichlet integral. If $f\in H^2$, then we shall use the symbol f to denote both the analytic function $f\in H^2(\mathbf{D})$ defined on the open unit disc $\mathbf{D}$ and its nontangential limit function $f\in H^2\subseteq L^2(\mathbf{T})$, which is defined a.e. on the unit circle $\mathbf{T}$. We shall start out with some elementary observations, which easily follow from the definitions.

Fix $\zeta\in\mathbf{T}$. For a complex number α and an H^1-function f define

$$I_\zeta(f,\alpha)=\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{f(e^{it})-\alpha}{e^{it}-\zeta}\right|^2 dt.$$

It is clear that $I_\zeta(f,\alpha)$ can be finite for at most one $\alpha\in\mathbf{C}$. If $I_\zeta(f,\alpha)=\infty$ for every $\alpha\in\mathbf{C}$, then $D_\zeta(f)=\infty$. On the other hand, if $I_\zeta(f,\alpha)$ is finite for $\alpha\in\mathbf{C}$, then the function g defined by

$$g(e^{it})=\frac{f(e^{it})-\alpha}{e^{it}-\zeta} \tag{2.1}$$

is in H^2. It follows that $f=\alpha+(e^{it}-\zeta)g\in H^2$, and we shall show in a moment that $f(\zeta)=\alpha$. Thus, we obtain $D_\zeta(f)=\inf\{I(f,\alpha):\alpha\in\mathbf{C}\}$.

Now recall that, if g is any H^2 function, then

$$(1-|z|^2)|g(z)|^2\to 0 \quad \text{as } |z|\to 1,\ z\in\mathbf{D}.$$

This can easily be verified by using the Szegö kernel function, $k_z(e^{it})=1/(1-\bar{z}e^{it})$, and the dominated convergence theorem.

If $g\in H^2$ satisfies (2.1), then for $z\in\mathbf{D}$ we have

$$|f(z)-\alpha|^2=|(z-\zeta)g(z)|^2=\frac{|z-\zeta|^2}{(1-|z|^2)}(1-|z|^2)|g(z)|^2.$$

It follows that $f(z)\to\alpha$ if $z\to\zeta$ in any oricyclic approach region

$$\mathcal{O}_\kappa(\zeta)=\{z\in\mathbf{D}:|z-\zeta|^2<\kappa(1-|z|^2)\}.$$

This implies, of course, that $\alpha=f(\zeta)$, the nontangential limit of f at ζ.

Summarizing the information gained above, we see that an H^1-function f has a finite local Dirichlet integral at $\zeta\in\mathbf{T}$ if and only if there is a complex number α such that $I_\zeta(f,\alpha)<\infty$. If $I_\zeta(f,\alpha)$ is finite, then $\alpha=f(\zeta)$ (the nontangential limit of f at ζ), the oricyclic limit of f exists at ζ, and $I_\zeta(f,\alpha)=D_\zeta(f)$.

We state and prove the following proposition for the sake of completeness. The proof is the same as in [7].

PROPOSITION 2.1. *Let $\zeta\in\mathbf{T}$. If $f\in H^2$ such that $D_\zeta(f)<\infty$, then the Fourier series of f at ζ converges to $f(\zeta)$.*

Proof. For an integer $k\geq 0$ we let $\hat{f}(k)$ denote the kth Fourier coefficient of f. If g is the H^2 function associated to f by (2.1), then $\hat{f}(0)=f(\zeta)-\zeta\hat{g}(0)$ and $\hat{f}(k)=\hat{g}(k-1)-\zeta\hat{g}(k)$ for $k>0$. Thus,

$$\begin{aligned}\sum_{k=0}^{n}\hat{f}(k)\zeta^k&=f(\zeta)-\zeta\hat{g}(0)+\sum_{k=1}^{n}\hat{g}(k-1)\zeta^k-\hat{g}(k)\zeta^{k+1}\\&=f(\zeta)-\hat{g}(n)\zeta^{n+1}\to f(\zeta)\quad\text{as } n\to\infty.\end{aligned}$$ □

In order to explain the connection between the local Dirichlet integral and the Dirichlet-type spaces that arose in [13] in the context of analytic 2-isometries, we need to recall a few definitions. For a nonnegative finite Borel measure μ on $\mathbf{T}$, define the harmonic function φ_μ by

$$\varphi_\mu(z)=\int_{\mathbf{T}}\frac{1-|z|^2}{|\zeta-z|^2}\,d\mu(\zeta).$$

If $\mu=0$ then let $D(\mu)=H^2$; otherwise define $D(\mu)$ to be the space of all H^2 functions f such that

$$\iint_{\mathbf{D}} |f'(z)|^2 \varphi_\mu(z)\, dA(z) < \infty.$$

Here we use dA to denote normalized area measure on $\mathbf{D}$, $dA(z)=(1/\pi)r\,dr\,dt$ if $z=re^{it}$. A norm on $D(\mu)$ is given by

$$\|f\|_\mu^2 = \|f\|_{H^2}^2 + \iint_{\mathbf{D}} |f'(z)|^2 \varphi_\mu(z)\, dA(z), \quad f\in D(\mu).$$

In particular, the classical Dirichlet space D equals the space $D(m)$, where m denotes normalized Lebesgue measure on $\mathbf{T}$. We already pointed out in the introduction that one obtains Douglas' formula for the Dirichlet integral by integrating the local Dirichlet integrals $D_\zeta(f)$ with respect to Lebesgue measure on $\mathbf{T}$. This fact generalizes to $D(\mu)$-integrals for arbitrary μ.

PROPOSITION 2.2. *Let μ be a nonnegative finite Borel measure on* $\mathbf{T}$. *If $f\in H^2$, then*

$$\int_{\mathbf{T}} D_\zeta(f)\, d\mu(\zeta) = \iint_{\mathbf{D}} |f'(z)|^2 \varphi_\mu(z)\, dA(z). \tag{2.2}$$

Proof. An application of Fubini's theorem to the right-hand side of (2.2) shows that it suffices to verify the identity for unit point masses δ_ζ. Furthermore, a simple change of variables implies that we may assume $\zeta=1$.

We shall first prove the proposition for polynomials q. Since q is a finite linear combination of monomials of the form z^n, (2.2) will follow if we establish that for each $n, m\in \mathbf{N}\cup\{0\}$,

$$\iint_{\mathbf{D}} (z^n)'\overline{(z^m)'}P_z(1)\, dA(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{ins}-1}{e^{is}-1}\,\overline{\frac{e^{ims}-1}{e^{is}-1}}\, ds.$$

Here $P_z(e^{it})=(1-|z|^2)/|e^{it}-z|^2$ denotes the Poisson kernel at $e^{it}\in\mathbf{T}$.

In fact, we shall show that each side of the equation equals $\min\{n,m\}$:

$$\begin{aligned}\iint_{\mathbf{D}} (z^n)'\overline{(z^m)'}P_z(1)\, dA(z) &= 2nm\int_0^1 r^{n+m-1}\frac{1}{2\pi}\int_0^{2\pi} e^{i(n-m)t}P_r(e^{it})\, dt\, dr\\ &= 2nm\int_0^1 r^{n+m-1}r^{|n-m|}\, dr\\ &= \frac{2nm}{n+m+|n-m|} = \min\{n,m\};\end{aligned}$$

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{ins}-1}{e^{is}-1}\,\overline{\frac{e^{ims}-1}{e^{is}-1}}\, ds &= \frac{1}{2\pi}\int_0^{2\pi}\sum_{k=0}^{n-1} e^{iks}\sum_{1=0}^{m-1} e^{-ils}\, ds\\ &= \min\{n,m\}.\end{aligned}$$

To finish the proof we must show that

$$D_1(f) = \iint_{\mathbf{D}} |f'(z)|^2 \frac{1-|z|^2}{|1-z|^2}\, dA(z)$$

for arbitrary H^2 functions f.

First assume that $f \in H^2$ such that $D_1(f) < \infty$. Then f has a finite radial limit $f(1)$ at 1, and we may define an H^2 function g by $g(z) = (f(z) - f(1))/(z-1)$. Choose a sequence of polynomials $\{q_n\}$ that converges to g in H^2. Then the sequence $\{p_n\}$, $p_n(z) = f(1) + (z-1)q_n(z)$, converges pointwise in $\mathbf{D}$ to the function f. Also, the definition of p_n implies that $D_1(p_n - f) = \|q_n - g\|_{H^2}^2 \to 0$. It now follows from the first part of the proof that $\{p_n\}$ is a Cauchy sequence in $D(\delta_1)$; thus $\{p_n\}$ must converge to $f \in D(\delta_1)$ and

$$\begin{aligned} \iint_{\mathbf{D}} |f'(z)|^2 P_z(1)\, dA(z) &= \lim_{n\to\infty} \iint_{\mathbf{D}} |p_n'(z)|^2 P_z(1)\, dA(z) \\ &= \lim_{n\to\infty} D_1(p_n) \\ &= D_1(f). \end{aligned}$$

Now suppose that $f \in D(\delta_1)$. By Corollary 3.8 of [13] (the polynomials are dense in $D(\mu)$) we may choose a sequence $\{p_n\}$ of polynomials such that

$$\|p_n - f\|_{\delta_1}^2 = \|p_n - f\|_{H^2}^2 + \iint_{\mathbf{D}} |p_n'(z) - f'(z)|^2 P_z(1)\, dA(z) \to 0.$$

Then it follows from the above that the polynomials q_n,

$$q_n(z) = \frac{p_n(z) - p_n(1)}{z-1},$$

form a Cauchy sequence in H^2. Hence the sequence $\{q_n\}$ converges to a function g in H^2. For each fixed $z \in \mathbf{D}$ we have

$$\frac{p_n(z) - p_n(1)}{z-1} = q_n(z) \to g(z) \quad \text{and} \quad p_n(z) \to f(z).$$

It follows that $p_n(1)$ converges to some complex number α, and we have $g(z) = (f(z) - \alpha)/(z-1) \in H^2$. Hence $I_1(f, \alpha) = \|g\|_{H^2}^2 < \infty$, and also $D_1(f) = I_1(f, \alpha) < \infty$. Thus, the above implies that

$$\iint_{\mathbf{D}} |f'(z)|^2 P_z(1)\, dA(z) = D_1(f).$$

This concludes the proof of Proposition 2.2. □

The following corollary improves the results of Section 4 of [13].

