A Functional Calculus for a Scalar Perturbation of $\partial/\partial z$

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1. Introduction

In this paper, we determine when a functional calculus exists for the operator

$$L = a_1\left(-i\frac{\partial}{\partial z}\right) + a_2\left(-i\frac{\partial}{\partial \bar{z}}\right), \quad a_1 \text{ close to } 1, \ a_2 \text{ close to } 0.$$

In other words, we consider when $\phi(L)$ can be defined as a bounded operator on $L^2(\mathbf{R}^2)$ for a certain class of functions ϕ. The operator L is not normal, thus the usual spectral theory cannot be applied. The spectrum of L is the whole complex plane, so resolvents need to be interpreted, and one cannot define functions of L by integrating on the boundary of the spectrum.

Extending the unpublished results of Coifman and Meyer ([CM2]; see also [CM1]), we construct a functional calculus for L and prove L^2 boundedness for a certain class of ϕ, and connect the study of the functional calculus to a certain surface in $\mathbf{C}^2$. The assumption of the boundedness on L^2 of some natural functions of L is equivalent to certain quantitative conditions on the surface. We also show how L can be obtained by conjugation from the Coifman–Meyer case. This gives another geometric interpretation: a connection via a change of variables to a simpler surface considered by Coifman and Meyer.

In Section 2, we discuss some general facts about functional calculi which lead to the definition of a surface Σ in $\mathbf{C}^2$ and the definition of the conjugate operator $\bar{L}$. Section 3 examines restrictions on the coefficients a_1 and a_2, and exhibits a class of functions satisfying these restrictions. In Section 4, we calculate $\bar{L}/L$ and $L/\bar{L}$, while in Section 5 we use the expression

$$\phi(L) = \frac{1}{\pi}\int_{\mathbf{C}} \frac{\partial\phi}{\partial\bar{\xi}}\frac{1}{L-\xi}\,d\sigma(\xi)$$

to define $\phi(L)$ for $\varphi \in C_0^\infty(\mathbf{C})$. In Section 6 we show that the product formula holds for the functional calculus, and in Section 7 we extend the class of ϕ

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to those functions which are bounded and holomorphic in a conical sector in $\mathbf{C}^2$. Section 8 considers the quantitative restrictions on the surface implied by assuming that $L/\bar{L}$ and $\bar{L}/L$ are bounded on L^2, and Section 9 exhibits the conjugation to the Coifman–Meyer case.

2. Definition of L and a Naturally Associated Surface

Let

$$\frac{\partial}{\partial z}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right),\ \frac{\partial}{\partial \bar{z}}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\quad\text{and}\quad \partial_z=-i\frac{\partial}{\partial z},\ \partial_{\bar{z}}=-i\frac{\partial}{\partial \bar{z}}.$$

Consider the operator

$$L=a_1\partial_z+a_2\partial_{\bar{z}}.$$

Here, $a_1=1/(1+\alpha)$, and α and a_2 have small L^∞ norm and are C^1. We will restrict them more later.

We would like to define $\phi(L)$ and to show that, for ϕ in a certain general class, $\phi(L)$ is a bounded operator on L^2. As is usually done when defining a functional calculus, one at first assumes ϕ is in some special class, uses an integral representation formula for ϕ, and then attempts to define $\phi(L)$ at least on a formal level. Then one shows that the proposed expression for $\phi(L)$ makes sense, verifies that $\phi_1(L)\phi_2(L)=(\phi_1\phi_2)(L)$, and extends the obtained formula to more general ϕ.

The first problem is thus to choose some representation formula for ϕ. A common candidate is the Cauchy integral formula, in which the integration is over the boundary of the domain containing the spectrum of L. Since the spectrum of L is the whole complex plane, we follow [CM2] and instead try the well-known integral representation formula

$$\phi(z)=\frac{1}{\pi}\int_{\mathbf{C}}\frac{\partial\phi}{\partial\bar{\xi}}\frac{1}{z-\xi}\,d\sigma(\xi), \tag{2.1}$$

which is certainly valid for smooth ϕ which decrease sufficiently rapidly at infinity. Thus we will attempt to make sense of the formula

$$\phi(L)=\frac{1}{\pi}\int_{\mathbf{C}}\frac{\partial\phi}{\partial\bar{\xi}}\frac{1}{L-\xi}\,d\sigma(\xi). \tag{2.2}$$

Note that $1/(L-\xi)$ still needs to be interpreted since $(L-\xi)^{-1}$ does not exist. The intuitive reason for choosing (2.1) instead of the Cauchy integral is that the singularity in (2.1) is integrable in two dimensions.

Suppose that $\chi(z)$ has the property that $L\chi=\xi\chi$. Then we can interpret $(L-\xi)^{-1}$ as $\chi L^{-1}\chi^{-1}$, on a formal level. Setting aside for the time being the problem of defining L^{-1}, we focus our attention on χ. Note that choosing $\chi_0=\exp(i(\xi z+\bar{\xi}\bar{z}))$, we see that $\partial_z\chi_0=\xi\chi_0$. Since L is a perturbation of ∂_z, a reasonable candidate for χ is

$$\chi=\exp(i(\xi(h(z)-g(z))+\bar{\xi}g(z))), \tag{2.3}$$

where $h(z)-g(z)$ should be close to z, and $g(z)$ should be close to $\bar{z}$ in some sense. The reason for choosing the notation $h-g$ to play the role of z rather than a function unrelated to g is for convenience in later formulas. Solving the equation $L\chi=\xi\chi$ for this χ leads to the system

$$(2.4)\qquad \begin{cases} a_1\dfrac{\partial h}{\partial z}+a_2\dfrac{\partial h}{\partial \bar{z}}=1, \\ a_1\dfrac{\partial g}{\partial z}+a_2\dfrac{\partial g}{\partial \bar{z}}=0. \end{cases}$$

These are variants of the Beltrami equation, which can be solved explicitly:

$$h=z+\bar{z}+\eta, \qquad g=\bar{z}+m.$$

The functions η and m should be thought of as perturbations of z and $\bar{z}$ (see the remarks at the end of this section).

The above discussion serves as motivation for examining the functions h and g which solve system (2.4), and thinking of $h-g$ as being similar to z and g similar to $\bar{z}$. We move away from interpreting formula (2.2) for the time being, until Section 5, and concentrate on h and g. We will use these functions to find an operator $\bar{L}$ commuting with L. Since L is a perturbation of ∂_z, we are looking for $\bar{L}$ to be a perturbation of $\partial_{\bar{z}}$. $\bar{L}$ is a very useful operator to find explicitly because, if we are given $\phi(\xi)$ and set $\phi_1(\xi)=\phi(\bar{\xi})$, then $\phi_1(L)=\phi(\bar{L})$ if $\phi_1(L)$ can be defined. To understand how to use h and g, consider the unperturbed operators ∂_z and $\partial_{\bar{z}}$. We define the surface $\Sigma_0=(z,\bar{z})$, $\Sigma_0\subset\mathbf{C}\times\mathbf{C}=\{(z_1,z_2)\}$. For F holomorphic in a neighborhood of Σ_0, define $(\Lambda_0F)(z)=F(z,\bar{z})$. It follows immediately that

$$\partial_z(\Lambda_0F)=\Lambda_0(\partial_{z_1}F).$$

So $\Lambda_0^{-1}\partial_z\Lambda_0=\partial_{z_1}$, and similarly $\Lambda_0^{-1}\partial_{\bar{z}}\Lambda_0=\partial_{z_2}$. Thus, on the surface Σ_0, ∂_z and $\partial_{\bar{z}}$ become the (commuting) operators ∂_{z_1} and ∂_{z_2}. Hence, following the idea of Coifman and Meyer in [CM2], to find $\bar{L}$ we consider the surface $\Sigma=((h-g)(z),g(z))$ and define, for F holomorphic in a neighborhood of Σ,

$$(\Lambda F)(z)=F((h-g)(z),g(z)).$$

Then

$$L(\Lambda F)=(a_1\partial_z+a_2\partial_{\bar{z}})F((h-g)(z),g(z))=\Lambda(\partial_{z_1}F),$$

using (2.4). So $\Lambda^{-1}L\Lambda=\partial z_1$, or $L=\Lambda\partial z_1\Lambda^{-1}$. From the previous remarks, to find a vector field $\bar{L}$ commuting with L, it is natural to set $\bar{L}=\Lambda\partial_{z_2}\Lambda^{-1}$. If $\bar{L}=\tilde{b}_1\partial_z+\tilde{b}_2\partial_{\bar{z}}$, then using a similar calculation to the one above, we will obtain the following system of equations for $\tilde{b}_1,\tilde{b}_2$:

$$(2.5a)\qquad \begin{cases} \tilde{b}_1\dfrac{\partial(h-g)}{\partial z}+\tilde{b}_2\dfrac{\partial(h-g)}{\partial \bar{z}}=0, \\ \tilde{b}_1\dfrac{\partial g}{\partial z}+\tilde{b}_2\dfrac{\partial g}{\partial \bar{z}}=1. \end{cases}$$

