An Approximation Theorem for Szegö Kernels and Applications

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1. Introduction

We begin by explaining the title of this paper. Consider the following question: Let K be a distributional kernel on a compact CR manifold, for example, a compact hypersurface in $\mathbf{C}^n$. We ask what kind of conditions on K are sufficient to guarantee that K differs from the Szegö kernel only by a smooth function? Our answer provides an approximation theorem for Szegö kernels. This question of approximation arises naturally from the study of the Szegö kernel, since finding the Szegö kernel explicitly is almost impossible for most CR manifolds. Before making a precise statement, we introduce two applications of the main theorem which constitute motivations for this work.

APPLICATION 1. The first application is to the question of a localization of the Szegö kernels. Let Ω_1 and Ω_2 be two bounded pseudoconvex domains in $\mathbf{C}^n$ with smooth boundaries $b\Omega_j$. Suppose that $\Omega_2 \subset \Omega_1$ and $b\Omega_1 \cap b\Omega_2 \neq \emptyset$. If S_j is the Szegö kernel for Ω_j $(j=1,2)$, is $S_1 - S_2$ smooth in the interior of $(b\Omega_1 \cap b\Omega_2) \times (b\Omega_1 \cap b\Omega_2)$? The answer is yes, under a certain type of condition (see Corollary 4.3 for a precise statement). An analogous question on the Bergman kernels was resolved by Fefferman by an elegant trick [4].

APPLICATION 2. The second application is related to the study of the Szegö kernel for domains in $\mathbf{C}^3$. Let Ω be a bounded pseudoconvex domain with a smooth boundary. Suppose that a portion of Ω is defined by the defining function

$$\rho(z) = p_1(z_1, \bar{z}_1) + p_2(z_2, \bar{z}_2) - \operatorname{Im} z_3,$$

where p_j is a subharmonic but not harmonic polynomial. Let

$$\Omega_1 = \Omega \cap \{(0, z_2, z_3)\} \quad \text{and} \quad \Omega_2 = \Omega \cap \{(z_1, 0, z_3)\}.$$

We want to exploit the relationship between the Szegö kernels for Ω_1, Ω_2, and Ω. It turns out that the Szegö kernel for Ω differs by a smooth function from a kind of convolution of the Szegö kernels for Ω_1 and Ω_2 (see Theorem 5.1).

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It is apparent that both applications are very much related to an approximation of the Szegö kernels.

We now briefly state the basic hypothesis and main result of this work. Let M be a compact CR manifold and let $\mathbf{H}_b$ denote the class of square integrable CR functions. Throughout this paper we will assume that there exists an open subset U of M such that, for any pair of functions $\{\psi_1, \psi_2\}$ in $C_0^\infty(U)$ with $\psi_2 \equiv 1$ in a neighborhood of the support of ψ_1 and for any $s > 0$, there exists a positive constant C_s such that if $u \in \operatorname{dom}(\bar\partial_b)$ and $u \perp \mathbf{H}_b$, then

$$\|\psi_1 u\|_s \leq C_s(\|\psi_2 \bar\partial_b u\|_s + \|\bar\partial_b u\|). \tag{2.1}$$

Among well-known examples of such CR manifolds are compact pseudoconvex CR manifolds of finite type on which $\bar\partial_b$ has a closed range (see Remark 2.2). Note that (2.1) is not a subelliptic estimate. We can even allow the right-hand side of (2.1) to have $\|\psi_2 \bar\partial_b u\|_{s+N}$ for some N instead of $\|\psi_2 \bar\partial_b u\|_s$. It would be interesting to see what kinds of geometric conditions on the manifolds will imply the estimate (2.1). If a 3-dimensional compact CR manifold is Levi flat in an open subset U (locally it is $\mathbf{C} \times \mathbf{R}$), then it is easy to see that (2.1) does not hold in U.

Suppose that M is a finite-dimensional compact CR manifold and that (2.1) holds in an open subset U of M. The distributional kernels K in which we are interested are of the following kind: K is in $C^\infty(U \times U \setminus \Delta)$ where Δ is the diagonal in $U \times U$. We assume that the operator T defined by K is bounded from $L^2(U)$ to $L^2(U)$. The main theorem of this paper provides conditions on such a distributional kernel K which imply that K differs from the Szegö kernel by a smooth function.