COROLLARY 2.3. *Let $f \in D(\mu)$. Then the oricyclic limit of f exists at μ-a.e. boundary point $\zeta \in \mathbf{T}$. Furthermore, the boundary value function f is in $L^2(\mu)$ and*

(2.3) $$\|zf\|_\mu^2-\|f\|_\mu^2=\int_{\mathbf{T}}|f(\zeta)|^2\,d\mu(\zeta).$$

Proof. It follows from Proposition 2.2 that any function f in $D(\mu)$ has a finite local Dirichlet integral $D_\zeta(f)$ for μ almost every $\zeta\in\mathbf{T}$. It now follows from the discussion preceding Proposition 2.1 that the oricyclic limit of f exists at all those $\zeta\in\mathbf{T}$ where $D_\zeta(f)$ is finite.

We observe that, for any f with $D_\zeta(f)<\infty$,

$$\frac{zf(z)-\zeta f(\zeta)}{z-\zeta}=z\frac{f(z)-f(\zeta)}{z-\zeta}+f(\zeta).$$

Hence, from the H^2-orthogonality of the functions on the right-hand side, we obtain

$$\left\|\frac{zf(z)-\zeta f(\zeta)}{z-\zeta}\right\|_{H^2}^2=\left\|\frac{f(z)-f(\zeta)}{z-\zeta}\right\|_{H^2}^2+|f(\zeta)|^2,$$

which is equivalent to $D_\zeta(zf)-D_\zeta(f)=|f(\zeta)|^2$. The identity (2.3) follows by integrating this with respect to μ and by noting that $\|zf\|_{H^2}^2-\|f\|_{H^2}^2=0$ for any $f\in H^2$. □

In the case where $\mu=m=$ normalized Lebesgue measure on $\mathbf{T}$, the theorem deals with the Dirichlet space $D=D(m)$. We note that in this case much better results than Corollary 2.3 were obtained in [11]. In fact, it is shown there that every function f in D has a finite tangential limit at m-a.e. $\zeta\in\mathbf{T}$, where the approach region can be taken to have exponential contact with $\mathbf{T}$ at ζ.

However, we point out that in the case where μ is a point mass, $\mu=\delta_\zeta$, our result is best possible in the following sense: If $\{\alpha_n\}\subseteq\mathbf{D}$ approaches ζ and $\{\alpha_n\}$ is not contained in $\mathcal{O}_\kappa(\zeta)$ for any $\kappa>0$, then there exists a function $f\in D(\delta_\zeta)$ such that $|f(\alpha_n)|\to\infty$ as $n\to\infty$.

Indeed, the existence of f will follow from the uniform boundedness principle if we show that the evaluation functionals λ_{α_n} are not pointwise bounded. To this end let $f_n(z)=(1-|\alpha_n|^2)^{1/2}(\zeta-z)/(1-\bar{\alpha}_n z)$. It now follows easily from Proposition 2.2 that $\|f_n\|_{\delta_\zeta}^2\le 3$, while the assumption on $\{\alpha_n\}$ implies that $\{f_n(\alpha_n)\}\to\infty$ as $n\to\infty$.

For a function $f\in D(\mu)$ we shall frequently use the nontangential limit function f and the fact that $f\in L^2(\mu)\cap L^2(m)$. As a notational help, we shall use $f(\zeta)$ to denote a limit that exists a.e. $[\mu]$, because $f\in D(\mu)$ (or, more specifically, because $f\in D(\delta_\zeta)$), rather than $f(e^{it})$, which will denote the limit that exists a.e. $[m]$, because $f\in H^2$.

We now return to the connection of the spaces $D(\delta_\zeta)$ to the de Branges spaces $\mathfrak{M}(\bar{\zeta}-S^*)$ mentioned in the introduction. Fix $\zeta\in\mathbf{T}$ and let S denote the unilateral shift, that is, the operator of multiplication by z on H^2. The space $\mathfrak{M}(\bar{\zeta}-S^*)\subseteq H^2$ is defined to be the range of the operator $\bar{\zeta}-S^*$. A Hilbert space norm $\|\;\|_{\mathfrak{M}}$ can be defined on $\mathfrak{M}(\bar{\zeta}-S^*)$ so that $\bar{\zeta}-S^*$ acts as an isometry from H^2 onto $\mathcal{M}(\bar{\zeta}-S^*)$. Thus, if $f=(\bar{\zeta}-S^*)g\in\mathcal{M}(\bar{\zeta}-S^*)$ for some $g\in H^2$, then

$$\|f\|_{\mathcal{M}}^2=\|(\bar{\zeta}-S^*)g\|_{\mathcal{M}}^2=\|g\|_{H^2}^2.$$

PROPOSITION 2.4. *The spaces $\mathcal{M}(\bar{\zeta}-S^*)$ and $D(\delta_\zeta)$ coincide with equivalence of norms. More precisely, we have*

$$\|f\|_{\mathcal{M}}^2=|f(\zeta)|^2+D_\zeta(f)=D_\zeta(zf) \quad \textit{for any } f\in H^2.$$

Proof. Let $f\in D(\delta_\zeta)$. Then we can define the function g by

$$g(z)=\zeta f(\zeta)+\zeta z\frac{f(z)-f(\zeta)}{z-\zeta}.$$

The assumption that $f\in D(\delta_\zeta)$ implies that $g\in H^2$. Hence

$$(\bar{\zeta}-S^*)g(z)=f(\zeta)+z\frac{f(z)-f(\zeta)}{z-\zeta}-\zeta\frac{f(z)-f(\zeta)}{z-\zeta}=f(z),$$

and thus $f\in\mathcal{M}(\bar{\zeta}-S^*)$ and

$$\begin{aligned}\|f\|_{\mathcal{M}}^2=\|(\bar{\zeta}-S^*)g\|_{\mathcal{M}}^2=\|g\|_{H^2}^2&=|g(0)|^2+\|S^*g\|_{H^2}^2\\ &=|f(\zeta)|^2+D_\zeta(f)=D_\zeta(zf).\end{aligned}$$

The last equality follows from (2.3).

Now, if $f=(\bar{\zeta}-S^*)g\in\mathcal{M}(\bar{\zeta}-S^*)$, then

$$f=(\bar{\zeta}-S^*)g=\bar{\zeta}(g(0)+(z-\zeta)S^*g).$$

Hence with $\alpha=\bar{\zeta}g(0)$ we have $(f-\alpha)/(z-\zeta)=S^*g$; thus $D_\zeta(f)=\|S^*g\|_{H^2}^2$, that is, $f\in D(\delta_\zeta)$. □

3. A Formula for the Local Dirichlet Integral

In this section we shall derive a formula for $D_\zeta(f)$ for an arbitrary H^2 function f. This formula is an analog of Carleson's formula [6], and we shall show that it implies Carleson's formula. However, our approach is quite different from Carleson's proof.

For the rest of this paper we have adopted the standard notation that $0\cdot(\mp\infty)=0$. Note that, in the notation for the Blaschke factor B of the H^2 function f, we certainly allow finite Blaschke products and understand that $\bar{\alpha}_j/|\alpha_j|=1$ if $\alpha_j=0$.

THEOREM 3.1. *Let $\zeta\in\mathbf{T}$, let $f\in H^2$, and let $f=BSf_0$, that is, let*

$$f(z)=\prod_{j=1}^{\infty}\frac{\bar{\alpha}_j}{|\alpha_j|}\frac{\alpha_j-z}{1-\bar{\alpha}_j z}\exp\left\{-\int\frac{e^{it}+z}{e^{it}-z}\,d\sigma(t)\right\}\exp\left\{\frac{1}{2\pi}\int\frac{e^{it}+z}{e^{it}-z}\log|f(e^{it})|\,dt\right\}$$

be the factorization of f into a Blaschke product, a singular inner factor, and an outer factor. Write $u(e^{it})=\log|f_0(e^{it})|$. Then

$$D_\zeta(f)=\sum_{j=1}^{\infty}P_{\alpha_j}(\zeta)|f_0(\zeta)|^2+\int_0^{2\pi}\frac{2}{|e^{it}-\zeta|^2}\,d\sigma(t)|f_0(\zeta)|^2$$
$$+\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{2u(e^{it})}-e^{2u(\zeta)}-2e^{2u(\zeta)}(u(e^{it})-u(\zeta))}{|e^{it}-\zeta|^2}\,dt.$$

If either of the canonical factors of f is absent, then the corresponding summand in the expression for $D_\zeta(f)$ will be 0.

First we note that the integrand of the last integral is always nonnegative. To see this we introduce a function that we shall also use in the proof of the theorem. Define $\phi:\mathbf{R}\to\mathbf{R}$ by

$$\phi(x)=e^{2x}-1-2x.$$

It is easy to check that ϕ takes only nonnegative values. Furthermore, if β is any real number, then

$$e^{2x}-e^{2\beta}-2e^{2\beta}(x-\beta)=e^{2\beta}\phi(x-\beta)\geq 0\quad\forall x\in\mathbf{R}. \tag{3.1}$$

Thus, it follows that the above integrand is nonnegative.