Using (2.4), and letting $\tilde{b}_1 = a_1 + b_1$ and $\tilde{b}_2 = a_2 + b_2$, it is equivalent to solve:

$$\text{(2.5b)} \qquad \begin{cases} b_1 \dfrac{\partial h}{\partial z} + b_2 \dfrac{\partial h}{\partial \bar{z}} = 0, \\[2ex] b_1 \dfrac{\partial g}{\partial z} + b_2 \dfrac{\partial g}{\partial \bar{z}} = 1. \end{cases}$$

Defining

$$D \equiv \frac{\partial h}{\partial z}\frac{\partial g}{\partial \bar{z}} - \frac{\partial g}{\partial z}\frac{\partial h}{\partial \bar{z}} = \frac{\partial (h-g)}{\partial z}\frac{\partial g}{\partial \bar{z}} - \frac{\partial g}{\partial z}\frac{\partial (h-g)}{\partial \bar{z}},$$

(2.5b) has solutions

$$b_1 = -\frac{\partial h}{\partial \bar{z}}\frac{1}{D}, \qquad b_2 = \frac{\partial h}{\partial z}\frac{1}{D}.$$

For the development of Sections 4 through 7, when discussing the operators $L/\bar{L}$ and $\bar{L}/L$, when defining more general functions of L, and when considering L^2 boundedness of a certain class of functions of L, we will always assume that η and m have small Lipschitz norms. This assumption (together with the assumptions that a_2 is small relative to a_1, as in (i) of Proposition 8.1) is sufficient to guarantee that $h-g$ is a quasiconformal, and g an (anti)quasiconformal, homeomorphism of the plane. Furthermore, all the required expressions (e.g., D) stay away from 0, $L/\bar{L}$ and $\bar{L}/L$ are bounded on L^2, and the rest of the results, including Proposition 7.1, go through. In Proposition 8.1, we see that we can estimate the distance of D from 0, the L^∞ norms of the partials of η and m, and the L^∞ norms of certain other functions connected with the quasiconformal nature of $h-g$ and g, using only the L^2 norms of $L/\bar{L}$ and $\bar{L}/L$ as well as certain L^∞ norms involving a_1 and a_2. However, Proposition 8.1 is not a complete converse to our (sufficient) assumption that η and m have small partials: we need to make assumptions about boundedness without estimates in Proposition 8.1, and then we obtain the estimates; we do not obtain that η and m have necessarily small Lipschitz norm, but merely bounded. In the next section, we solve (2.4) explicitly, and give examples which guarantee that η and m have small partials.

3. Beurling Transform; Restrictions on a_1 and a_2

We first make a small digression to discuss the Beurling transform. Let $B = \partial_z \partial_{\bar{z}}^{-1}$ be the Beurling transform, and $B^{-1} = \partial_{\bar{z}} \partial_z^{-1}$ be its inverse. B^{-1} is convolution with $p.v. -1/\pi \bar{z}^2$, has symbol $\xi/\bar{\xi}$, and is bounded from L^2 to L^2.

Let f be a radial function, $f(r)$, defined on $\mathbf{C}$, and let $F(z) = e^{im\theta} f(r)$. Following Garcia-Cuerva's derivation of $B(f)$ in [GC], one can show that

$$\text{(3.1)} \quad (B^{-1}F)(re^{i\psi}) = \begin{cases} -e^{i(m+2)\psi}\{2(m+1)/r^{m+2}\int_0^r s^{m+1} f(s)\,ds - f(r)\}, & m \ge -1, \\ e^{i(m+2)\psi}\{2(m+1)/r^{m+2}\int_r^\infty s^{m+1} f(s)\,ds + f(r)\}, & m \le -3, \\ -(2\int_r^\infty f(s)/s\,ds + 2f(0)\ln r - f(r)), & m = -2. \end{cases}$$

Using (3.1), if $F=e^{im\theta}f(r)$ for any integral m, if $f(r)$ is supported in $[1/M,M]$ with M some fixed constant, and if $f\in C^\infty$, then

$$\|B^{-1}F\|_\infty\leq 5\ln M\|f\|_\infty.$$

So, in particular, if $\|f\|_\infty\leq\delta/5\ln M$ $(\delta<1)$, we have that the series

$$(1-B^{-1}f)^{-1}(1)=1+B^{-1}(f)+B^{-1}(fB^{-1}(f))+\cdots \tag{3.2}$$

converges in L^∞ with norm bounded by $1/(1-\delta)$.

In what follows, f will be either a_2, αa_2, or their linear combination (here $a_1=1/(1+\alpha)$), so we must take f in L^∞. This places restrictions on the functions involved since B^{-1}, a Calderon–Zygmund operator, will not usually map L^∞ to L^∞.

Now we return to the system (2.4). We treat each equation separately. Each is similar to the Beltrami equation, except one has a nonzero right-hand term, and the roles of ∂_z and $\partial_{\bar z}$ are reversed. So we can solve them in the usual way (a minor modification to the solution in Ahlfors [A]) to obtain:

$$\begin{aligned} g&=\bar z+m,\\ m&=-\left(\frac{a_2}{a_1}\left(1+B^{-1}\frac{a_2}{a_1}\right)^{-1}(1)\right)*\frac{1}{\pi\bar z},\\ \frac{\partial m}{\partial z}&=-\frac{a_2}{a_1}\left(1+B^{-1}\frac{a_2}{a_1}\right)^{-1}(1),\\ \frac{\partial m}{\partial\bar z}&=\left(1+B^{-1}\frac{a_2}{a_1}\right)^{-1}(1)-1.\end{aligned} \tag{3.3}$$

Here, a_2/a_1 denotes the corresponding multiplication operator. We also have:

$$\begin{aligned} h&=z+\bar z+\eta,\\ \eta&=\left\{\alpha-\frac{a_2}{a_1}\left(1+B^{-1}\frac{a_2}{a_1}\right)^{-1}\left(1+B^{-1}\frac{1}{a_1}\right)\right\}*\frac{1}{\pi\bar z},\\ \frac{\partial\eta}{\partial z}&=\alpha-\frac{a_2}{a_1}\left(1+B^{-1}\frac{a_2}{a_1}\right)^{-1}\left(1+B^{-1}\frac{1}{a_1}\right),\\ \frac{\partial\eta}{\partial\bar z}&=\left(1+B^{-1}\frac{a_2}{a_1}\right)^{-1}\left(1+B^{-1}\frac{1}{a_1}\right)-1\\ &=\left(1+B^{-1}\frac{a_2}{a_1}\right)^{-1}\left(B^{-1}\left(\frac{1}{a_1}-\frac{a_2}{a_1}\right)\right)\end{aligned} \tag{3.4}$$

(here, $a_1=1/(1+\alpha)$). Noting that $B^{-1}(1)=0$, and using (3.2), we see that if we choose α and a_2 to be C^∞, compactly supported on $1/M\leq r\leq M$ (M fixed), of the form $\sum_{j=1}^N e^{im_j\theta}f_j(r)$ (the sum finite), and such that each f_j is of sufficiently small L^∞ norm, then all the series defining m, η, and their derivatives will converge, and m,η will be Lipschitz of norm as small as we want.

These restrictions on the form of α and a_2 are probably unnecessarily severe. They merely serve to illustrate that there do exist many functions α and a_2 such that the operator series in (3.3) and (3.4) converge and the partials of m and η can be taken small.

We now return to (2.5b) to calculate $\tilde{b}_1 = a_1 + b_1$ and $\tilde{b}_2 = a_2 + b_2$. Note that $D = (1/a_1)(\partial g/\partial \bar{z})$, using (2.4). D is never 0 and is, in fact, bounded away from 0, given our choices of α and a_2. Thus,

$$\tilde{b}_1 = -\frac{a_1(1+B^{-1}(a_2/a_1))^{-1}(B^{-1}(1/a_1))}{(1+B^{-1}(a_2/a_1))^{-1}(1)} \tag{3.5a}$$

and

$$\tilde{b}_2 = \frac{1-a_2(1+B^{-1}(a_2/a_1))^{-1}(B^{-1}(1/a_1))}{(1+B^{-1}(a_2/a_1))^{-1}(1)}. \tag{3.5b}$$

So $\bar{L} = \tilde{b}_1 \partial_z + \tilde{b}_2 \partial_{\bar{z}}$, where $\tilde{b}_1$ and $\tilde{b}_2$ (given by the formulas above) are in L^∞ and $\bar{L}$ is obtained by simply finding the conjugate vector field, corresponding to $\partial/\partial z_2$ on Σ, to the vector field L (which corresponds to $\partial/\partial z_1$). If we write out the partial differential equations obtained by considering $L\bar{L} = \bar{L}L$, we obtain:

$$\begin{cases} a_1 \dfrac{\partial \tilde{b}_1}{\partial z} + a_2 \dfrac{\partial \tilde{b}_1}{\partial \bar{z}} = \tilde{b}_1 \dfrac{\partial a_1}{\partial z} + \tilde{b}_2 \dfrac{\partial a_1}{\partial \bar{z}}, \\ a_1 \dfrac{\partial \tilde{b}_2}{\partial z} + a_2 \dfrac{\partial \tilde{b}_2}{\partial \bar{z}} = \tilde{b}_1 \dfrac{\partial a_2}{\partial z} + \tilde{b}_2 \dfrac{\partial a_2}{\partial \bar{z}}. \end{cases} \tag{3.6}$$

One can check, without too much difficulty, that (3.5a) and (3.5b) satisfy system (3.6). It would be hard to guess a solution to (3.6) (other than the trivial one, viz., a multiple of L) without using the surface Σ.