THEOREM 4.2. *Let M and U be as above and suppose that K is a distributional kernel which satisfies the above conditions. Let $\{\psi_j\}_{j=1}^2$ be real-valued functions in $C_0^\infty(U)$ such that $\psi_2 \equiv 1$ in a neighborhood of the support of ψ_1. Suppose that K and the operator T defined by K satisfy*

(1) $\bar\partial_b K \in C^\infty(U \times U)$ *where $\bar\partial_b$ acts on the first variable,*
(2) $K(z, w) - \overline{K(w, z)} \in C^\infty(U \times U)$, *and*
(3) *there exists a number k (either positive or negative) and a constant C such that*

$$\|\psi_1(I - T)(\psi_2 f)\|_{L^2} \leq C\|\bar\partial_b(\psi_2 f)\|_k$$

for any $f \in \mathbf{H}_b$.

Then K differs from the Szegö kernel for M by a smooth function in the interior of the support of ψ_1.

Roughly speaking, Theorem 4.2 says that the "almost conjugate symmetry" property (2) and the "almost reproducing" property (3) almost determine the Szegö kernel as the conjugate symmetry and the reproducing property completely determine the Szegö kernel.

This paper is organized as follows: We first formulate the basic hypothesis of this paper (Section 2) and then prove that the Szegö kernel is smooth off the diagonal (Section 3).. In Section 4, an approximation theorem for the Szegö kernel is proved, and as an easy consequence we show that Szegö kernel can be localized. Section 5 is devoted to Application 2.

One word about notation: constants denoted by C or C_s may differ at each occurrence, but the dependence does not change.

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2. The Basic Hypothesis and Preliminaries

We first formulate the following definition for its repeated use throughout the paper.

DEFINITION 2.1. Let M be any smooth manifold. For $j=1,2,\dots,n$ let $\{U_j\}$ be open sets in M and let $\{\psi_j\}$ be real-valued functions in $C_0^\infty(M)$. We call $\{(U_j,\psi_j)\}$ *nested* in M if

(1) $U_1 \subset\subset U_2 \subset\subset \cdots \subset\subset M$,
(2) $\psi_j \equiv 1$ on U_j for $j=1,2,\dots,n$, and
(3) $\mathrm{supp}(\psi_j) \subset U_{j+1}$ for $j=1,2,\dots,n-1$.

$\{U_j\}$ is called nested in M if (1) holds and $\{\psi_j\}$ is called nested in M if $\psi_{j+1}\equiv 1$ in a neighborhood of the support of ψ_j.

Let M be a compact CR manifold. We denote by $\mathbf{H}_b$ the class of all functions in $L^2(M)$ which are annihilated by the CR operator $\bar\partial_b$ (i.e., $\mathbf{H}_b$ is the space of the square integrable CR functions). We refer to [6] for the definition of $\bar\partial_b$. We also denote by $\|\ \|_s$ a Sobolev norm of orer s on M. Throughout this paper P represents the Szegö projection and S the Szegö kernel.

We now formulate the basic hypothesis of this paper.

BASIC HYPOTHESIS. Let M be a compact CR manifold. We assume that there exists an open set U in M such that, for a pair $\{\psi_1,\psi_2\}$ nested in U and for any number $s>0$, there exists a positive constant C_s such that if $u\in \mathrm{dom}(\bar\partial_b)$ and $u\perp \mathbf{H}_b$, then

$$\|\psi_1 u\|_s \le C_s(\|\psi_2\bar\partial_b u\|_s+\|\bar\partial_b u\|). \tag{2.1}$$

REMARK 2.2. If M is a compact pseudoconvex CR manifold of dimension $2n+1$ on which $\bar\partial_b$ has a closed range, and if for any $x\in U$ there exists a positive integer k such that $1\in I_k^1(x)$ (we refer to [10] for the definitions of I_k^1), then Theorem 2.6 of [8] on the subelliptic estimates for $\bar\partial_b$ guarantees the estimate (2.1). We should mention that the estimate (2.1) is not a subelliptic estimate; that is, the estimate does not require any gain of regularities.

LEMMA 2.3. *Let M be as above. Suppose that the basic hypothesis holds in an open subset U of M. If $\{\psi_1,\psi_2\}$ is nested in U, then for any $s>0$ there exists a positive constant C_s such that*

$$\|\psi_1 Pf\|_s \le C_s(\|\psi_1 f\|_s+\|\psi_2\bar\partial_b f\|_s+\|\bar\partial_b f\|) \tag{2.2}$$

for any $f \in H_s$, where H_s is a Sobolev space of order s on M and P is the Szegö projection for M.