If f is an arbitrary H^2 function, then since $\{\zeta\}\subseteq\mathbf{T}$ is a set of Lebesgue measure zero the value $f_0(\zeta)$ may or may not exist as the nontangential limit of the outer factor of f at ζ. In this case the above formula must be understood in the following sense: If there is some way to define $|f_0(\zeta)|$ so that the right-hand side of the equation is finite, then $D_\zeta(f)$ is finite and the equation holds with this choice of $|f_0(\zeta)|$. In this case it follows from the above formula that $D_\zeta(f_0)$ is finite as well, and this implies that $f_0(\zeta)$ exists as the oricyclic limit of $f_0(z)$ at ζ. On the other hand, if the nontangential limit of $f_0(z)$ does not exist at ζ, or if it exists and equals $\alpha\in\mathbf{C}$, and the right-hand side of the above equation is infinite with $|f_0(\zeta)|=|\alpha|$, then it is infinite for every choice of $|f_0(\zeta)|$ and $D_\zeta(f)=\infty$.

Finally, we point out that if $f_0(\zeta)=0$ then the whole formula reduces to

$$D_\zeta(f)=\frac{1}{2\pi}\int_0^{2\pi}\frac{|f_0(e^{it})|^2}{|e^{it}-\zeta|^2}\,dt,$$

which immediately follows from the definition of the local Dirichlet integral.

Proof. We shall prove Theorem 3.1 via a sequence of lemmas and propositions. The strategy is as follows: First we shall show that $D_\zeta(f)=\|(f(z)-f(\zeta))/(z-\zeta)\|_{H^2}^2$ is the limit of $\|(f(z)-f(\lambda))/(z-\lambda)\|_{H^2}^2$ as $\lambda\to\zeta$ nontangentially. From this it will follow that for any H^2 function of the form φf, where φ is inner, one has

$$D_\zeta(\varphi f)=D_\zeta(\varphi)|f(\zeta)|^2+D_\zeta(f).$$

Thus, Theorem 3.1 may be proved by considering the inner and the outer factor of f separately.

Furthermore, the approximation of $D_\zeta(f)$ by $\|(f(z)-f(\lambda))/(z-\lambda)\|_{H^2}^2$ will show that, if φ is an inner function, then $D_\zeta(\varphi)$ equals the absolute value of the angular derivative of φ at ζ (definition below). The angular derivatives of inner functions have been determined by M. Riesz [15] and Frostman [10]; thus, for that part of the above formula involving the inner factor, we shall refer the reader to the literature.

Finally, as the last step, we shall verify the formula for outer functions f in H^2.

We start with a lemma, whose proof we omit, as it is an elementary (and standard) computation.

LEMMA 3.2. *Let $\lambda \in \mathbf{D}$ and $f \in H^2$. Then*

$$\left\| \frac{f(z)-f(\lambda)}{z-\lambda} \right\|_{H^2}^2 = \frac{1}{2\pi}\int_0^{2\pi} \frac{|f(e^{it})|^2}{|1-\bar{\lambda}e^{it}|^2}\,dt - \frac{|f(\lambda)|^2}{1-|\lambda|^2}$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{|f(e^{it})|^2-|f(\lambda)|^2}{|1-\bar{\lambda}e^{it}|^2}\,dt.$$

Next we fix some notation. For $\zeta \in \mathbf{T}$ we define the nontangential approach region $\Gamma_\kappa(\zeta) = \{\lambda \in \mathbf{D} : |\lambda-\zeta| < \kappa(1-|\lambda|)\}$. We say that $\lambda \to \zeta \in \mathbf{T}$ *nontangentially* if there exists $\kappa > 0$ such that $\lambda \to \zeta$, $\lambda \in \Gamma_\kappa(\zeta)$.

LEMMA 3.3. *Let $\zeta \in \mathbf{T}$ and $f \in H^2$. For $\lambda \in \mathbf{D}$ set*

$$g_\lambda(z) = \frac{f(z)-f(\lambda)}{z-\lambda}.$$

(a) *If $f \in D(\delta_\zeta)$, then $\|g_\lambda\|_{H^2}^2 \le (1+\kappa)^2 D_\zeta(f)$ for each $\kappa > 0$ and $\lambda \in \Gamma_\kappa(\zeta)$, and $\|g_\lambda\|_{H^2}^2 \to D_\zeta(f)$ as $\lambda \to \zeta$ nontangentially.*

(b) *If there is a sequence $\{\lambda_n\} \subseteq \mathbf{D}$, $\{\lambda_n\} \to \zeta$ (unrestrictedly) such that $\{\|g_{\lambda_n}\|_{H^2}\}$ is bounded, then $f \in D(\delta_\zeta)$.*

Consequently, $\|g_\lambda\|_{H^2}^2 \to D_\zeta(f)$ as $\lambda \to \zeta$ nontangentially, whether $D_\zeta(f)$ is finite or infinite.

Proof. (a) Let $f \in D(\delta_\zeta)$; then the function $g(z) = (f(z)-f(\zeta))/(z-\zeta)$ is in H^2 and $\|g\|_{H^2}^2 = D_\zeta(f)$. Thus, $f(z) = f(\zeta) + (z-\zeta)g(z)$ and, for $\lambda \in \mathbf{D}$,

$$g_\lambda(z) = \frac{f(z)-f(\lambda)}{z-\lambda} = \frac{(z-\zeta)g(z)-(\lambda-\zeta)g(\lambda)}{z-\lambda}$$
$$= g(z) + (\lambda-\zeta)\frac{g(z)-g(\lambda)}{z-\lambda}. \tag{3.2}$$

By Lemma 3.2 we have

$$\left\| \frac{g-g(\lambda)}{z-\lambda} \right\|_{H^2}^2 = \frac{1}{2\pi}\int_0^{2\pi} \frac{|g(e^{it})|^2}{|1-\bar{\lambda}e^{it}|^2}\,dt - \frac{|g(\lambda)|^2}{1-|\lambda|^2}$$
$$\le \frac{1}{(1-|\lambda|)^2}\|g\|_{H^2}^2.$$

Hence, for $\lambda \in \Gamma_\kappa(\zeta)$,

$$\|g_\lambda\|_{H^2} \le \left(1+\frac{|\lambda-\zeta|}{1-|\lambda|}\right)\|g\|_{H^2} \le (1+\kappa)\sqrt{D_\zeta(f)}.$$

Furthermore,

$$|\zeta-\lambda|^2\left\|\frac{g-g(\lambda)}{z-\lambda}\right\|_{H^2}^2 \le \frac{1}{2\pi}\int_0^{2\pi}\frac{|\zeta-\lambda|^2}{|1-\bar{\lambda}e^{it}|^2}|g(e^{it})|^2\,dt \to 0.$$

as $\lambda \to \zeta$ nontangentially by the dominated convergence theorem. Thus, $g_\lambda \to g$ as $\lambda \to \zeta$ nontangentially, and so $\|g_\lambda\|_{H^2}^2 \to \|g\|_{H^2}^2 = D_\zeta(f)$. This proves part (a) of the lemma.

(b) Now assume that $\{\lambda_n\} \subseteq \mathbf{D}$, $\{\lambda_n\} \to \zeta$ (unrestrictedly), and $\{\|g_{\lambda_n}\|_{H^2}\}$ is bounded. Then $\{g_{\lambda_n}\}$ will have a subsequence which converges weakly to a function $g \in H^2$. By possibly renaming the sequences we may assume that $\{g_{\lambda_n}\} \to g$ (weakly). Weak convergence implies pointwise convergence, thus we see that $\{f(\lambda_n)\}$ must converge to some $\alpha \in \mathbf{C}$. From this we see that $g(z) = (f(z)-\alpha)/(z-\zeta) \in H^2$. This is equivalent to $f \in D(\delta_\zeta)$. □

By combining Lemmas 3.2 and 3.3 we can now show that the proof of Theorem 3.1 can be split into two parts: one corresponding to the inner factor and one to the outer factor of f.

LEMMA 3.4. *Let $\zeta \in \mathbf{T}$ and let φ be an inner function and $f \in H^2$.*

(a) *If $D_\zeta(f) < \infty$, then*

$$D_\zeta(\varphi f) = D_\zeta(\varphi)|f(\zeta)|^2 + D_\zeta(f). \tag{3.3}$$

(b) *If $D_\zeta(f) = \infty$, then $D_\zeta(\varphi f) = \infty$.*

We point out that according to our convention $\infty \cdot 0 = 0$; thus, if $f(\zeta) = 0$, (3.3) then becomes $D_\zeta(\varphi f) = D_\zeta(f)$, whether $D_\zeta(\varphi)$ is finite or infinite.

Proof. For $\lambda \in \mathbf{D}$ consider

$$\begin{aligned}
&\left\|\frac{\varphi f - \varphi f(\lambda)}{z-\lambda}\right\|_{H^2}^2 \\
&= \frac{1}{2\pi}\int_0^{2\pi}\frac{|\varphi f(e^{it})|^2 - |\varphi f(\lambda)|^2}{|1-\bar{\lambda}e^{it}|^2}\,dt \quad \text{(by Lemma 3.2)} \\
&= \frac{1}{2\pi}\int_0^{2\pi}\frac{|\varphi f(e^{it})|^2 - |f(\lambda)|^2 + |f(\lambda)|^2 - |\varphi f(\lambda)|^2}{|1-\bar{\lambda}e^{it}|^2}\,dt \\
&= \frac{1}{2\pi}\int_0^{2\pi}\frac{|f(e^{it})|^2 - |f(\lambda)|^2}{|1-\bar{\lambda}e^{it}|^2}\,dt + |f(\lambda)|^2\frac{1}{2\pi}\int_0^{2\pi}\frac{|\varphi(e^{it})|^2 - |\varphi(\lambda)|^2}{|1-\bar{\lambda}e^{it}|^2}\,dt \\
&\qquad \text{(because } |\varphi(e^{it})| = 1 \text{ a.e. } [m]) \\
&= \left\|\frac{f-f(\lambda)}{z-\lambda}\right\|_{H^2}^2 + |f(\lambda)|^2\left\|\frac{\varphi-\varphi(\lambda)}{z-\lambda}\right\|_{H^2}^2 \quad \text{(by Lemma 3.2).}
\end{aligned}$$

In particular, we have

$$\left\|\frac{\varphi f-\varphi f(\lambda)}{z-\lambda}\right\|_{H^2}^2 \geq \left\|\frac{f-f(\lambda)}{z-\lambda}\right\|_{H^2}^2,$$

thus (b) follows from Lemma 3.3 by letting $\lambda\to\zeta$ nontangentially.