4. Calculation of L^{-1}, $\bar{L}^{-1}$, $L/\bar{L}$, $\bar{L}/L$

In this section, we calculate some particular operators which have already been mentioned and which will also be used in what follows. In all that follows, we suppose $f \in C_0^\infty(\mathbf{C})$. We first state a lemma of Coifman-Meyer, a proof of which appears in the appendix.

LEMMA 4.1. *If $\rho : \mathbf{C} \mapsto \mathbf{C}$ is a quasi-conformal mapping, and $f \in C_0^\infty$, then*

$$\frac{\partial}{\partial z} \int_{\mathbf{C}} \frac{1}{\rho(z)-\rho(w)} f(w)\, d\sigma(w) = -p.v. \int_{\mathbf{C}} \frac{\partial \rho/\partial z}{(\rho(z)-\rho(w))^2} f(w)\, d\sigma(w)$$

while

$$\frac{\partial}{\partial \bar{z}} \int_{\mathbf{C}} \frac{1}{\rho(z)-\rho(w)} f(w)\, d\sigma(w)$$
$$= \pi \frac{f(z)}{\partial \rho/\partial z} - p.v. \int_{\mathbf{C}} \frac{\partial \rho/\partial \bar{z}}{(\rho(z)-\rho(w))^2} f(w)\, d\sigma(w).$$

Now, let

$$T_1 f = \int_{\mathbf{C}} \frac{i}{\pi} \frac{1}{g(z)-g(w)} f(w) D(w)\, d\sigma(w).$$

Then, $LT_1 f = f(z)$: we apply Lemma 4.1, but since $\bar{g}$ (not g) is q.c., we must interchange $\partial/\partial z$ and $\partial/\partial \bar{z}$ in that lemma. So,

$$\begin{aligned} LT_1 f &= a_1 \frac{f(z)}{\partial g/\partial \bar{z}} D(z) - p.v. \frac{i}{\pi} \int_{\mathbf{C}} \frac{a_1(\partial g/\partial z) f(w) D(w)}{(g(z)-g(w))^2}\, d\sigma(w) \\ &\quad - p.v. \frac{i}{\pi} \int_{\mathbf{C}} \frac{a_2(\partial g/\partial \bar{z}) f(w) D(w)}{(g(z)-g(w))^2}\, d\sigma(w) \\ &= f(z), \end{aligned}$$

using (3.1) and the equality $D(z) = (1/a_1)(\partial g/\partial \bar{z})$. One can also show, using (2.4) and a bit more manipulation, that $T_1 L f = f(z)$. So, $T_1 = 1/L$.

Similarly,

$$\frac{1}{\bar{L}} = \int_{\mathbf{C}} \frac{i}{\pi} \frac{1}{(h-g)(z)-(h-g)(w)} f(w) D(w)\, d\sigma(w).$$

Here we use (2.5) and the fact that also $D(z) = (\partial(h-g)/\partial z)/\tilde{b}_2$.

Using Lemma 4.1 and the expressions for L^{-1} and $\bar{L}^{-1}$,

$$\begin{aligned} \frac{\bar{L}}{L} f &= (\tilde{b}_1 \partial_z + \tilde{b}_2 \partial_{\bar{z}}) \left\{ \int_{\mathbf{C}} \frac{i}{\pi(g(z)-g(w))} f(w) D(w)\, d\sigma(w) \right\} \\ &= -p.v. \int_{\mathbf{C}} \frac{1}{\pi(g(z)-g(w))^2} f(w) D(w)\, d\sigma(w) + \frac{\tilde{b}_1(z)}{a_1(z)} f(z). \end{aligned}$$

Similarly,

$$\begin{aligned} \frac{L}{\bar{L}} f &= (a_1 \partial_z + a_2 \partial_{\bar{z}}) \left\{ \int_{\mathbf{C}} \frac{i}{\pi((h-g)(z)-(h-g)(w))} f(w) D(w)\, d\sigma(w) \right\} \\ &= -p.v. \int_{\mathbf{C}} \frac{1}{\pi((h-g)(z)-(h-g)(w))^2} f(w) D(w)\, d\sigma(w) + \frac{a_2(z)}{\tilde{b}_2(z)} f(z). \end{aligned}$$

5. Functional Calculus

We now try to make sense of the formula (2.2) which was discussed in Section 2:

$$\phi(L) = \frac{1}{\pi} \int_{\mathbf{C}} \frac{\partial \phi}{\partial \bar{\xi}} \frac{1}{L-\xi}\, d\sigma(\xi).$$

The first part of this section follows closely the work of Coifman–Meyer. Let us first suppose that ϕ is decreasing more rapidly than a linear exponential. Now, we know what L^{-1} is. If we let $\chi = \exp(i(\xi(h(z)-g(z)) + \bar{\xi} g(z)))$, as before, then $L\chi = \xi\chi$ and so

$$(L-\xi)^{-1} = \chi L^{-1} \frac{1}{\chi}.$$

Since the kernel of L^{-1} is $(i/\pi(g(z)-g(w)))D(w)$,

$$\frac{\partial}{\partial\bar{\xi}}(L-\xi)^{-1}=\frac{i^2}{\pi}\int_{\mathbf{C}}\exp(i(\xi((h-g)(z)-(h-g)(w)) +\bar{\xi}(g(z)-g(w))))f(w)D(w)\,d\sigma(w)$$

and

$$\phi(L)f = \frac{1}{\pi^2}\int_{\mathbf{C}}k((h-g)(z)-(h-g)(w),g(z)-g(w))f(w)D(w)\,d\sigma(w), \tag{5.1}$$

where

$$k(u,v)=\int_{\mathbf{C}}\phi(\xi)\exp(i(u\xi+v\bar{\xi}))\,d\sigma(\xi).$$

We now attempt to define more general functions of L, and to tie in $\bar{L}$ (which was obtained by geometric considerations) to the functional calculus. To this end, define $\phi(\xi)=\exp(-|\xi|^2)$ and $\phi_t(\xi)=\exp(-t^2|\xi|^2)$. Then

$$\int_{\mathbf{C}}\phi_t(\xi)\exp(i(z\xi+\bar{z}\bar{\xi}))\,d\sigma(\xi)=\frac{\pi}{t^2}\exp\left(-\frac{1}{t^2}\left(\frac{z+\bar{z}}{2}\right)^2-\frac{1}{t^2}\left(\frac{z-\bar{z}}{2i}\right)^2\right).$$

We can extend this uniquely as an entire function in $(u,v)\in\mathbf{C}\times\mathbf{C}$, to obtain

$$\theta_t(u,v)\equiv\frac{\pi}{t^2}\exp\left(-\frac{1}{t^2}\left(\frac{u+v}{2}\right)^2-\frac{1}{t^2}\left(\frac{u-v}{2i}\right)^2\right).$$

Let $u=(h-g)(z)-(h-g)(w)$ and $v=g(z)-g(w)$. Note that

$$|\theta_t(u,v)|\le\frac{\pi}{t^2}\exp\left(-\frac{c}{t^2}\left(\frac{(z-w)+\overline{(z-w)}}{2}\right)^2-\frac{c}{t^2}\left(\frac{(z-w)-\overline{(z-w)}}{2i}\right)^2\right).$$

We want to calculate $\lim_{t\to0}\phi_t(L)f$, where $\phi_t(L)f$ is defined by (5.1). Now, making a change of variables $w\to z-u'$ and then $u'\to vt$, we obtain

$$\phi_t(L)f(z) =\frac{1}{\pi}\int_{\mathbf{C}}\exp\left(-\left(\frac{h(z)-h(z-vt)}{2t}\right)^2-\left(\frac{(h-2g)(z)-(h-2g)(z-vt)}{2it}\right)^2\right) f(z-vt)D(z-vt)\,d\sigma(v).$$

Taking the limit as $t\to0$, we obtain (since we can take the limit inside):

$$\lim_{t\to0}\phi_t(L)f(z) =f(z)D(z) \times\left\{\frac{1}{\pi}\int_{\mathbf{C}}\exp\left(-\left(\left(\frac{\partial h}{\partial\bar{z}}(z)\bar{v}+\frac{\partial h}{\partial z}(z)v\right)\Big/2\right)^2 -\left(\left(\frac{\partial(h-2g)}{\partial\bar{z}}(z)\bar{v}+\frac{\partial(h-2g)}{\partial z}(z)v\right)\Big/2i\right)^2\right)d\sigma(v)\right\}.$$

The expression inside { } is an unpleasant (but straightforward) Gaussian, which yields $1/D(z)$ after calculation. So,

$$\lim_{t\to 0}\phi_t(L)f(z)=f(z). \tag{5.2}$$

Hence we have an entire (and rapidly decreasing on Σ) approximation to the identity.