Proof. Since $(I-P)f \perp \mathbf{H}_b$, this lemma follows immediately from (2.1). □

COROLLARY 2.4. $P(C^\infty(U) \cap \operatorname{dom}(\bar\partial_b)) \subset C^\infty(U)$.

3. Smoothness of the Szegö Kernel

Let M be a compact CR manifold and assume that the Basic Hypothesis holds in an open subset U of M. The purpose of this section is to show that the Szegö kernel for M is smooth in $U \times U \setminus \Delta$ where $\Delta = \{(z,z) \mid z \in U\}$. The same result when M is a 3-dimensional pseudoconvex compact CR manifold of finite type has been proved in various papers (e.g., [3] and [11]). We include a proof here for the sake of completeness.

THEOREM 3.1. *Let M, U and Δ be as above. Let S be the Szegö kernel for M. Then*

$$S \in C^\infty(U \times U \setminus \Delta).$$

Proof. Let $\{\phi_j\}_{j=1}^2$ and $\{\psi_j\}_{j=1}^2$ be two nested pairs in U such that

$$\operatorname{supp}(\phi_2) \cap \operatorname{supp}(\psi_2) = \emptyset.$$

Put $T = M_{\psi_1}(I-P)M_{\phi_1}$, where P is the Szegö projection for M and M_ψ is the multiplication operator by ψ. Note that $T = -M_{\psi_1}PM_{\phi_1}$ since

$$\operatorname{supp}(\psi_1) \cap \operatorname{supp}(\phi_1) = \emptyset.$$

Let $f \in H_1$. Since $(I-P)(\phi_1 f) \perp \mathbf{H}_b$, it follows from (2.1) that for each $s > 0$ there exists $C_s > 0$ independent of f such that

$$\|Tf\|_s \le C_s(\|\psi_2 \bar\partial_b(I-P)(\phi_1 f)\|_s + \|\bar\partial_b(I-P)(\phi_1 f)\|).$$

Since

$$\psi_2 \bar\partial_b(I-P)(\phi_1 f) = \psi_2 \bar\partial_b(\phi_1 f) \quad \text{and} \quad \operatorname{supp}(\psi_2) \cap \operatorname{supp}(\phi_1) = \emptyset,$$

$\psi_2 \bar\partial_b(I-P)(\phi_1 f) = 0$ and hence

$$\|Tf\|_s \le C_s \|\bar\partial_b(\phi_1 f)\| \le C_s \|f\|_1.$$

Therefore, T is bounded from H_1 to H_s for any $s > 0$.

On the other hand, the adjoint operator T^* of T is $M_{\phi_1}(I-P)M_{\psi_1}$ and hence, in the same way as above, we can show that T^* is bounded from H_1 to H_s for any $s > 0$. By taking the adjoint of T^* we can see that T is bounded from H_{-s} to H_{-1} for any $s > 0$. It then follows from an interpolation that T is bounded from H_s to H_k for any s and k. So the kernel $-\psi_1(z)S(z,w)\phi_1(w)$ of T belongs to $C^\infty(M \times M)$. Since ϕ_1 and ψ_1 are arbitrary functions in $C^\infty(U)$ such that $\operatorname{supp}(\phi_1) \cap \operatorname{supp}(\psi_1) = \emptyset$, we can conclude that

$$S \in C^\infty(U \times U \setminus \Delta),$$

and the proof is completed. □

Theorem 3.1 and Lemma 2.3 lead us to the following corollary.

COROLLARY 3.2. *Let M and U be as above. If $\{\psi_1, \psi_2\}$ is nested in U, then for any $s \geq 0$ and t there exists a positive constant $C_{s,t}$ such that*

$$\|\psi_1 Pf\|_s \leq C_{s,t}(\|\psi_2 f\|_{s+1} + \|f\|_{-t})$$

for any $f \in H_s$.

In fact, better estimates can be found in [3] and [5]. But for our purpose, Corollary 3.2 is enough.

4. Approximation of the Szegö Kernel

In this section we will derive a theorem on an approximation of the Szegö kernels, and as a corollary we prove a theorem on a localization of Szegö kernels which was introduced at the beginning of Section 1.