To prove (a) we assume that $D_\zeta(f)<\infty$ and distinguish two cases. First, if $f(\zeta)\neq 0$ then $f(\lambda)\to f(\zeta)$ as $\lambda\to\zeta$ nontangentially. Hence in this case (3.3) follows again from the above equation and Lemma 3.3 by letting $\lambda\to\zeta$ nontangentially. Finally, if $f(\zeta)=0$ then we have

$$\begin{aligned} D_\zeta(f) &= \frac{1}{2\pi}\int\left|\frac{f(e^{it})-f(\zeta)}{e^{it}-\zeta}\right|^2 dt = \frac{1}{2\pi}\int\left|\frac{f(e^{it})}{e^{it}-\zeta}\right|^2 dt \\ &= \frac{1}{2\pi}\int\left|\frac{\varphi f(e^{it})}{e^{it}-\zeta}\right|^2 dt = D_\zeta(\varphi f). \end{aligned}$$

□

Now recall that a function $f:\mathbf{D}\to\mathbf{D}$ is said to have an *angular derivative* at a point $\zeta\in\mathbf{T}$ if there exists a $\omega\in\mathbf{T}$ such that $(f(\lambda)-\omega)/(\lambda-\zeta)$ tends to a finite limit as λ tends to ζ nontangentially. If it exists, then this limit is called the angular derivative of f at ζ, and we shall denote it by $f'(\zeta)$. It is a part of the Julia–Carathéodory theorem (see [5, §§298–299]) that f has an angular derivative at ζ if and only if

$$\liminf_{\lambda\to\zeta}\frac{1-|f(\lambda)|}{1-|\lambda|}=d<\infty,$$

and that in this case $d=|f'(\zeta)|$. Thus, it is consistent to set $|f'(\zeta)|=\infty$ if f does not have a finite angular derivative.

PROPOSITION 3.5. *If an analytic function $f:\mathbf{D}\to\mathbf{D}$ has a finite angular derivative at $\zeta\in\mathbf{T}$, then $f\in D(\delta_\zeta)$ and $D_\zeta(f)\leq|f'(\zeta)|$.*

If φ is an inner function, then $\varphi\in D(\delta_\zeta)$ if and only if φ has a finite angular derivative at ζ and $D_\zeta(f)=|\varphi'(\zeta)|$.

Proof. If f maps the unit disc into itself, then $|f(e^{it})|\leq 1$ a.e. $[m]$; thus,

$$\begin{aligned} \left\|\frac{f-f(\lambda)}{z-\lambda}\right\|_{H^2}^2 &= \frac{1}{2\pi}\int_0^{2\pi}\frac{|f(e^{it})|^2-|f(\lambda)|^2}{|1-\bar\lambda e^{it}|^2}\,dt \\ &\leq \frac{1}{2\pi}\int_0^{2\pi}\frac{1-|f(\lambda)|^2}{|1-\bar\lambda e^{it}|^2}\,dt \\ &= \frac{1-|f(\lambda)|^2}{1-|\lambda|^2} \\ &= \frac{1-|f(\lambda)|}{1-|\lambda|}\,\frac{1+|f(\lambda)|}{1+|\lambda|}. \end{aligned} \tag{3.4}$$

If f has a finite angular derivative at ζ, that is, if

$$\liminf_{\lambda\to\zeta}\frac{1-|f(\lambda)|}{1-|\lambda|}=d<\infty,$$

then clearly $|f(\lambda_n)| \to 1$ for some sequence $\{\lambda_n\} \to \zeta$. Thus, the first part of the proposition follows from Lemma 3.3(b).

The second part follows as well, because if φ is an inner function then we actually have equality in (3.4). □

Note that Proposition 3.5 implies that if an analytic function $f: \mathbf{D} \to \mathbf{D}$ has a finite angular derivative at ζ, then the Fourier series of f converges at ζ (see Proposition 2.1).

As mentioned above, the angular derivatives of inner functions have been determined by M. Riesz and Frostman ([15], [10]). For a simple proof of these results see [2, Thm. 2]. In fact, let φ be an inner function, that is, let

$$\varphi(z) = \prod_{j=1}^{\infty} \frac{\bar{\alpha}_j}{|\alpha_j|} \frac{\alpha_j - z}{1 - \bar{\alpha}_j z} \exp\left\{-\int \frac{e^{it}+z}{e^{it}-z}\, d\sigma(t)\right\},$$

where σ is singular with respect to Lebesgue measure. Then

$$|\varphi'(\zeta)| = \sum_{j=1}^{\infty} P_{\alpha_j}(\zeta) + \int_0^{2\pi} \frac{2}{|e^{is}-\zeta|^2}\, d\sigma(s).$$

This holds whether the above expression is finite or infinite.

In light of the equality of the space $D(\delta_\zeta)$ and the de Branges space $\mathcal{M}(\bar{\zeta} - S^*)$, Proposition 3.5 is similar to Theorem 6 of [16]. The difference lies in the proofs and in the fact that we establish actual equality of terms rather than just equivalence.

To finish the proof of Theorem 3.1, we only have to verify the formula for outer functions. Thus, from now on we shall assume that $f \in H^2$, $f \neq 0$, and that $|f(\lambda)| = e^{u(\lambda)}$, where

$$u(\lambda) = \frac{1}{2\pi} \int P_\lambda(e^{it}) \log|f(e^{it})|\, dt.$$

The harmonic function u has nontangential limits $u(e^{it})$ at a.e. $e^{it} \in \mathbf{T}$, $u \in L^1(m)$, and $u(e^{it}) = \log|f(e^{it})|$ a.e. $[m]$.

For $|\lambda| \leq 1$ define

$$I(\lambda) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{2u(e^{it})} - e^{2u(\lambda)} - 2e^{2u(\lambda)}(u(e^{it}) - u(\lambda))}{|e^{it}-\lambda|^2}\, dt.$$

This expression exists in $[0,\infty]$ for each $\lambda \in \mathbf{D}$ and for a.e. $\lambda \in \mathbf{T}$ $[m]$.

We need to show that $I(\zeta) = D_\zeta(f)$. We note that for $\lambda \in \mathbf{D}$,

$$\text{(3.5)} \qquad \frac{1}{2\pi} \int_0^{2\pi} \frac{u(e^{it})}{|e^{it}-\lambda|^2}\, dt = \frac{u(\lambda)}{1-|\lambda|^2} = \frac{1}{2\pi} \int_0^{2\pi} \frac{u(\lambda)}{|e^{it}-\lambda|^2}\, dt.$$

Hence it follows that

$$\text{(3.6)} \qquad \begin{aligned} I(\lambda) &= \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{2u(e^{it})} - e^{2u(\lambda)}}{|e^{it}-\lambda|^2}\, dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(e^{it})|^2 - |f(\lambda)|^2}{|e^{it}-\lambda|^2}\, dt = \left\| \frac{f - f(\lambda)}{z-\lambda} \right\|_{H^2}^2, \quad \lambda \in \mathbf{D}. \end{aligned}$$

The integrand in the definition of $I(\lambda)$ is always nonnegative (see (3.1)); thus, it follows immediately from Fatou's lemma and Lemma 3.3 that, if $u(\lambda)\to u(\zeta)$ as $\lambda\to\zeta$ (nontangentially), then

$$I(\zeta)\le \underline{\lim}\, I(\lambda)=D_\zeta(f). \tag{3.7}$$

Furthermore, if $D_\zeta(f)<\infty$, then $f(\lambda)\to f(\zeta)$ as $\lambda\to\zeta$ nontangentially, hence also $u(\lambda)\to u(\zeta)$. This means that if $u(\lambda)$ does not converge to $u(\zeta)$ then $D_\zeta(f)=\infty$, and we see that (3.7) holds without restriction. The reverse inequality is trivially true if $I(\zeta)=\infty$. Hence the proof of the following claim will conclude the proof of Theorem 3.1.

CLAIM. *If $I(\zeta)<\infty$ for some choice of $u(\zeta)$, then $u(\lambda)\to u(\zeta)$ as $\lambda\to\zeta$ nontangentially and $D_\zeta(f)\le I(\zeta)$.*

Proof. If $I(\zeta)$ is finite for $u(\zeta)=-\infty$, then

$$\begin{aligned}
I(\zeta)&=\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{2u(e^{it})}-e^{2u(\zeta)}-2e^{2u(\zeta)}(u(e^{it})-u(\zeta))}{|e^{it}-\zeta|^2}\,dt\\
&=\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{2u(e^{it})}}{|e^{it}-\zeta|^2}\,dt\\
&=\left\|\frac{f}{z-\zeta}\right\|_{H^2}^2\\
&=D_\zeta(f).
\end{aligned}$$

Now assume $I(\zeta)<\infty$ for $u(\zeta)>-\infty$. We shall first show that $u(\lambda)\to u(\zeta)$ as $\lambda\to\zeta$ nontangentially.