For ρ rapidly decreasing, we define $f_\rho\equiv\rho(L)f$.

PROPOSITION 5.1.

$$f_{\xi^n\phi_t}=L^n(f_{\phi_t})=(L^nf)_{\phi_t}$$

and

$$f_{\bar\xi^n\phi_t}=\bar L^n(f_{\phi_t})=(\bar L^nf)_{\phi_t}$$

for $n=1,2,3,\ldots$.

This proposition connects $\bar L$, obtained geometrically, to the functional calculus. Equation (5.2) shows that $\lim_{t\to 0}(f_{\xi^n\phi_t})=L^nf$, giving the expected result, and similarly for $\bar L^nf$. Before proving the proposition, we first note that, by a calculation using (2.5), we have the following lemma.

LEMMA 5.2.

$$\partial_z(a_1D)+\partial_{\bar z}(a_2D)=0 \quad \textit{and} \quad \partial_z(\tilde b_1D)+\partial_{\bar z}(\tilde b_2D)=0.$$

Here, as before,

$$D=\frac{\partial h}{\partial z}\frac{\partial g}{\partial \bar z}-\frac{\partial g}{\partial z}\frac{\partial h}{\partial \bar z}.$$

Proof of Proposition 5.1. We will show that

$$f_{\xi^n\phi}=L^n(f_\phi)=(L^nf)_\phi \quad \text{and} \quad f_{\bar\xi^n\phi}=\bar L^n(f\phi)=(\bar L^nf)_\phi$$

for any suitable ϕ. ϕ_t is certainly suitable. We define

$$u\equiv(h-g)(z)-(h-g)(w) \quad \text{and} \quad v\equiv g(z)-g(w).$$

Now,

$$L(\theta(u,v))=\frac{\partial\theta}{\partial u}L(u)+\frac{\partial\theta}{\partial v}L(v), \quad L(u)=-i,\ L(v)=0.$$

Thus $L(\theta(u,v))=-i(\partial\theta/\partial u)$. So

$$L(f_\phi)=\frac{1}{\pi^2}\int_{\mathbf{C}}(-i)\frac{\partial\theta(u,v)}{\partial u}f(w)D(w)\,d\sigma(w)$$

and

$$L^n(f_\phi)=\frac{1}{\pi^2}\int_{\mathbf{C}}(-i)^n\frac{\partial^n\theta(u,v)}{\partial u^n}f(w)D(w)\,d\sigma(w).$$

Now,

$$(Lf)_\phi=\frac{1}{\pi^2}\int_{\mathbf{C}}\theta(u,v)(Lf)(w)D(w)\,d\sigma(w).$$

But

$$\frac{1}{\pi^2}\int_{\mathbf{C}}\theta(u,v)\left\{\left(a_1\left(-i\frac{\partial}{\partial w}\right)+a_2\left(-i\frac{\partial}{\partial \bar{w}}\right)\right)f(w)\right\}D(w)\,d\sigma(w)$$
$$=-\frac{1}{\pi^2}\int_{\mathbf{C}}\left\{\left(a_1\left(-i\frac{\partial}{\partial w}\right)+a_2\left(-i\frac{\partial}{\partial \bar{w}}\right)\right)\theta(u,v)\right\}f(w)D(w)\,d\sigma(w),$$

using integration by parts and Lemma 5.2.

But this last expression may be rewritten as

$$\frac{1}{\pi^2}\int_{\mathbf{C}}(-i)\frac{\partial\theta(u,v)}{\partial u}f(w)D(w)\,d\sigma(w).$$

(We have the extra minus sign since $L_w(u)=-L_z(u)=+i$.) So

$$(L^n f)_\phi=\frac{1}{\pi^2}\int_{\mathbf{C}}(-i)^n\frac{\partial^n\theta(u,v)}{\partial u^n}f(w)D(w)\,d\sigma(w).$$

Also, for $f_{\xi\phi}$,

$$\int_{\mathbf{C}}\xi\phi(\xi)\exp(i(\xi u+\bar{\xi}v))\,d\sigma(\xi)=(-i)\frac{\partial}{\partial u}\theta(u,v),$$

and

$$f_{\xi^n\phi}=\frac{1}{\pi^2}\int_{\mathbf{C}}(-i)^n\frac{\partial^n\theta(u,v)}{\partial u^n}f(w)D(w)\,d\sigma(w).$$

Hence, the first part of the proposition holds. In an analogous way, one can obtain

$$(\bar{L}^n f)_\phi=\bar{L}^n(f_\phi)=f_{\bar{\xi}^n\phi}=\frac{1}{\pi^2}\int_{\mathbf{C}}(-i)^n\frac{\partial^n\theta(u,v)}{\partial v^n}f(w)D(w)\,d\sigma(w).$$

This concludes the proof of Proposition 5.1. □

6. More Functional Calculus

Returning to formula (5.1), we would like to show that if $\phi_1,\phi_2\in C_0^\infty(\mathbf{R}^2)$, or are of the form $x^j y^k\exp(-x^2-y^2)$, then

(6.1) $$\phi_1(L)\phi_2(L)=(\phi_2\phi_2)(L).$$

First we have the following proposition.

PROPOSITION 6.1. *If $F(z_1,z_2)$ is an entire function which decreases rapidly in a conic sector $|z_1-\bar{z}_2|<M|z_1+\bar{z}_2|+M'$ (if $((h-g)(z),g(z))=(z_1,z_2)$, then Σ is contained in this sector), then*

$$\int_{\mathbf{C}}F((h-g)(z),g(z))D(z)\,d\sigma(z)=\int_{\mathbf{C}}F(z,\bar{z})\,d\sigma(z).$$

Proof. We first claim that if $\phi(w)=(h-g)\circ g^{-1}(\bar{w})$ then

(i) $$\int_{\mathbf{C}} F(\phi(w),\bar{w})\frac{\partial\phi}{\partial w}\,d\sigma(w)=\int_{\mathbf{C}} F((h-g)(z),g(z))D(z)\,d\sigma(z).$$

Indeed,

$$\phi(w)=h\circ g^{-1}(\bar{w})-\bar{w},\qquad \frac{\partial\phi}{\partial w}=\frac{\partial(h\circ g^{-1})(\bar{w})}{\partial w}.$$

Let $\bar{w}=g(z)$, and substitute into the left-hand side of (i). We have $\phi(w)=(h-g)(z)$ and $d\sigma(w)=|-J_g(z)|\,d\sigma(z)=-J_g(z)\,d\sigma(z)$, since $|\partial_{\bar{z}}g|^2-|\partial_z g|^2>0$. Also note that

$$\left.\frac{\partial(h\circ g^{-1})(\bar{w})}{\partial w}\right|_{\bar{w}=g(z)}=\left.\frac{\partial(h\circ g^{-1})(w)}{\partial\bar{w}}\right|_{w=g(z)}.$$

Using the chain rule, and letting

$$X=\left.\frac{\partial(h\circ g^{-1})}{\partial w}\right|_{w=g(z)},\qquad Y=\left.\frac{\partial(h\circ g^{-1})}{\partial\bar{w}}\right|_{w=g(z)},$$

we have the following system of equations:

$$\begin{cases}\dfrac{\partial h}{\partial z}=X\dfrac{\partial g}{\partial z}+Y\dfrac{\partial\bar{g}}{\partial z},\\[2ex]\dfrac{\partial h}{\partial\bar{z}}=X\dfrac{\partial g}{\partial\bar{z}}+Y\dfrac{\partial\bar{g}}{\partial\bar{z}},\end{cases}$$

which implies that $Y=-(D(z)/J_g(z))$. Substituting these expressions for Y, $\phi(w)$, and $d\sigma(w)$ into the left-hand side of (i), we obtain the right-hand side.