LEMMA 4.1. *Let M be a compact CR manifold and U be an open subset of M. Suppose that the Basic Hypothesis holds in U. Let $\{\psi_1, \psi_2\}$ be a nested pair in U. Then for any $s > 0$ there exists a positive constant C_s such that*

$$\|\psi_1(I-P)\psi_2 f\|_s \leq C_s \|\bar{\partial}_b(\psi_2 f)\| \tag{4.1}$$

for any $f \in L^2$ such that $\bar{\partial}_b f \equiv 0$ in U.

Proof. Choose a function ζ in $C_0^\infty(U)$ so that $\{\psi_1, \zeta, \psi_2\}$ is nested in U. Because $(I-P)(\psi_2 f) \perp \mathbf{H}_b$, it follows from (2.1) that

$$\|\psi_1(I-P)\psi_2 f\|_s \leq C_s(\|\zeta\bar{\partial}_b(\psi_2 f)\|_s + \|\bar{\partial}_b(\psi_2 f)\|)$$

for some constant C_s independent of f. Since $\psi_2 \equiv 1$ in a neighborhood of the support of ζ, we have $\zeta\bar{\partial}_b(\psi_2 f) \equiv 0$ if $\bar{\partial}_b f = 0$ in U. Therefore,

$$\|\psi_1(I-P)\psi_2 f\|_s \leq C_s \|\bar{\partial}_b(\psi_2 f)\|.$$

This completes the proof. □

The following is the main theorem of this paper, and is a converse of Lemma 4.1. It shows that if an almost self-adjoint integral operator (property (2)) enjoys the property (4.2) (we called it "almost reproducing property"), then its distributional kernel differs from the Szegö kernel by a smooth function.

THEOREM 4.2. *Let M and U be the same as above. Suppose $\{(U_j, \psi_j)\}_{j=1}^2$ is nested in U. Let K be a distributional kernel such that $K \in C^\infty(U \times U \setminus \Delta)$ where $\Delta = \{(z,z) \mid z \in U\}$. Define an operator T by*

$$Tf(z) = \psi_2(z) \int_M K(z,w)\psi_2(w)f(w)\, d\sigma(w)$$

where $d\sigma$ is the surface area on M. Assume that T and K satisfy the following properties:

(1) $\bar{\partial}_b K \in C^\infty(U_2 \times U_2)$, *where* $\bar{\partial}_b$ *act on the first variable.*
(2) $K(z, w) - \overline{K(z, w)} \in C^\infty(U_2 \times U_2)$.
(3) *Let* $Ef(z) = \psi_1(I - T)f(z)$. *There exist a number k and a positive constant C such that*

$$\|Ef\| \leq C\|\bar{\partial}_b(\psi_2 f)\|_k \tag{4.2}$$

for any $f \in \mathbf{H}_b$.

Then $K - S \in C^\infty(\bar{U}_1 \times \bar{U}_1)$.

REMARK 4.3. (i) Note that (4.2) is a much more relaxed condition than (4.1).

(ii) (4.2) implies that $EPM_{\psi_1} = M_{\psi_1}(I - T)PM_{\psi_1}$ is a smoothing operator. More precisely, for any real number s there exists a constant $C_s > 0$ such that

$$\|EPM_{\psi_1} f\| \leq C_s \|f\|_s.$$

To prove this we note that

(a) the kernel of $\bar{\partial}_b M_{\psi_2} PM_{\psi_1}$ is $(\bar{\partial}_b \psi_2)(z)S(z, w)\psi_1(w)$,
(b) $\psi_2 \equiv 1$ in a neighborhood of the support of ψ_1, and
(c) $\text{supp}(\psi_j) \subset U$.

It then follows from Theorem 3.1 that the kernel of the operator $\bar{\partial}_b M_{\psi_2} PM_{\psi_1}$ is smooth, and hence

$$\|EPM_{\psi_1} f\| \leq C\|\bar{\partial}_b M_{\psi_2} PM_{\psi_1} f\|_k \leq C_s \|f\|_s$$

for any number s. This is an essential observation for the proof of Theorem 4.2.

Proof of Theorem 4.2. Consider the operator $M_{\psi_1} TPM_{\psi_1}$. If $f, g \in C^\infty(M)$, then by (2) and Remark 4.3(ii) we have

$$(M_{\psi_1} TPM_{\psi_1} f, g) = (M_{\psi_1} PM_{\psi_1} f, g) + (E_1 PM_{\psi_1} f, g), \tag{4.3}$$

where

$$\|E_1 PM_{\psi_1} f\| \leq C\|\bar{\partial}_b M_{\psi_2} PM_{\psi_1} f\|_k \leq C_s \|f\|_s$$

for any real number s.