To this end recall the definition of the function $\phi:\mathbf{R}\to\mathbf{R}$,

$$\phi(x)=e^{2x}-1-2x.$$

We note that ϕ is convex, $\phi(x)\ge 0$ for all $x\in\mathbf{R}$, and $\phi(x)=0$ if and only if $x=0$.

Now factor $e^{2u(\zeta)}$ out of the integral for $I(\zeta)$, and observe that the assumption implies that

$$\int_0^{2\pi}\frac{\phi(u(e^{it})-u(\zeta))}{|e^{it}-\zeta|^2}\,dt<\infty.$$

We have

$$\begin{aligned}
\phi(u(\lambda)-u(\zeta))&=\phi\left(\frac{1}{2\pi}\int P_\lambda(e^{it})(u(e^{it})-u(\zeta))\,dt\right)\\
&\le\frac{1}{2\pi}\int P_\lambda(e^{it})\phi(u(e^{it})-u(\zeta))\,dt \quad \text{(by Jensen's inequality)}\\
&=(1-|\lambda|^2)\frac{1}{2\pi}\int_0^{2\pi}\frac{\phi(u(e^{it})-u(\zeta))}{|e^{it}-\zeta|^2}\frac{|e^{it}-\zeta|^2}{|e^{it}-\lambda|^2}\,dt\\
&\le(1+\kappa)^2(1-|\lambda|^2)\frac{1}{2\pi}\int_0^{2\pi}\frac{\phi(u(e^{it})-u(\zeta))}{|e^{it}-\zeta|^2}\,dt
\end{aligned}$$

for λ in the nontangential approach region $\Gamma_\kappa(\zeta)$. Thus $u(\lambda)\to u(\zeta)$ as $\lambda\to\zeta$ nontangentially.

In order to see that $D_\zeta(f)\le I(\zeta)$ we consider

$$\begin{aligned}
&I(\zeta)-I(\lambda)\\
&=\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{2u(e^{it})}-e^{2u(\zeta)}-2e^{2u(\zeta)}(u(e^{it})-u(\zeta))}{|e^{it}-\zeta|^2}\,dt\\
&\quad-\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{2u(e^{it})}-e^{2u(\lambda)}}{|e^{it}-\lambda|^2}\,dt\quad\text{(by (3.6))}\\
&=\frac{1}{2\pi}\int_0^{2\pi}(e^{2u(e^{it})}-e^{2u(\zeta)}-2e^{2u(\zeta)}(u(e^{it})-u(\zeta)))\left(\frac{1}{|e^{it}-\zeta|^2}-\frac{1}{|e^{it}-\lambda|^2}\right)dt\\
&\quad+\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{2u(e^{it})}-e^{2u(\zeta)}-2e^{2u(\zeta)}(u(e^{it})-u(\zeta))}{|e^{it}-\lambda|^2}-\frac{e^{2u(e^{it})}-e^{2u(\lambda)}}{|e^{it}-\lambda|^2}\,dt\\
&=\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{2u(e^{it})}-e^{2u(\zeta)}-2e^{2u(\zeta)}(u(e^{it})-u(\zeta))}{|e^{it}-\zeta|^2}\left(1-\frac{|e^{it}-\zeta|^2}{|e^{it}-\lambda|^2}\right)dt\\
&\quad+\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{2u(\lambda)}-e^{2u(\zeta)}-2e^{2u(\zeta)}(u(\lambda)-u(\zeta))}{|e^{it}-\lambda|^2}\,dt\quad\text{(by (3.5))}.
\end{aligned}$$

The integrand in the last integral is nonnegative and the first integral converges to zero by the dominated convergence theorem as $\lambda\to\zeta$ nontangentially. Furthermore, by Lemma 3.3 we have $I(\lambda)\to D_\zeta(f)$ as $\lambda\to\zeta$ nontangentially. Thus $I(\zeta)-D_\zeta(f)\ge 0$, and the claim follows. This also concludes the proof of Theorem 3.1. □

As a first corollary we obtain Carleson's formula [6].

COROLLARY 3.6 (Carleson). *Let $f\in H^2$ and let $f=BSf_0$, that is, let*

$$f(z)=\prod_{j=1}^{\infty}\frac{\bar\alpha_j}{|\alpha_j|}\frac{\alpha_j-z}{1-\bar\alpha_j z}\exp\left\{-\int\frac{e^{it}+z}{e^{it}-z}\,d\sigma(t)\right\}$$
$$\times\exp\left\{\frac{1}{2\pi}\int\frac{e^{it}+z}{e^{it}-z}\log|f(e^{it})|\,dt\right\}$$

be the factorization of f into a Blaschke product, a singular inner factor, and an outer factor. Write $u(e^{it})=\log|f(e^{it})|$. Then

$$\begin{aligned}
&\iint_{\mathbf{D}}|f'(z)|^2\,dA(z)\\
&=\frac{1}{2\pi}\int_0^{2\pi}\sum_{j=1}^{\infty}P_{\alpha_j}(e^{is})|f(e^{is})|^2\,ds+\frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi}\frac{2}{|e^{it}-e^{is}|^2}\,d\sigma(t)|f(e^{is})|^2\,ds\\
&\quad+\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2\pi}\int_0^{2\pi}\frac{(e^{2u(e^{it})}-e^{2u(e^{is})})(u(e^{it})-u(e^{is}))}{|e^{it}-e^{is}|^2}\,dt\,ds.
\end{aligned}$$

Proof. We shall use Theorem 3.1. The Dirichlet integral of f,

$$\iint_{\mathbf{D}} |f'(z)|^2 \, dA(z),$$

equals the $L^1(\mathbf{T}, m)$ norm of $D_\zeta(f)$ (by Douglas' formula; see also Proposition 2.2). Thus, since $|f(e^{is})| = |f_0(e^{is})|$ a.e. we need only check the summand dealing with the outer factor of f. If f is an outer function, then by Theorem 3.1 we have

$$\begin{aligned}
&\iint_{\mathbf{D}} |f'(z)|^2 \, dA(z) \\
&= \frac{1}{2\pi}\int_0^{2\pi} D_{e^{is}}(f)\, ds \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{2u(e^{it})} - e^{2u(e^{is})} - 2e^{2u(e^{is})}(u(e^{it}) - u(e^{is}))}{|e^{it} - e^{is}|^2}\, dt\, ds \\
&= \frac{1}{2}\frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{2u(e^{it})} - e^{2u(e^{is})} - 2e^{2u(e^{is})}(u(e^{it}) - u(e^{is}))}{|e^{it} - e^{is}|^2}\, dt\, ds \\
&\quad + \frac{1}{2}\frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{2u(e^{is})} - e^{2u(e^{it})} - 2e^{2u(e^{it})}(u(e^{is}) - u(e^{it}))}{|e^{is} - e^{it}|^2}\, dt\, ds \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2\pi}\int_0^{2\pi} \frac{(e^{2u(e^{it})} - e^{2u(e^{is})})(u(e^{it}) - u(e^{is}))}{|e^{it} - e^{is}|^2}\, dt\, ds. \qquad \square
\end{aligned}$$

In [14] it was shown that every function in the Dirichlet space D can be written as the quotient of two bounded functions in D. In fact, this result holds for analytic functions with finite Dirichlet integral on any connected open set in $\mathbf{C}$. For the case of the disc one can prove this result using cut-off functions (see below). These functions have been used in [4] (see Lemma 7, also compare [14, Remark 3, p. 154]). We shall now show that one can use Theorem 3.1 to show that the local Dirichlet integral of the cut-off functions is bounded by the local Dirichlet integral of the original function. This implies that every function in $D(\mu)$ can be written as the quotient of two bounded functions in $D(\mu)$ for any nonnegative finite Borel measure μ.

For $x \in [-\infty, \infty)$ let $x_+ = \max\{x, 0\}$ and $x_- = \min\{x, 0\}$. We leave the proof of the following lemma to the reader; by separating several cases it can be proved using techniques from elementary calculus.

LEMMA 3.7. *Let* $F(x,y) = e^x - e^y - e^y(x-y)$ *for* $(x,y) \in \mathbf{R}^2$, *and let* $F(x, -\infty) = e^x$ *for* $x \in \mathbf{R}$. *Then* $F(x_+, y_+) \le F(x,y)$ *and* $F(x_-, y_-) \le F(x,y)$ *for all* $x \in (-\infty, \infty)$ *and* $y \in [-\infty, \infty)$.

To define the cut-off functions mentioned earlier, recall that any function f on $\mathbf{T}$ with $\log|f| \in L^1$ determines an outer function f_0 by

$$f_0(z)=\exp\left\{\frac{1}{2\pi}\int \frac{e^{it}+z}{e^{it}-z}\log|f(e^{it})|\,dt\right\}$$

with $|f_0(e^{it})|=|f(e^{it})|$ a.e. If $f\in H^2$, then by a *cut-off* function of f we mean the outer functions determined by $\min\{|f|,\alpha\}$ or $\min\{|f|^{-1},\alpha\}$ for some $\alpha\in\mathbf{R}$.

COROLLARY 3.8. *Let $\alpha\in(0,\infty)$ and $f\in D(\delta_\zeta)$. Suppose $f=If_0$ is the inner–outer factorization of f. Let φ_0 be the outer function determined by $|\varphi_0|=\min\{|f_0|,\alpha\}$, let $\varphi=I\varphi_0$, and let $\psi=\varphi_0/f_0$. Then $f=\varphi/\psi$, $\varphi,\psi\in H^\infty$,*

$$\|\varphi\|_\infty\le\alpha,\qquad \|\psi\|_\infty\le 1,$$

and $\varphi,\psi,1/\psi\in D(\delta_\zeta)$ with

$$D_\zeta(\varphi)\le D_\zeta(f),$$
$$D_\zeta(\psi)\le D_\zeta(1/\psi)\le(1/\alpha^2)D_\zeta(f_0).$$

Consequently, every function in $D(\mu)$ can be written as the quotient of two bounded functions in $D(\mu)$.