For the second part of the proof, we want to show that

$$\int_{\mathbf{C}} F(\phi(w),\bar{w})\frac{\partial\phi}{\partial w}\,d\sigma(w)=\int_{\mathbf{C}} F(w,\bar{w})\,d\sigma(w)$$

(this is done in Coifman–Meyer [CM2]). We write $\phi(w)=w+r(w)$ for some r with small Lipschitz norm. Let $\phi_t(w)=w+tr(w)$ $(0\le t\le 1)$, and let

$$J(t)=\int_{\mathbf{C}} F(\phi_t(w),\bar{w})\frac{\partial\phi_t}{\partial w}\,d\sigma(w).$$

Then

$$\begin{aligned}\frac{d}{dt}J(t)&=\int_{\mathbf{C}} F_1(\phi_t(w),\bar{w})r(w)\frac{\partial\phi_t}{\partial w}\,d\sigma(w)+\int_{\mathbf{C}} F(\phi_t(w),\bar{w})\frac{\partial r}{\partial w}\,d\sigma(w)\\&=\int_{\mathbf{C}}\frac{\partial}{\partial w}\{F(\phi_t(w),\bar{w})r(w)\}\,d\sigma(w)\\&=0.\end{aligned}$$

So $J(0)=J(1)$, which is what we needed. (The first part of the proof seems to be the nonlinear part of the calculation, while the second part is the linear one.) □

Now, we would like to show (6.1). We need to show that

$$(6.2)\quad \begin{aligned} \frac{1}{\pi^4}\int_{\mathbf{C}} k_1((h-g)(z)-(h-g)(w), g(z)-g(w))D(w) \\ k_2((h-g)(w)-(h-g)(z'), g(w)-g(z'))\,d\sigma(w) \\ =\frac{1}{\pi^2}k_3((h-g)(z)-(h-g)(z'), g(z)-g(z')). \end{aligned}$$

Here, for $i=1,2$,

$$k_i(u,v)=\int_{\mathbf{C}}\phi_i(\xi)\exp(i(u\xi+v\bar{\xi}))\,d\sigma(\xi),$$

and k_3 is associated in an analogous way to $\phi_1\phi_2$. The left-hand side of (6.2) is

$$\frac{1}{\pi^4}\int_{\mathbf{C}} k_1((h-g)(z)-w, g(z)-\bar{w})k_2(w-(h-g)(z'), \bar{w}-g(z'))\,d\sigma(w),$$

by Proposition 6.1. Now: $k_1((h-g)(z)-w, g(z)-\bar{w})=(\tilde{\phi}_1)\hat{}\,(2x,-2y)$; $u=x+iy$, where $\tilde{\phi}_1=\phi_1(\xi)\exp(i(\xi(h-g)(z)+\bar{\xi}g(z)))$; and

$$k_2(w-(h-g)(z'), \bar{w}-g(z'))=(\tilde{\phi}_2)\check{}\,(2x,-2y)4\pi^2,$$

where $\tilde{\phi}_2=\phi_2(\xi)\exp(-i(\xi(h-g)(z')+\bar{\xi}g(z')))$. Thus, the left-hand side of (6.2) is:

$$\begin{aligned} &\frac{4\pi^2}{4\pi^4}\int_{\mathbf{C}}(\tilde{\phi}_1)\hat{}\,(x,y)(\tilde{\phi}_2)\check{}\,(x,y)\,dx\,dy \\ &=\frac{1}{\pi^2}\int_{\mathbf{C}}\tilde{\phi}_1\tilde{\phi}_2\,dx\,dy \\ &=\frac{1}{\pi^2}\int_{\mathbf{C}}\phi_1(\xi)\phi_2(\xi)\exp(i(\xi((h-g)(z)-(h-g)(z')) \\ &\qquad\qquad +\bar{\xi}(g(z)-g(z'))))\,d\sigma(z) \\ &=\frac{1}{\pi^2}k_3((h-g)(z)-(h-g)(z'), g(z)-g(z')). \end{aligned}$$

This finishes the demonstration of (6.1). □

7. L^2 Boundedness and Extension to More General ϕ

In this section we define $\phi(L)$ for more general ϕ, and extend the results in [CM2] about L^2 boundedness. We define Ω_M $(0<M<1)$ to be the sector on $\mathbf{C}^2$ given by

$$(\operatorname{Im}\tilde{u})^2+(\operatorname{Im}\tilde{v})^2<M^2((\operatorname{Re}\tilde{u})^2+(\operatorname{Re}\tilde{v})^2),$$

with $\tilde{u}=\xi+i\xi'$ and $\tilde{v}=\eta+i\eta'$.

PROPOSITION 7.1. *Let L be as before. Suppose a_1 and a_2 are as in Section 3, or (more generally) that η and m have small Lipschitz norms (see* *also Section 8). Then, for any* $0<M<1$, *if* $\chi:\Omega_M\mapsto\mathbf{C}$ *is bounded and holomorphic then the operator* $\chi(L)$ *is bounded on* $L^2(\mathbf{R}^2)$. $\chi(L)$ *is defined in the following way: We let*

$$\chi_\epsilon=\chi\exp(-\epsilon(\tilde{u}^2+\tilde{v}^2)),\quad (\tilde{u},\tilde{v})\in\Omega_M,$$

and $\chi(L)(f)=\lim_{\epsilon\to0}\chi_\epsilon(L)(f)$ *in the weak sense.*

Proof. We will show that $\|\chi_\epsilon(L)\|_{L^2,L^2}\le c\|\chi\|_{H^\infty(\Omega_M)}$, c independent of ϵ. We use formula (5.1) for $\chi_\epsilon(L)$. Let $k_\epsilon(z,w)$ be the kernel of $\chi_\epsilon(L)$, and let $k_\epsilon^0(z,w)$ be such that $k_\epsilon(z,w)=(1/\pi^2)k_\epsilon^0(z,w)D(w)$. Let T_0^ϵ be the operator defined by k_ϵ^0. We have:

(i) $|k_\epsilon^0(z,w)|\le c_1/|z-w|^2$;
(ii) $|(\partial/\partial z)k_\epsilon^0(z,w)|+|(\partial/\partial\bar{z})k_\epsilon^0(z,w)|\le c_2/|z-w|^3$, and similarly for $|(\partial/\partial w)k_\epsilon^0|+|(\partial/\partial\bar{w})k_\epsilon^0|$;
(iii) $T_\epsilon^0(D)=c\chi(0)$;
(iv) $(T_\epsilon^0)^t(D)=c\chi(0)$; and
(v) $\chi_\epsilon(L)f(z)=\int A(z,w)(Lf)(w)D(w)\,d\sigma(w)$, where $|A(z,w)|\le c_3/|z-w|$.

The constants c_1, c_2, c_3, c depend only on M (in particular, not on ϵ). Once we have (i) through (v) we are done. (v) shows weak boundedness of DT_ϵ^0D; if f,g are supported in a cube $Q\subseteq\mathbf{C}$ then

$$|\langle D(z)\chi_\epsilon(L)f(z),g(z)\rangle|\le c|Q|^{3/2}\|g\|_\infty\|\nabla f\|_\infty.$$

(i) and (ii) show that k_ϵ^0 is standard, and (iii) and (iv) complete the requirements of the $T(b)$ theorem [DJS] for T_ϵ^0, since $\operatorname{Re}D(z)\approx1$ if m and η have small Lipschitz norms.

To show (i) and (ii), we repeat the argument in [CM2]:

$$\chi_\epsilon=\chi\exp(-\epsilon(\tilde{u}^2+\tilde{v}^2)),$$

and let $\phi_\epsilon(\zeta)=\chi_\epsilon(\bar{\zeta}/2)$. Calculate the inverse Fourier–Laplace transform of $\phi_\epsilon(\xi,\eta)$:

$$\Phi_\epsilon(\tilde{u},\tilde{v})=\frac{1}{4\pi^2}\int_{\mathbf{R}^2}\exp(i(\tilde{u}\xi+\tilde{v}\eta))\phi_\epsilon(\xi,\eta)\,d\xi\,d\eta.$$

Then $\Phi_\epsilon(\tilde{u},\tilde{v})$ is holomorphic in the sector $\Omega_{M'}$, $M'<M$, and we have

$$|\Phi_\epsilon(\tilde{u},\tilde{v})|\le c\|\chi\|_{H^\infty(\Omega_M)}(|\tilde{u}|^2+|\tilde{v}|^2)^{-1}.$$

Also,

$$|\partial_u^\alpha\partial_v^\beta\Phi_\epsilon(\tilde{u},\tilde{v})|\le c_{\alpha,\beta}\|\chi\|_{H^\infty(\Omega_M)}(|\tilde{u}|^2+|\tilde{v}|^2)^{-1-\alpha/2-\beta/2}.$$

This can be obtained in the usual way. We change the contour of integration to change $\tilde{u}$ and $\tilde{v}$ in $\exp(i(\tilde{u}\xi+\tilde{v}\eta))$ to be real. By the definition of χ_ϵ, we are permitted to do this in each variable separately. Once we have done this rotation, using Cauchy's theorem again, we see that the symbol σ associated with $\Phi_\epsilon(\tilde{u},\tilde{v})$ satisfies

$$|\partial^\alpha \partial^\beta \sigma| \le \frac{c_{\alpha,\beta}}{(|\xi|^2+|\eta|^2)^{(\alpha+\beta)/2}}.$$

We then do the usual real variable argument. Now,

$$\begin{aligned} k_\epsilon^0(z,w) &= \Phi_\epsilon\left(\frac{h(z)-h(w)}{2}, \frac{(h-2g)(z)-(h-2g)(w)}{2i}\right) \\ &= \Phi_\epsilon\left(x-s+\frac{\eta(z)-\eta(w)}{2}, y-t+\frac{(\eta-2m)(z)-(\eta-2m)(w)}{2i}\right), \end{aligned}$$

where $z=x+iy$ and $w=s+it$. (From the first section on functional calculus, $\tilde{u}$ corresponds to $(u+v)/2$ and $\tilde{v}$ to $(u-v)/2$.) We obtain (i) and (ii) from the fact that η and m are Lipschitz with small norm.