On the other hand,

$$\begin{aligned} (M_{\psi_1} T^* PM_{\psi_1} f, g) &= (f, M_{\psi_1} PTM_{\psi_1} g) \\ &= (f, M_{\psi_1} TM_{\psi_1} g) + (f, E_2 TM_{\psi_1} g) \\ &= (M_{\psi_1} TM_{\psi_1} f, g) + ((E_2 TM_{\psi_1})^* f, g), \end{aligned} \tag{4.4}$$

where $E_2 TM_{\psi_1} = -M_{\psi_1}(I - P)TM_{\psi_1}$. We claim that the operator E_2 satisfies the estimate $\|E_2 TM_{\psi_1} g\|_s \leq C_{s,k} \|g\|_k$ for some constant $C_{s,k}$ for any s and k. In fact, by (2.1), there is a constant C_s such that

$$\|E_2 TM_{\psi_1} g\|_s \leq C_s(\|\psi_2 \bar{\partial}_b TM_{\psi_1} g\|_s + \|\bar{\partial}_b TM_{\psi_1} g\|),$$

and the kernel of the operator $\bar{\partial}_b TM_{\psi_1}$ is

$$(4.5)\qquad (\bar\partial_b\psi_2)(z)K(z,w)\psi_1(w)+\psi_2(z)\bar\partial_b K(z,w)\psi_1(w).$$

Since $\psi_2\equiv 1$ in a neighborhood of the support of ψ_1 and $K\in C^\infty(U\times U\setminus\Delta)$, and since $\bar\partial_b K\in C^\infty(U_2\times U_2)$, the function given in (4.5) is a smooth function. Therefore,

$$\|E_2TM_{\psi_1}g\|_s\le C_{s,k}\|g\|_k$$

for any s and k. In particular, $(E_2TM_{\psi_1})^*$ is a bounded operator from H_s to L^2 for any s. By equating (4.3) and (4.4), we see that

$$\begin{aligned}M_{\psi_1}(P-T)M_{\psi_1}&=M_{\psi_1}(P-T^*)M_{\psi_1}+M_{\psi_1}(T^*-T)M_{\psi_1}\\&=M_{\psi_1}(T^*-T)(I-P)M_{\psi_1}+(E_2TM_{\psi_1})^*-E_1PM_{\psi_1}.\end{aligned}$$

Since the kernel of T^*-T is smooth by hypothesis (2), the operator

$$M_{\psi_1}(I-P)(T^*-T)M_{\psi_1}$$

is bounded from H_s to H_k for any s and k, and hence $M_{\psi_1}(T^*-T)(I-P)M_{\psi_1}$ is bounded from H_s to H_k for any s and k. Therefore,

$$\begin{aligned}\|M_{\psi_1}(P-T)M_{\psi_1}f\|&=\|(E_2TM_{\psi_1})^*f\|+\|(E_1PM_{\psi_1})f\|\\&\quad+\|M_{\psi_1}(T^*-T)(I-P)M_{\psi_1}f\|\\&\le C_s\|f\|_s.\end{aligned}$$

So far, we have proved that the operator $M_{\psi_1}(P-T)M_{\psi_1}$ is bounded from H_s to L^2 for any s. In the same way we can show that $M_{\psi_1}(P-T^*)M_{\psi_1}$ is bounded from H_s to L^2 for any s. It then follows from interpolations that the operator $M_{\psi_1}(P-T)M_{\psi_1}$ is bounded from H_k to H_s for any s and k. So, we can conclude that the kernel $\psi_1(z)(S(z,w)-K(z,w))\psi_1(w)$ of the operator $M_{\psi_1}(P-T)M_{\psi_1}$ belongs to $C^\infty(M\times M)$. Since $\psi_1\equiv 1$ on U_1, the theorem follows. □

COROLLARY 4.3. *Let M_1 and M_2 be two compact CR manifolds such that $M_1\cap M_2\neq\emptyset$. Let U be an open subset of $M_1\cap M_2$ and suppose that the Basic Hypothesis holds in U. If S_j is the Szegö kernel for M_j for $j=1,2$, then*

$$S_1-S_2\in C^\infty(U\times U).$$

Proof. By Lemma 4.1, both S_1 and S_2 satisfy the hypothesis of Theorem 4.3 for any pair $\{\psi_j\}_{j=1}^2$ nested in U. Hence the proof is completed. □