Proof. It is enough to consider the case $\alpha=1$. The general case follows from this one by applying it to the function f/α. Furthermore, if φ_0 is the outer function determined by $\min\{|f_0|,1\}$, then

$$\varphi_0(z)=\exp\left\{\frac{1}{2\pi}\int \frac{e^{it}+z}{e^{it}-z}\log_-|f_0(e^{it})|\,dt\right\}.$$

Thus, with $\varphi=I\varphi_0$ and $\psi=\varphi_0/f_0$, we clearly have $f=\varphi/\psi$, $\|\varphi\|_\infty\le 1$, and $\|\psi\|_\infty\le 1$. It now follows from Lemma 3.7 and Theorem 3.1 that $D_\zeta(\varphi_0)\le D_\zeta(f_0)$. In particular, the expression for $D_\zeta(\varphi_0)$ in Theorem 3.1 is finite for the boundary value choice $|\varphi_0(\zeta)|=\min\{|f(\zeta)|,1\}$. By the remark following Theorem 3.1, this implies that the oricyclic limit of φ_0 exists at ζ and that its absolute value equals $|\varphi_0(\zeta)|=\min\{|f(\zeta)|,1\}$. Hence we see that $|\varphi_0(\zeta)|\le|f(\zeta)|$. Thus, by applying Theorem 3.1 (or Lemma 3.4) to $\varphi=I\varphi_0$, we obtain

$$D_\zeta(\varphi)=D_\zeta(I)|\varphi_0(\zeta)|^2+D_\zeta(\varphi_0)\le D_\zeta(f).$$

Finally, the definition of ψ implies that

$$\frac{1}{\psi}(z)=\exp\left\{\frac{1}{2\pi}\int \frac{e^{it}+z}{e^{it}-z}\log_+|f_0(e^{it})|\,dt\right\}.$$

Thus, from Lemma 3.7 and Theorem 3.1 we see that $D_\zeta(1/\psi)\le D_\zeta(f)$. Also, since $\|\psi\|_\infty\le 1$ we have

$$|\psi'(z)|=|\psi^2(z)|\,|(1/\psi)'(z)|\le|(1/\psi)'(z)|.$$

Hence, from Proposition 2.2 applied with $\mu=\delta_\zeta$, we obtain

$$D_\zeta(\psi)\le D_\zeta(1/\psi)\le D_\zeta(f). \qquad \square$$

4. Inner Functions and 2-Isometries

It follows from the results of Section 3 that, whenever φ is an inner function and $f \in H^2$ such that $\varphi f \in D(\mu)$, then f and $\varphi^2 f$ are also in $D(\mu)$ and the 2-isometric relationship is satisfied, that is,

$$\|\varphi^2 f\|_\mu^2 - \|\varphi f\|_\mu^2 = \|\varphi f\|_\mu^2 - \|f\|_\mu^2. \tag{4.1}$$

In fact, since $\|f\|_\mu^2 = \|f\|_{H^2}^2 + \int D_\zeta(f)\, d\mu(\zeta)$ we need only check the above identity for D_ζ. If $\varphi f \in D(\delta_\zeta)$, then by Lemma 3.4

$$D_\zeta(\varphi f) = D_\zeta(\varphi)|f(\zeta)|^2 + D_\zeta(f).$$

Thus it follows immediately that $f \in D(\delta_\zeta)$. If $f(\zeta) = 0$, then clearly

$$D_\zeta(\varphi^2 f) - D_\zeta(\varphi f) = D_\zeta(\varphi f) - D_\zeta(f) = 0.$$

If $f(\zeta) \neq 0$, then $D_\zeta(\varphi)$ must be finite. This means that the inner function φ has a finite angular derivative at ζ (Proposition 3.5) and, in particular, $|\varphi(\zeta)| = 1$ so that $D_\zeta(\varphi^2) = 2D_\zeta(\varphi)$ and

$$D_\zeta(\varphi^2 f) = 2D_\zeta(\varphi)|f(\zeta)|^2 + D_\zeta(f).$$

Thus $\varphi^2 f \in D(\delta_\zeta)$ and

$$D_\zeta(\varphi^2 f) - D_\zeta(\varphi f) = D_\zeta(\varphi)|\varphi f(\zeta)|^2 = D_\zeta(\varphi)|f(\zeta)|^2 = D_\zeta(\varphi f) - D_\zeta(f).$$

In Theorem 4.2 we shall establish a converse of this statement. First we prove a lemma stating that functions in $D(\delta_\zeta)$ approach their boundary value at a fast rate, at least from within nontangential approach regions.

LEMMA 4.1. *If $g \in D(\delta_\zeta)$ then*

$$\frac{|g(\lambda) - g(\zeta)|^2}{1 - |\lambda|^2} \to 0$$

as $\lambda \to \zeta$ nontangentially.

Proof. There exists a function $h \in H^2$ such that $g(\lambda) = g(\zeta) + (\lambda - \zeta)h(\lambda)$. Then, for λ in the nontangential approach region $\Gamma_\kappa(\zeta)$, we have

$$\frac{|g(\lambda) - g(\zeta)|^2}{1 - |\lambda|^2} \leq \kappa(1 - |\lambda|)|h(\lambda)|^2 \to 0,$$

because $h \in H^2$. □

In the following, unless stated otherwise, all limits will be understood to mean nontangential limits.

THEOREM 4.2. *Let μ be a nonnegative finite Borel measure on* $\mathbf{T}$, *and let φ be a complex-valued function on* $\mathbf{D}$ *such that $f, \varphi f, \varphi^2 f \in D(\mu)$ for some nonzero function f. Then φ is the quotient of two inner functions if and only if* (4.1) *holds for f and zf.*

In particular, a multiplier φ on $D(\mu)$ acts as a 2-isometry if and only if φ is an inner function.

Proof. We almost proved the "only if" part of the theorem above. In fact, if $\varphi=\varphi_1/\varphi_2$ (where φ_1 and φ_2 are two relatively prime inner functions), then it follows from the assumption $\varphi^2 f\in D(\mu)\subseteq H^2$ that $f=\varphi_2^2 g$ for some $g\in H^2$. The inner function φ_2^2 expands the norm in $D(\mu)$, so $g\in D(\mu)$. We now repeatedly apply the result that multiplication by an inner function acts as a 2-isometry, obtaining

$$\begin{aligned}\|\varphi^2 f\|_\mu^2-\|\varphi f\|_\mu^2 &= \|\varphi_1^2 g\|_\mu^2-\|\varphi_1\varphi_2 g\|_\mu^2=\|\varphi_1 g\|_\mu^2-\|\varphi_2 g\|_\mu^2\\ &=\|\varphi_2\varphi_1 g\|_\mu^2-\|\varphi_2^2 g\|_\mu^2=\|\varphi f\|_\mu^2-\|f\|_\mu^2.\end{aligned}$$

Of course, the same argument also works with zf in place of f; hence,

$$\|\varphi^2 zf\|_\mu^2-\|\varphi zf\|_\mu^2=\|\varphi zf\|_\mu^2-\|zf\|_\mu^2.$$

Now assume that φ and f satisfy the two 2-isometric identities of the theorem. Since $f,\varphi f\in H^2$ we see that φ must be a function in the Nevannlinna class, thus it will have a boundary value function on $\mathbf{T}$ that is well defined a.e. Then

$$\begin{aligned}0&=\|\varphi^2 f\|_\mu^2-2\|\varphi f\|_\mu^2+\|f\|_\mu^2\\ &=\|\varphi^2 f\|_{H^2}^2-2\|\varphi f\|_{H^2}^2+\|f\|_{H^2}^2+\int_{\mathbf{T}} D_\zeta(\varphi^2 f)-2D_\zeta(\varphi f)+D_\zeta(f)\,d\mu(\zeta)\\ &=\frac{1}{2\pi}\int_0^{2\pi}(|\varphi|^2-1)^2|f|^2\,dt+\int_{\mathbf{T}} D_\zeta(\varphi^2 f)-2D_\zeta(\varphi f)+D_\zeta(f)\,d\mu(\zeta).\end{aligned}$$

We shall show that the integrand of the second integral is nonnegative for μ a.e. ζ. This will conclude the proof, because it will imply that both integrals are zero and, as the integrand of the first integral is nonnegative it must vanish a.e., that is, $|\varphi|=1$ a.e.

Before we can show that $D_\zeta(\varphi^2 f)-2D_\zeta(\varphi f)+D_\zeta(f)\ge 0$ a.e. $[\mu]$, we need to make a few preliminary observations.

We know that $f,\varphi f,\varphi^2 f\in D(\delta_\zeta)$ for μ a.e. $\zeta\in\mathbf{T}$. For any such ζ the numbers $f(\zeta)$, $(\varphi f)(\zeta)$, and $(\varphi^2 f)(\zeta)$ exist as the oricyclic limits of $f(\lambda)$, $(\varphi f)(\lambda)$, and $(\varphi^2 f)(\lambda)$. We thus have

$$\begin{aligned}|(\varphi^2 f)(\zeta)|^2-2|(\varphi f)(\zeta)|^2+|f(\zeta)|^2&=\lim|(\varphi^2 f)(\lambda)|^2-2|(\varphi f)(\lambda)|^2+|f(\lambda)|^2\\ &=\lim(|\varphi(\lambda)|^2-1)^2|f(\lambda)|^2\ge 0,\end{aligned}$$

where the limits are oricyclic limits as $\lambda\to\zeta$.