(iii) and (iv) follow from Proposition 6.1.

Finally, (v) follows from considering the symbol $\chi_\epsilon(\zeta)/\zeta$ instead of $\chi_\epsilon(\zeta)$ and applying a similar argument from pseudodifferential operators to the one used to obtain (i) and (ii).

It remains to show that $\lim_{\epsilon\to 0}\chi_\epsilon(L)f$ exists in the weak sense, that is, given $f, p \in L^2(\mathbf{R}^2)$, $\lim_{\epsilon\to 0}\int \chi_\epsilon(L)fp$ exists. This is equivalent to showing that

$$\lim_{\epsilon\to 0}\int \chi_\epsilon(L)f(z)p(z)D(z)\,d\sigma(z)$$

exists. We will show that the above sequence is Cauchy, that is,

$$(*) \qquad \int \chi_{\epsilon_1}(L)f(z)p(z)D(z) - \int \chi_{\epsilon_2}(L)f(z)p(z)D(z) \to 0$$

as $\epsilon_1, \epsilon_2 \to 0$. Since we know that $\|\chi_\epsilon(L)\|_{L^2,L^2}$ is uniformly bounded, to show $(*)$ it suffices to take f, p in some dense class in L^2.

In order to define an appropriate dense class, we digress a bit to define an operator τ:

$$\tau u(z) = \frac{1}{2\pi}\int_{\mathbf{R}^2}\exp\left(\frac{i}{2}((h-g)(z)\bar{\xi}+g(z)\xi)\right)u(\xi)\,d\sigma(\xi).$$

Note that τu is well defined if u has compact support. It is easy to show that the set of τu, where u has compact support, is dense in $L^2(\mathbf{R}^2)$.

We now return to $(*)$. We set $f=\tau\phi$ and $p=\tau\psi$, where ϕ, ψ have compact support and are in L^2. Then

$$\begin{aligned} (*) = \frac{1}{\pi^2}\iint k((h-g)(z)-(h-g)(w), g(z)-g(w)) \\ (\tau\phi)(w)(\tau\psi)(z)D(z)D(w)\,d\sigma(z)\,d\sigma(w), \end{aligned}$$

where

$$k(u,v) = \int_{\mathbf{C}}(\chi_{\epsilon_1}-\chi_{\epsilon_2})(\xi)\exp(i(u\xi+v\bar{\xi}))\,d\sigma(\xi).$$

By Proposition 6.1 (applied twice),

$$(*) = c\int_{\mathbf{C}}\int_{\mathbf{C}}\int_{\mathbf{C}}(\chi_{\epsilon_1}-\chi_{\epsilon_2})(\xi)\exp(i((z-w)\xi+\overline{(z-w)\xi}))\,\hat{\phi}(w)\hat{\psi}(z)\,d\sigma(w)\,d\sigma(z)\,d(\xi).$$

So

$$(*) \le c\int_{\mathbf{C}}|(\chi_{\epsilon_1}-\chi_{\epsilon_2})(\xi)||\phi(-2\bar{\xi})||\psi(2\bar{\xi})|\,d\sigma(\xi)$$
$$\le c\left(\int_{\mathbf{C}}|(\chi_{\epsilon_1}-\chi_{\epsilon_2})(\xi)|^2|\phi(-2\bar{\xi})|^2\right)^{1/2}\|\psi\|_2 \to 0$$

as $\epsilon_1, \epsilon_2 \to 0$, by dominated convergence theorem. This concludes the proof of Proposition 7.1. □

8. Restrictions Forced on m and η Assuming $L/\bar{L}$ and $\bar{L}/L$ Are Bounded on L^2

We have obtained a functional calculus for $L = a_1\partial_z + a_2\partial_{\bar{z}}$, assuming that the partials of m and η are small, and making implicitly the assumptions of (i) in Proposition 8.1 below. All of the above are satisfied if, for example, we place the restrictions on a_2 and a_1 described in Section 3. In this section, we wish to show that bounds on a_1, a_2 and L^2 norms of $L/\bar{L}$ and $\bar{L}/L$ are all that is needed to estimate the bi-Lipschitz norms of $h-g$, g, the Lipschitz norms of m and η, and the distance of D away from 0.

In Proposition 8.1 below, we assume there are constants $c_1 > 0$, $c_2 > 0$, $c_3 > 0$, and $1 > c_4 > 0$ such that

$$\|a_1\|_\infty < c_1, \quad \|a_2\|_\infty < c_2, \quad \left\|\frac{1}{a_1}\right\|_\infty < c_3, \quad \text{and} \quad \left\|\frac{a_2}{a_1}\right\|_\infty < 1-c_4.$$

We define $\tilde{b}_1$ and $\tilde{b}_2$ using (3.5a) and (3.5b) (see Proposition 8.1 for more details), and assume that there exist $c_5 > 0$ and $c_6 > 0$ such that $L/\bar{L}$ and $\bar{L}/L$ have L^2 norms bounded by c_5 and c_6 (resp.). We use the following form for $L/\bar{L}$ and $\bar{L}/L$:

$$\frac{L}{\bar{L}} = (a_1(z)B + a_2(z))\frac{1}{\tilde{b}_1(z)B+\tilde{b}_2(z)},$$
$$\frac{\bar{L}}{L} = (\tilde{b}_1(z)B+\tilde{b}_2(z))\frac{1}{a_1(z)B+a_2(z)}$$

(not the integral kernel form obtained in Section 4). Now, let's solve (2.4) and (2.5) to obtain:

$$\frac{\partial g}{\partial z} = -\frac{a_2}{a_1}\frac{\partial g}{\partial \bar{z}},$$
$$\frac{\partial(h-g)}{\partial \bar{z}} = -\frac{\tilde{b}_1}{\tilde{b}_2}\frac{\partial(h-g)}{\partial z},$$
$$\frac{\partial(h-g)}{\partial z} = \frac{1}{a_1}\left(1-\frac{a_2}{a_1}\frac{\tilde{b}_1}{\tilde{b}_2}\right)^{-1},$$

$$\frac{\partial g}{\partial \bar z}=\frac{1}{\tilde b_1}\left(1-\frac{\tilde b_1}{\tilde b_2}\frac{a_2}{a_1}\right)^{-1}.$$

Thus

$$\left(\frac{\partial(h-g)}{\partial z}\pm\frac{\partial(h-g)}{\partial \bar z}\right)^{\pm 1}=\left[\frac{1}{a_1}\left(1\mp\frac{\tilde b_1}{\tilde b_2}\right)\left(1-\frac{a_2}{a_1}\frac{\tilde b_1}{\tilde b_2}\right)^{-1}\right]^{\pm 1}$$

and

$$\left(\frac{\partial g}{\partial \bar z}\pm\frac{\partial g}{\partial z}\right)^{\pm 1}=\left[\frac{1}{\tilde b_2}\left(1\mp\frac{a_2}{a_1}\right)\left(1-\frac{a_2}{a_1}\frac{\tilde b_1}{\tilde b_2}\right)^{-1}\right]^{\pm 1}.$$

A similar expression can be obtained for D. Proposition 8.1 proves that $\|1/\tilde b_2\|_\infty<d_3$ and $\|\tilde b_1/\tilde b_2\|_\infty<1-d_4$ $(0<d_4<1)$, where d_3 and d_4 depend only on the constants $c_1,\dots,c_6$ mentioned above. Hence, we conclude that $h-g$ and g are bi-Lipschitz, m and η are Lipschitz, and D stays away from 0, with all quantities estimated in terms of $c_1,\dots,c_6$.

PROPOSITION 8.1.

(i) *Suppose there are constants* (*not all independent*) $c_1>0$, $c_2>0$, $c_3>0$, *and* $1>c_4>0$ *such that* $\|a_1\|_\infty<c_1$, $\|a_2\|_\infty<c_2$, $\|1/a_1\|_\infty<c_3$, *and* $\|a_2/a_1\|_\infty<1-c_4$.

(ii) *Consider the operator* $Q=(1+B^{-1}M(a_2/a_1))^{-1}$, *where* $M(a_2/a_1)$ *denotes multiplication by* a_2/a_1. *Assume that* $Q(1)$ *and* $Q(B^{-1}(1/a_1))$ *are in* L^∞, *and that* $\|Q(1)\|_\infty>0$.

(iii) *Using assumption* (ii), *we can define* $\tilde b_1$ *and* $\tilde b_2$ *by* (3.5a) *and* (3.5b), *and* $\tilde b_1$ *and* $\tilde b_2$ *are in* L^∞. *Assume that* $\|\tilde b_1/\tilde b_2\|_\infty<1$ *and* $\|1/\tilde b_2\|_\infty<\infty$.