5. Szegö Kernels on Certain Domains in $\mathbf{C}^3$

Let $\Omega=\{z\in\mathbf{C}^3\mid\rho(z)<0\}$ be a bounded domain in $\mathbf{C}^3$ with a smooth boundary $b\Omega$. Suppose also that $0\in b\Omega$ and that

$$(5.1)\qquad \rho(z)=p_1(z_1,\bar z_1)+p_2(z_2,\bar z_2)-\operatorname{Im} z_3$$

for z in some neighborhood O of 0 in $\mathbf{C}^3$, where the functions p_j are subharmonic but not harmonic polynomials. Let

$$\Omega_1=\{(z_1,z_3)\in\mathbf{C}^2\mid\rho(z_1,0,z_3)<0\}\ \text{ and }\ \Omega_2=\{(z_2,z_3)\in\mathbf{C}^2\mid\rho(0,z_2,z_3)<0\}.$$

The purpose of this section is to exploit relationships of the Szegö kernels for Ω, Ω_1, and Ω_2. It turns out that the Szegö kernel for Ω can be written as a kind of convolution (it is a convolution with respect to the $\operatorname{Re} z_3$ variable) of the Szegö kernels for Ω_j because of the "product structure" of $b\Omega \cap O$.

If the domain Ω is globally defined by (5.1), then it has been shown [7] that

$$S(z,t;w,s) = \int_0^\infty e^{-2\pi\lambda A} K_{\lambda p}(z,w)\, d\lambda,$$

where $A = i(s-t) + (p_1(z_1)+p_2(z_2)+p_1(w_1)+p_2(w_2))$ and K_p is the Bergman kernel on $L^2(\mathbf{C}^2)$ weighted by $e^{-p(z)}$. A simple observation shows that

$$K_p(z,w) = K_{p_1}(z_1,w_1)K_{p_2}(z_2,w_2)$$

and hence

$$S(z,t;w,s) = \int_{-\infty}^\infty S_1(z_1,t;w_1,r)S_2(z_2,r;w_2,s)\, dr. \tag{5.2}$$

Also, (5.2) implies that

$$P = P_1P_2. \tag{5.3}$$

We would like to localize this formula.

Let $U = O \cap b\Omega$. Then, on U we can use (z_1, z_2, t) as our coordinate system, and the tangential Cauchy-Riemann operator $\bar\partial_b$ is given by

$$\bar\partial_b f = \sum_{j=1}^2 (\bar L_j f)\, d\bar z_j,$$

where $L_j = \partial_{z_j} + i(\partial_{z_j} p_j)\partial_t$ $(j=1,2)$.

Here and throughout this section we use the following notations: $S(P)$ is the Szegö kernel(projection) for Ω; $S_j(P_j)$ is the Szegö kernel(projection) for Ω_j; and $U_j = O \cap b\Omega_j$ $(j=1,2)$. For convenience, we use Z for (z_1,z_2,t), W for (w_1,w_2,s), and so forth.

Let U_1 be a relatively compact open subset of U and let ϕ and ζ be functions in $C_0^\infty(U)$ such that $\{\zeta, \phi\}$ is nested and $\zeta \equiv 1$ on U_1. Inspired by (5.2) and (5.3), we will consider the following operator:

$$T = \phi P_1 \zeta P_2 \phi. \tag{5.4}$$

If K is the kernel of T, then K is formally given by

$$\begin{aligned} K(Z,W) &= \phi(Z)\phi(W) \\ &\quad \times \int_{-\infty}^\infty \zeta(w_1,z_2,r)S_1(z_1,t;w_1,r)S_2(z_2,r;w_2,s)\, dr. \end{aligned} \tag{5.5}$$

We want to prove the following theorem.

THEOREM 5.1. *Let Ω be the domain as above. Let K be the kernel given in* (5.5) *and let S be the Szegö kernel for Ω. Then*

$$K - S \in C^\infty(U_1 \times U_1).$$

COROLLARY 5.2. *Let T be the operator given in* (5.4) *and let P be the Szegö projection for Ω. Then $P-T$ is an infinitely smoothing operator.*

We prove Theorem 5.1 in the following sequence of lemmas.