Recall from Theorem 2.3 that $\|zg\|_\mu^2-\|g\|_\mu^2=\int|g(\zeta)|^2\,d\mu(\zeta)$ for any $g\in D(\mu)$. By applying this to f, φf, and $\varphi^2 f$, our two assumptions imply that

$$\begin{aligned}0&=(\|\varphi^2 zf\|_\mu^2-2\|\varphi zf\|_\mu^2+\|zf\|_\mu^2)-(\|\varphi^2 f\|_\mu^2-2\|\varphi f\|_\mu^2+\|f\|_\mu^2)\\ &=\int|(\varphi^2 f)(\zeta)|^2-2|(\varphi f)(\zeta)|^2+|f(\zeta)|^2\,d\mu(\zeta).\end{aligned}$$

We saw above that the integrand of the last integral is nonnegative a.e. $[\mu]$; thus $|(\varphi^2 f)(\zeta)|^2 - 2|(\varphi f)(\zeta)|^2 + |f(\zeta)|^2 = 0$ a.e. $[\mu]$.

Now recall from Lemma 3.2 and 3.3 that

$$D_\zeta(g) = \lim_{\lambda\to\zeta}\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{|g(e^{it})|^2}{|1-\bar{\lambda}e^{it}|^2}\,dt - \frac{|g(\lambda)|^2}{1-|\lambda|^2}\right\}$$

for any $g \in D(\delta_\zeta)$. We fix $\zeta \in \mathbf{T}$ such that $f, \varphi f, \varphi^2 f \in D(\delta_\zeta)$ and

$$|(\varphi^2 f)(\zeta)|^2 - 2|(\varphi f)(\zeta)|^2 + |f(\zeta)|^2 = 0. \tag{4.2}$$

Then

$$\begin{aligned} &D_\zeta(\varphi^2 f) - 2D_\zeta(\varphi f) + D_\zeta(f) \\ &\quad = \lim_{\lambda\to\zeta}\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{(|\varphi(e^{it})|^2-1)^2|f(e^{it})|^2}{|1-\bar{\lambda}e^{it}|^2}\,dt - \frac{(|\varphi(\lambda)|^2-1)^2|f(\lambda)|^2}{1-|\lambda|^2}\right\}. \end{aligned}$$

We shall show that the term on the right converges to zero as $\lambda \to \zeta$ nontangentially. This will conclude the proof, because it will show that the left-hand side of the equation is nonnegative for μ a.e. ζ.

We must consider two cases.

(a) $f(\zeta) \neq 0$. In this case, it follows from (4.2) that $\varphi(\lambda)$ approaches a limit of modulus 1 as $\lambda \to \zeta$ nontangentially, say $\varphi(\lambda) \to \varphi(\zeta)$, $|\varphi(\zeta)| = 1$. By our assumption $f, \varphi f \in D(\delta_\zeta)$, and hence from

$$\frac{\varphi f - \varphi f(\zeta)}{z-\zeta} = \frac{\varphi - \varphi(\zeta)}{z-\zeta} f + \varphi(\zeta)\frac{f - f(\zeta)}{z-\zeta}$$

it follows that $((\varphi - \varphi(\zeta))/(z-\zeta))f = g \in H^2$. We evaluate at λ and obtain

$$\varphi(\lambda) = \varphi(\zeta) + (\lambda-\zeta)\frac{g(\lambda)}{f(\lambda)}.$$

Thus,

$$\frac{(|\varphi(\lambda)|^2-1)^2|f(\lambda)|^2}{1-|\lambda|^2} \le \frac{|\lambda-\zeta|^2|g(\lambda)|^2(2+|\lambda-\zeta||g(\lambda)|/|f(\lambda)|)^2}{1-|\lambda|^2} \to 0$$

as $\lambda \to \zeta$ nontangentially, because $f(\lambda) \to f(\zeta) \neq 0$, and $(1-|\lambda|)|g(\lambda)|^2 \to 0$ since $g \in H^2$.

(b) $f(\zeta) = 0$. In this case $f(\lambda) \to 0$, and thus from $|\varphi^2 f(\lambda)| \to |(\varphi^2 f)(\zeta)| < \infty$ it follows that $|\varphi f(\lambda)| \to 0 = |(\varphi f)(\zeta)|$. Hence

$$|(\varphi^2 f)(\zeta)|^2 = |(\varphi^2 f)(\zeta)|^2 - 2|(\varphi f)(\zeta)|^2 + |f(\zeta)|^2 = 0$$

by (4.2). We can now use Lemma 4.1 to finish the proof:

$$\frac{(|\varphi(\lambda)|^2-1)^2|f(\lambda)|^2}{1-|\lambda|^2} = \frac{|\varphi^2 f(\lambda)|^2}{1-|\lambda|^2} - 2\frac{|\varphi f(\lambda)|^2}{1-|\lambda|^2} + \frac{|f(\lambda)|^2}{1-|\lambda|^2} \to 0$$

as $\lambda \to \zeta$ nontangentially. □

5. Approximation of Functions in $D(\delta_\zeta)$

In this section we shall answer some questions of Brown and Shields (see [4], or [17]) about cyclic vectors in the Dirichlet space. All our results are valid in the generality of all $D(\mu)$ spaces. The results will follow from Theorem 5.2 below, which states that $D_\zeta(f_r) \le 4D_\zeta(f)$, where $f_r(z) = f(rz)$, $0 < r < 1$. This can be used to show that, if $f \in D(\mu)$ and $\varphi \in H^\infty$ such that $\varphi f \in D(\mu)$, then $\varphi_r f \to \varphi f$ (weakly) in $D(\mu)$ (Lemma 5.4).

Before proving the next lemma we include a word of caution: If $D_\zeta(f) < \infty$, then $g = (f - f(\zeta))/(z-\zeta) \in H^2$ and $D_\zeta(f) = \|g\|_{H^2}^2$. However, the relationship between $D_\zeta(f_r)$ and $\|g_r\|_{H^2}^2$ is more complicated (see the proof of Theorem 5.2).

LEMMA 5.1. *Let $\zeta \in \mathbf{T}$, $g \in H^2$, and $r < 1$. Then*

$$(1-r^2)^2 D_\zeta(g_r) \le \|g\|_{H^2}^2.$$

Proof. This will follow from an elementary computation with Taylor coefficients. Let $g(z) = \sum \hat{g}(n) z^n$; then

$$\begin{aligned}
\frac{g_r(z) - g_r(\zeta)}{z-\zeta} &= \sum_{n=0}^{\infty} \hat{g}(n) r^n \frac{z^n - \zeta^n}{z-\zeta} \\
&= \sum_{n=1}^{\infty} \hat{g}(n) r^n \sum_{k=0}^{n-1} z^k \zeta^{n-1-k} \\
&= \sum_{k=0}^{\infty} \left(\sum_{n=k+1}^{\infty} \hat{g}(n) r^n \zeta^n \right) \bar{\zeta}^{k+1} z^k \\
&= \sum_{k=0}^{\infty} (P_{k+1} g)(r\zeta) \bar{\zeta}^{k+1} z^k,
\end{aligned}$$

where P_k denotes the orthogonal projection of H^2 onto $z^k H^2$ for $k \in \mathbf{N}$. For $\lambda \in \mathbf{D}$ let k_λ denote the Szegö kernel at λ, that is, $k_\lambda(z) = 1/(1-\bar{\lambda} z)$ and k_λ satisfies $(f, k_\lambda)_{H^2} = f(\lambda)$ for each $f \in H^2$. Then $(P_k k_\lambda)(z) = \bar{\lambda}^k z^k k_\lambda(z)$, so that

$$\begin{aligned}
D_\zeta(g_r) &= \left\| \frac{g_r(z) - g_r(\zeta)}{z-\zeta} \right\|_{H^2}^2 \\
&= \sum_{k=0}^{\infty} |((P_{k+1} g), k_{r\zeta})_{H^2}|^2 \\
&= \sum_{k=0}^{\infty} |(g, P_{k+1} k_{r\zeta})_{H^2}|^2 \\
&\le \|g\|_{H^2}^2 \sum_{k=0}^{\infty} \|P_{k+1} k_{r\zeta}\|_{H^2}^2 =
\end{aligned}$$

$$= \|g\|_{H^2}^2 \sum_{k=0}^{\infty} r^{2k+2} \|z^{k+1} k_{r\zeta}\|_{H^2}^2$$

$$\leq \|g\|_{H^2}^2 \frac{1}{(1-r^2)^2}. \qquad \square$$

THEOREM 5.2. *Let* $f \in D(\delta_\zeta)$ *and* $0 < r < 1$. *Then* $D_\zeta(f_r) \leq 4D_\zeta(f)$.