(iv) *Let* c_5 *be the* L^2 *norm of the operator*

$$\frac{L}{\bar L}=(a_1(z)B+a_2(z))\frac{1}{\tilde b_1(z)B+\tilde b_2(z)}$$

and let c_6 *be the* L^2 *norm of the operator*

$$\frac{\bar L}{L}=(\tilde b_1(z)B+\tilde b_2(z))\frac{1}{a_1(z)B+a_2(z)}.$$

Then there exist $d_1>0$, $d_2>0$, $d_3>0$, *and* $0<d_4<1$, *depending only on* $c_1,\dots,c_6$, *such that* $\|\tilde b_1\|_\infty<d_1$, $\|\tilde b_2\|_\infty<d_2$, $\|1/\tilde b_2\|_\infty<d_3$, *and* $\|\tilde b_1/\tilde b_2\|_\infty<1-d_4$.

Note that the point of this proposition is that the quantities $d_1,\dots,d_4$, whose existence is assumed in the hypotheses (ii) and (iii), depend only on $c_1,\dots,c_6$.

We first prove the following lemma.

LOCALIZATION LEMMA. *Let* $e_1(z),e_2(z),e_3(z),e_4(z)$ *be bounded functions, and either* $\|e_4/e_3\|_\infty<1$ *and* $\|1/e_3\|_\infty<\infty$ *or* $\|e_3/e_4\|_\infty<1$ *and* $\|1/e_4\|_\infty<\infty$. *Define the operators* $S=e_1(z)B+e_2(z)$, $T=1/(e_3(z)B+e_4(z))$ *on* L^2. *Then* $\|ST\|_2\ge\|S_{z_0}T_{z_0}\|_2$ *for any fixed* z_0 *in* $\mathbf{C}$ *at which* $e_1,\dots,e_4$ *are continuous, where* $S_{z_0}=e_1(z_0)B+e_2(z_0)$ *and* $T_{z_0}=1/(e_3(z_0)B+e_4(z_0))$.

Proof. Choose f in L^2 of norm 1 and define $f_k(z)=kf(k(z-z_0))$, where the factor k preserves the L^2 norm. Write

$$STf_k-S_{z_0}T_{z_0}f_k=STf_k-ST_{z_0}f_k+ST_{z_0}f_k-S_{z_0}T_{z_0}f_k.$$

However,

$$|ST_{z_0}f_k-S_{z_0}T_{z_0}f_k\|_2=\|(S-S_{z_0})(T_{z_0}f)_k\|_2\to 0$$

as $k\to\infty$. We use the fact that B commutes with translation and dilation and hence so does T_{z_0}, for z_0 kept constant. The first part of $STf_k-S_{z_0}T_{z_0}f_k$ can be handled in a similar way, so we have

$$\|STf_k-S_{z_0}T_{z_0}f_k\|_2\to 0$$

as $k\to\infty$, and the conclusion of the lemma follows. □

The proof of Proposition 8.1 is now almost immediate. From (iv) and the localization lemma, we have:

(a) $$\left|\frac{a_1(z_0)(\bar{\xi}/\xi)+a_2(z_0)}{\tilde{b}_1(z_0)(\bar{\xi}/\xi)+\tilde{b}_2(z_0)}\right|<c_5;$$

(b) $$\left|\frac{\tilde{b}_1(z_0)(\bar{\xi}/\xi)+\tilde{b}_2(z_0)}{a_1(z_0)(\bar{\xi}/\xi)+a_2(z_0)}\right|<c_6.$$

Here we have used Plancherel's theorem. (a) and (b) hold for almost all z_0 and ξ in **C**. From (b), we can find the desired d_1 and d_2 to control $\|\tilde{b}_1\|_\infty$ and $\|\tilde{b}_2\|_\infty$. From (a), we obtain $1/||\tilde{b}_1|-|\tilde{b}_2||<d_5$, and these constants depend only on $c_1,\ldots,c_6$. The rest of Proposition 8.1 follows easily. □

9. Conjugation to the Coifman–Meyer Case

In this section, we see how the operator L arises by conjugation from the operator $a\partial_z$ considered by Coifman and Meyer in [CM1] and [CM2].

Let $Tf=U_{\bar{g}}a\partial_z U_{\bar{g}^{-1}}$, where $U_{\bar{g}}h\equiv h\circ\bar{g}$; g and a will be determined in a moment, so that $T=L$. Now

$$\partial_z U_{\bar{g}^{-1}}(f)=(-i)\left(\frac{\partial f}{\partial z}\circ\bar{g}^{-1}\frac{\partial\bar{g}^{-1}}{\partial z}+\frac{\partial f}{\partial\bar{z}}\circ\bar{g}^{-1}\frac{\overline{\partial\bar{g}^{-1}}}{\partial\bar{z}}\right),$$

so

$$Tf=-ia\circ\bar{g}\times\left(\frac{\partial f}{\partial z}\frac{\partial\bar{g}^{-1}}{\partial z}\circ\bar{g}+\frac{\partial f}{\partial\bar{z}}\frac{\overline{\partial\bar{g}^{-1}}}{\partial\bar{z}}\circ\bar{g}\right).$$

But

$$\frac{\partial\bar{g}^{-1}}{\partial z}\circ\bar{g}=\frac{1}{\rho}\frac{\partial g}{\partial\bar{z}}\quad\text{and}\quad\frac{\overline{\partial\bar{g}^{-1}}}{\partial\bar{z}}\circ\bar{g}=-\frac{1}{\rho}\frac{\partial g}{\partial z},\quad\text{where }\rho\equiv\left|\frac{\partial g}{\partial\bar{z}}\right|^2-\left|\frac{\partial g}{\partial z}\right|^2.$$

Therefore

$$Tf=-ia\circ\bar{g}\times\left(\frac{1}{\rho}\frac{\partial g}{\partial\bar{z}}\frac{\partial}{\partial z}-\frac{1}{\rho}\frac{\partial g}{\partial z}\frac{\partial}{\partial\bar{z}}\right)f.$$

Hence, $Tf = Lf$ if and only if

$$\begin{cases} a_1 = \dfrac{1}{\rho}\dfrac{\partial g}{\partial \bar{z}} \times a \circ \bar{g}, \\ a_2 = -\dfrac{1}{\rho}\dfrac{\partial g}{\partial z} \times a \circ \bar{g}. \end{cases}$$

From the above system,

$$\frac{a_2}{a_1} = -\left(\frac{\partial g}{\partial z} \Big/ \frac{\partial g}{\partial \bar{z}}\right),$$

so

$$\frac{\partial g}{\partial z} = -\frac{a_2}{a_1}\frac{\partial g}{\partial \bar{z}},$$

which shows that this g is the same g we had before. Thus

$$a = \left(a_1 \rho \Big/ \frac{\partial g}{\partial \bar{z}}\right) \circ \bar{g}^{-1}$$

makes $T = L$.

To show that the formula (5.1) is obtained by conjugating the Coifman-Meyer formula, $U_{\bar{g}}(\phi(a\partial_z))U_{\bar{g}^{-1}}$, we do the following. Let

$$\frac{\partial q}{\partial \bar{z}} = \mu \frac{\partial q}{\partial z}, \quad \text{where } \mu = aB^{-1}\left(\frac{1}{a}\right).$$

Then

$$\phi(a\partial_z)f = \int_{\mathbf{C}}\int_{\mathbf{C}} \phi(\xi)\exp(i((q(z)-q(u))\xi + (\bar{z}-\bar{w})\bar{\xi}))\, d\sigma(\xi)\frac{\partial q}{\partial w} f(w)\, d\sigma(w).$$

Hence

$$\begin{aligned} &U_{\bar{g}}(\phi(a\partial_z))U_{\bar{g}^{-1}}f \\ &\quad = \int_{\mathbf{C}}\int_{\mathbf{C}} \phi(\xi)\exp(i((q\circ\bar{g}(z)-q\circ\bar{g}(u))\xi \\ &\qquad\qquad + (g(z)-g(u))\bar{\xi}))\, d\sigma(\xi)\left(\frac{\partial q}{\partial w}\right)\circ\bar{g}(u)\rho f\, d\sigma(u), \end{aligned}$$

using the change of variables $u = \bar{g}^{-1}(w)$. Now $\partial q/\partial w = 1/a$, so

$$\frac{\partial q}{\partial w}\circ\bar{g} = \frac{1}{a\circ\bar{g}} = \frac{1}{\rho a_1}\frac{\partial g}{\partial \bar{z}}.$$

Hence

$$\rho \times \left(\frac{\partial q}{\partial w}\right)\circ\bar{g} = \frac{1}{a_1}\frac{\partial g}{\partial \bar{z}}.$$

Now, remembering D,

$$D \equiv \frac{\partial(h-g)}{\partial z}\frac{\partial g}{\partial \bar{z}} - \frac{\partial g}{\partial z}\frac{\partial(h-g)}{\partial \bar{z}} = \frac{1}{a_1}\frac{\partial g}{\partial \bar{z}},$$

so the weight factors in the expression for $U_{\bar{g}}\phi(a\partial_z)U_{\bar{g}^{-1}}$ and formula (5.1) match.