Proof. f has a finite local Dirichlet integral, hence there is a function $g \in H^2$ such that $\|g\|_{H^2}^2 = D_\zeta(f)$ and $f(z) = f(\zeta) + (z-\zeta)g(z)$. Thus for any $0 < r < 1$ we have

$$D_\zeta(f_r) = \left\| \frac{f_r - f_r(\zeta)}{z-\zeta} \right\|_{H^2}^2 = \left\| rg_r - \zeta(1-r)\frac{g_r - g_r(\zeta)}{z-\zeta} \right\|_{H^2}^2$$

$$\leq 2\|g\|_{H^2}^2 + 2(1-r)^2 D_\zeta(g_r) \leq 4\|g\|_{H^2}^2 = 4D_\zeta(f)$$

by Lemma 5.1. $\square$

LEMMA 5.3. *Let* $\varphi \in H^\infty$ *and* $f \in D(\delta_\zeta)$. *Then* $\varphi f \in D(\delta_\zeta)$ *if and only if* $f(\zeta) = 0$ *or* $\varphi \in D(\delta_\zeta)$. *Furthermore,*

$$D_\zeta(\varphi f) \leq 2(\|\varphi\|_\infty^2 D_\zeta(f) + |f(\zeta)|^2 D_\zeta(\varphi))$$

and

$$|f(\zeta)|^2 D_\zeta(\varphi) \leq 2(\|\varphi\|_\infty^2 D_\zeta(f) + D_\zeta(\varphi f)).$$

If $f(\zeta) = 0$ *then one even has* $D_\zeta(\varphi f) \leq \|\varphi\|_\infty^2 D_\zeta(f)$, *while the second inequality can be replaced with the trivial observation that the right-hand side is nonnegative.*

Proof. We use the identity

$$\frac{\varphi f - \varphi f(\zeta)}{z-\zeta} = \varphi \frac{f - f(\zeta)}{z-\zeta} + f(\zeta)\frac{\varphi - \varphi(\zeta)}{z-\zeta}.$$

Our assumptions imply that $\varphi(f - f(\zeta))/(z-\zeta) \in H^2$. Hence

$$\frac{\varphi f - \varphi f(\zeta)}{z-\zeta} \in H^2 \quad \text{if and only if} \quad f(\zeta)\frac{\varphi - \varphi(\zeta)}{z-\zeta} \in H^2.$$

This is equivalent to the first statement of the lemma. Also,

$$D_\zeta(\varphi f) = \left\| \frac{\varphi f - \varphi f(\zeta)}{z-\zeta} \right\|_{H^2}^2 \leq 2\|\varphi\|_\infty^2 \left\| \frac{f - f(\zeta)}{z-\zeta} \right\|_{H^2}^2 + 2|f(\zeta)|^2 \left\| \frac{\varphi - \varphi(\zeta)}{z-\zeta} \right\|_{H^2}^2$$

$$= 2(\|\varphi\|_\infty^2 D_\zeta(f) + |f(\zeta)|^2 D_\zeta(\varphi)).$$

Similarly,

$$|f(\zeta)|^2 D_\zeta(\varphi) = \left\| \frac{\varphi f - \varphi f(\zeta)}{z-\zeta} - \varphi \frac{f - f(\zeta)}{z-\zeta} \right\|_{H^2}^2 \leq 2(D_\zeta(\varphi f) + \|\varphi\|_\infty^2 D_\zeta(f)).$$

Finally, if $f(\zeta)=0$ then $\varphi f(\zeta)=0$, because φ is bounded. Thus,

$$D_\zeta(\varphi f)=\left\|\varphi\frac{f}{z-\zeta}\right\|_{H^2}^2\le\|\varphi\|_\infty^2\left\|\frac{f}{z-\zeta}\right\|_{H^2}^2=\|\varphi\|_\infty^2 D_\zeta(f). \qquad \square$$

We can now easily prove the result stated in the introduction to this section.

LEMMA 5.4. *Let $\varphi\in H^\infty$ and $f\in D(\mu)$ such that $\varphi f\in D(\mu)$. Then $\varphi_r f\to\varphi f$ (weakly) in $D(\mu)$.*

Proof. Clearly $\varphi_r f\to\varphi f$ pointwise in $\mathbf{D}$, hence it will suffice to show that $\|\varphi_r f\|_\mu$ is bounded independently of $r<1$. Notice that $\|\varphi_r f\|_{H^2}\le\|\varphi\|_\infty\|f\|_{H^2}$ and that μ is a finite measure, so it will be enough to check that $D_\zeta(\varphi_r f)$ is bounded independently of r and ζ. The case when $f(\zeta)=0$ follows immediately from Lemma 5.3. If $f(\zeta)\neq 0$, we then have

$$\begin{aligned}D_\zeta(\varphi_r f)&\le 2(\|\varphi_r\|_\infty^2 D_\zeta(f)+|f(\zeta)|^2 D_\zeta(\varphi_r)) && \text{(Lemma 5.3)}\\ &\le 2(\|\varphi\|_\infty^2 D_\zeta(f)+4|f(\zeta)|^2 D_\zeta(\varphi)) && \text{(Theorem 5.2)}\\ &\le 2(\|\varphi\|_\infty^2 D_\zeta(f)+8(\|\varphi\|_\infty^2 D_\zeta(f)+D_\zeta(\varphi f)) && \text{(Lemma 5.3)}\\ &\le c(\|\varphi\|_\infty^2 D_\zeta(f)+D_\zeta(\varphi f)). && \square\end{aligned}$$

For our convenience we proved only weak convergence in the previous lemma. However, the polynomials are dense in $D(\mu)$ for any measure μ, so one can use a standard argument to show that $f_r\to f$ even in the $D(\mu)$ norm.

Using the preceding lemmas one can now answer some of the questions posed in [4] and [17]. We shall illustrate this with two examples.

If $f\in D(\mu)$, then we shall denote by $[f]$ the smallest invariant subspace of the operator of multiplication by z. Thus $[f]$ is the closure of the polynomial multiples of f. A function $f\in D(\mu)$ is called a *cyclic vector* if the polynomial multiples of f are dense in $D(\mu)$, that is, if $[f]=D(\mu)$. It is clear that any cyclic vector in $D(\mu)$ must be an outer function, but for many measures μ there are examples of noncyclic outer functions in $D(\mu)$. Presently there is no known necessary and sufficient condition for the cyclicity of a function f even in the case of the Dirichlet space D. For more information about cyclic vectors in the Dirichlet space we refer the reader to [4] or the survey [17].

COROLLARY 5.5. *Let $f,g\in D(\mu)$. If $|f(z)|\ge|g(z)|$ for all $z\in\mathbf{D}$, then $[g]\subseteq[f]$. In particular, if g is cyclic then f is cyclic.*

This answers Question 3 of [4] for the $D(\mu)$ spaces. It also generalizes Theorem 1 of [4].

Proof. The assumption implies that $\varphi=g/f\in H^\infty$. If $r<1$, then clearly $\varphi_r f\in[f]$ (approximate φ_r with polynomials uniformly in a neighborhood of $\mathbf{D}$). Also $\varphi f=g\in D(\mu)$, so by Lemma 5.4 it follows that $g\in[f]$. But this implies that $[g]\subseteq[f]$. $\square$

It is easy to check that, for any two functions f and g in $D(\mu)\cap H^\infty$, their product fg is also contained in $D(\mu)\cap H^\infty$. In [4] it was shown that, if $f, g \in D\cap H^\infty$ such that fg is cyclic in D, then both f and g must be cyclic. For the converse the authors obtain a partial result, leaving the general case as an open question (Question 8 of [4]).

COROLLARY 5.6. *Let $f, g \in D(\mu)\cap H^\infty$. Then fg is cyclic if and only if both f and g are cyclic.*

Proof. Suppose fg is cyclic. For each $z \in \mathbf{D}$ we have $|fg(z)| \le \|f\|_\infty |g(z)|$; thus Corollary 5.5 implies that g must be cyclic. By symmetry, f must be cyclic as well.

Now assume that both f and g are cyclic vectors. Since $[f]=D(\mu)$ it is enough to show that $f\in[fg]$. Since g is cyclic we may choose a sequence of polynomials $\{p_n\}$ such that $p_n g \to 1$ in $D(\mu)$. Let $p_n g = \varphi_n/\psi_n$ be the cut-off functions of Corollary 3.8 ($\alpha=1$) satisfying $\|\varphi_n\|_\infty, \|\psi_n\|_\infty \le 1$ and $D_\zeta(\varphi_n), D_\zeta(\psi_n) \le D_\zeta(p_n g)$. We note that $|\psi_n p_n g f(z)| \le |p_n g f(z)|$; thus, by Corollary 5.5 it follows that $\varphi_n f = \psi_n p_n g f \in [p_n g f] \subseteq [gf]$.

We shall show that $\varphi_n f \to f$ (weakly). Since $p_n g = \varphi_n/\psi_n \to 1$ in H^2, one can show that $\varphi_n(z)\to 1$ for each $z\in\mathbf{D}$. This was done in [4, Lemma 6, p. 283] for $\psi_n(z)$, and this implies the result for $\varphi_n(z)$. It now follows that

$$\begin{aligned}\|\varphi_n f\|_\mu^2 &= \|\varphi_n f\|_{H^2}^2 + \int D_\zeta(\varphi_n f)\, d\mu(\zeta)\\ &\le \|f\|_{H^2}^2 + \int D_\zeta(f) + \|f\|_\infty^2 D_\zeta(\varphi_n)\, d\mu(\zeta)\\ &\le \|f\|_\mu^2 + \|f\|_\infty^2 \|p_n g\|_\mu^2.\end{aligned}$$

The last term is bounded, because $p_n g \to 1$ in $D(\mu)$. Thus $\{\varphi_n f\}$ converges weakly to f, and so $f\in[gf]$. □

We note that one can relax the hypothesis of Corollary 5.6 by merely assuming that f, g, and $fg\in D(\mu)$. Using the cut-off functions of Corollary 3.8 and the results of this section, one can show that for a general $D(\mu)$ function $f=\varphi/\psi$, where $\|\varphi\|_\infty, \|\psi\|_\infty \le 1$ and $D_\zeta(\varphi), D_\zeta(\psi), D_\zeta(1/\psi) \le D_\zeta(f)$ as in Corollary 3.8, one has $[f]=[\varphi]$, and that ψ and $1/\psi$ are cyclic in $D(\mu)$. Thus, the general problem of the cyclicity of f, g, and fg is equivalent to the case handled in Corollary 5.6.

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