It remains to show that $q \circ \bar{g} = h - g$. It suffices to show that the other equation in system (2.4) is satisfied (here, $q \circ \bar{g} \approx z$, so we need $h-g$, not h itself). We need to show that

$$1 = a_1 \frac{\partial(q \circ \bar{g})}{\partial z} + a_2 \frac{\partial(q \circ \bar{g})}{\partial \bar{z}}.$$

The right-hand side is:

$$(!) \qquad a_1\left(\frac{\partial q}{\partial z} \circ \bar{g} \frac{\partial \bar{g}}{\partial z} + \frac{\partial q}{\partial \bar{z}} \circ \bar{g} \frac{\partial g}{\partial z}\right) + a_2\left(\frac{\partial q}{\partial \bar{z}} \circ \bar{g} \frac{\partial g}{\partial \bar{z}} + \frac{\partial q}{\partial z} \circ \bar{g} \frac{\partial \bar{g}}{\partial \bar{z}}\right).$$

Now $(\partial q/\partial z) \circ \bar{g} = 1/(a \circ \bar{g})$, so

$$\begin{aligned}(!) &= a_1 \frac{\partial q}{\partial z} \circ \bar{g} \frac{\partial \bar{g}}{\partial z} + a_2 \frac{\partial q}{\partial z} \circ \bar{g} \frac{\partial \bar{g}}{\partial \bar{z}} + \mu \circ \bar{g}\left(a_1 \frac{\partial q}{\partial z} \circ \bar{g} \frac{\partial g}{\partial z} + a_2 \frac{\partial q}{\partial z} \circ \bar{g} \frac{\partial g}{\partial \bar{z}}\right) \\ &= \frac{1}{\rho} \frac{a \circ \bar{g}(\partial g/\partial \bar{z})}{a \circ \bar{g}} \frac{\overline{\partial g}}{\partial \bar{z}} + \frac{a \circ \bar{g}(-(1/\rho))}{a \circ \bar{g}} \frac{\partial g}{\partial z} \frac{\overline{\partial g}}{\partial z} + \mu \circ \bar{g}\left(\frac{a_1}{a \circ \bar{g}} \frac{\partial g}{\partial z} + \frac{a_2}{a \circ \bar{g}} \frac{\partial g}{\partial \bar{z}}\right)\end{aligned}$$

(using the equations connecting a, a_1, and a_2), and hence

$$\begin{aligned}(!) &= \frac{1}{\rho}\left(\left|\frac{\partial g}{\partial \bar{z}}\right|^2 - \left|\frac{\partial g}{\partial z}\right|^2\right) + \mu \circ \bar{g}\left(\frac{1}{\rho} \frac{\partial g}{\partial \bar{z}} \frac{\partial g}{\partial z} - \frac{1}{\rho} \frac{\partial g}{\partial z} \frac{\partial g}{\partial \bar{z}}\right) \\ &= \frac{1}{\rho} \rho = 1.\end{aligned}$$

10. Appendix

To prove Lemma 4.1, we first calculate the following. Let R_1 be a circle of radius $a > 0$, centered at the origin of $\mathbf{R}^2$, and let R_2 be the circumscribed ellipse with major axis b and minor axis a, $b \geq a$. Then

$$\int_{R_2 \setminus R_1} z^{-2} \, d\sigma(z) = \int_{R_2 \setminus R_1} \bar{z}^{-2} \, d\sigma(z) = \pi \frac{b-a}{b+a}.$$

Now, suppose ρ is quasiconformal. Let

$$G(u) = \int_{\mathbf{C}} \frac{1}{u - \rho(w)} f(w) \, d(w).$$

Then

$$\left.\frac{\partial G}{\partial \bar{u}}\right|_{\rho(z)} = \frac{\pi f(z)}{|\partial_z \rho|^2 - |\partial_{\bar{z}} \rho|^2},$$

and

$$\begin{aligned}\left.\frac{\partial G}{\partial u}\right|_{\rho(z)} &= \frac{\partial}{\partial u} \int_{\mathbf{C}} \frac{1}{u - \rho(w)} f(w) \, d\sigma(w) \\ &= \frac{\partial}{\partial u} \int_{\mathbf{C}} \frac{1}{u - v} f(\rho^{-1}(v)) J_{\rho^{-1}}(v) \, d\sigma(v) \\ &= -\lim_{\epsilon \to 0} \int_{|u-v| \geq \epsilon} \frac{1}{(u-v)^2} f(\rho^{-1}(v)) J_{\rho^{-1}}(v) \, d\sigma(v) =\end{aligned}$$

$$= -\lim_{\epsilon \to 0} \int_{\rho^{-1}(v) \notin \rho^{-1}\{B_\epsilon(u)\}} \frac{1}{(u-v)^2} f(\rho^{-1}(v)) J_{\rho^{-1}}(v)\, d\sigma(v)$$
$$+ \lim_{\epsilon \to 0} \int_{\rho^{-1}(v) \notin B_{\tilde{\epsilon}}(\rho^{-1}(u))} \frac{1}{(u-v)^2} f(\rho^{-1}(v)) J_{\rho^{-1}}(v)\, d\sigma(v) \qquad (A)$$
$$- \lim_{\epsilon \to 0} \int_{\rho^{-1}(v) \notin B_{\tilde{\epsilon}}(\rho^{-1}(u))} \frac{1}{(u-v)^2} f(\rho^{-1}(v)) J_{\rho^{-1}}(v)\, d\sigma(v). \qquad (B)$$

Now

$$B = -\lim_{\epsilon \to 0} \int_{w \notin B_{\tilde{\epsilon}}(z)} \frac{1}{(\rho(z) - \rho(w))^2} f(w)\, d\sigma(w)$$
$$= -p.v. \int_{\mathbf{C}} \frac{f(w)}{(\rho(z) - \rho(w))^2}\, d\sigma(w)$$

and

$$A = \lim_{\epsilon \to 0} \int_{v \in B_\epsilon(u) - \rho\{B_{\tilde{\epsilon}}(\rho^{-1}(u))\}} \frac{1}{(u-v)^2} f(\rho^{-1}(v)) J_{\rho^{-1}}(v)\, d\sigma(v).$$

In the above, B_ϵ and $B_{\tilde{\epsilon}}$ refer to balls of appropriate radii. Now, if (in the expression for A) the circle and the ellipse were aligned along the x and y axes, then we could use our preliminary calculation to obtain:

$$A = \left| \frac{\partial \bar{\rho}}{\partial z} \Big/ \frac{\partial \rho}{\partial z} \right| \frac{\pi f(z)}{J_\rho(z)}.$$

But our picture is rotated, so we must let $\tilde{z} = e^{-i\theta} z$ to rotate to standard picture, where

$$\theta = \frac{1}{2} \arg\left(\frac{\partial \rho}{\partial \bar{z}} \Big/ \frac{\partial \rho}{\partial z} \right)$$

(see [A]). Hence

$$A = -\left(\frac{\partial \bar{\rho}}{\partial z} \Big/ \frac{\partial \rho}{\partial z} \right) \frac{\pi f(z)}{J_\rho}.$$

Since

$$\frac{\partial}{\partial \bar{z}} (G \circ \rho) = \frac{\partial G}{\partial u} \frac{\partial \rho}{\partial \bar{z}} + \frac{\partial G}{\partial \bar{u}} \frac{\partial \bar{\rho}}{\partial \bar{z}},$$

we have

$$\frac{\partial}{\partial \bar{z}} \int_{\mathbf{C}} \frac{1}{\rho(z) - \rho(w)} f(w)\, d\sigma(w) = -p.v. \int_{\mathbf{C}} \frac{f(w)}{(\rho(z) - \rho(w))^2}\, d\sigma(w)$$
$$+ \pi f(z) \left\{ \frac{\overline{\partial \rho / \partial z}}{J_\rho} - \frac{\partial \bar{\rho}/\partial z}{\partial \rho / \partial z} \frac{\partial \rho / \partial \bar{z}}{J_\rho} \right\},$$

and the expression inside $\{\ \}$ is $1/(\partial \rho / \partial z)$. The second part of Lemma 4.1 follows in the same way. □

References

[A] L. V. Ahlfors, *Lectures on quasiconformal mappings,* Van Nostrand, Princeton, N.J., 1966.

[CM1] R. R. Coifman and Y. Meyer, *Non-linear harmonic analysis, operator theory, and PDE,* Ann. of Math. Stud., 112, Princeton Univ. Press, Princeton, N.J., 1986.

[CM2] ———, *Le calcul symbolique sur* $a(z)(\partial/\partial z)$, unpublished manuscript (two versions).

[DJS] G. David, J. L. Journé, and S. Semmes, *Opérateurs de Calderón-Zygmund, fonctions para-accrétives et interpolation,* Rev. Mat. Iberoamericana 1 (1985), 1–56.

[GC] J. García-Cuerva, *Weighted* H^p *spaces,* Dissertationes Math. 162 (1979).

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