LEMMA 5.3. *Let Ω and U be as above. Then $\bar{\partial}_b$ has a closed range, and for any $Z\in U$ there exists a positive integer k such that $1\in I_k^1(Z)$ and hence the Basic Hypothesis holds in U.*

Proof. The first assertion is a special case of Theorem 4.12 of [9] (also see [2]). For the second assertion it is enough (because of [10, Lemma 5.27]) to note that $[L_j,\bar{L}_j]=4(\Delta p_j)\partial_t$ for $j=1,2$ and that $[L_i,\bar{L}_j]=0$ if $i\neq j$. □

LEMMA 5.4. *Let Ω and U be as above and let T and K be as defined in* (5.4) *and* (5.5). *Then T and K satisfy the following:*

(1) $\bar{\partial}_b K\in C^\infty(U_1\times U_1)$, *and*
(2) $K(Z,W)-\overline{K(W,Z)}\in C^\infty(U_1\times U_1)$.

Proof. (2) is trivial since Szegö kernels are conjugate symmetric. If we invoke Theorem 3.1, (1) follows from integrations by parts and the following general lemma. □

LEMMA 5.5. *Let $G=\{z\in \mathbf{C}^{n+1}\mid r(z)<0\}$ be a bounded pseudoconvex domain with a smooth boundary bG. Suppose that $0\in bG$ and that there exists an open neighborhood U of 0 such that, for any $p\in U\cap bG$, there exists an integer k such that $1\in I_k^1(p)$. Suppose that $U\cap bG$ can be represented by $\operatorname{Im} z_{n+1}=\phi(z_1,z_2,\dots,z_n)$ for a smooth function ϕ. Let $(z',t)=(z_1,z_2,\dots,z_n,\operatorname{Re} z_{n+1})\in \mathbf{C}^n\times\mathbf{R}$ be a coordinate system on $U\cap bG$. If $S(z',t;w',s)$ is the Szegö kernel for G, then*

$$\frac{\partial}{\partial t}S(z',t;w',s)=-\frac{\partial}{\partial s}S(z',t;w',s)$$
$$+\textit{smooth function in } (U\cap bG)\times(U\cap bG).$$

Proof. We use the same idea as in the proof of Theorem 4.2. We first note that on $U\cap bG$, $\bar{\partial}_b u=\sum_{j=1}^n(\bar{L}_j u)d\bar{z}_j$ where $L_j=\partial_{z_j}+i(\partial_{z_j}\phi)\partial_t$. Therefore, $\bar{\partial}_b\partial_t f=\partial_t\bar{\partial}_b f=0$ in $U\cap bG$ if $f\in\mathbf{H}_b$. Let $\{\psi_1,\psi_2\}$ be a nested pair in $U\cap bG$. Let us consider the operator $T=M_{\psi_2}P\partial_t M_{\psi_2}$, where P is the Szegö projection for G. By Lemma 4.1,

$$\begin{aligned}M_{\psi_1}TPM_{\psi_1}&=M_{\psi_1}P\partial_t M_{\psi_2}PM_{\psi_1}\\&=M_{\psi_1}\partial_t M_{\psi_2}PM_{\psi_1}+E_1,\end{aligned}\tag{5.6}$$

where E_1 satisfies an estimate

$$\|E_1 f\|_s\le C_s\|\bar{\partial}_b(\partial_t\psi_2P(\psi_1 f))\|\le C_{s,k}\|f\|_k$$

for any $s>0$ and k (either positive or negative) and for some constants C_s and $C_{s,k}$ independent of f.

On the other hand, by Lemma 4.1 again,

$$(5.7)\qquad \begin{aligned}(M_{\psi_1}TPM_{\psi_1})^* &= M_{\psi_1}PT^*M_{\psi_1} = M_{\psi_1}PM_{\psi_2}\partial_t PM_{\psi_1}\\ &= M_{\psi_1}\partial_t PM_{\psi_1} + E_2,\end{aligned}$$

where E_2 satisfies an estimate

$$\|E_2 f\| \le C_s \|\bar{\partial}_b(\psi_2(\partial_t P(\psi_1 f)))\| \le C_{s,k}\|f\|_k.$$

By equating (5.6) and (5.7), we have

$$(5.8)\qquad M_{\psi_1}\partial_t M_{\psi_2}PM_{\psi_1} - M_{\psi_1}P\partial_t M_{\psi_1} = E_2^* - E_1,$$

and $E_2^* - E_1$ is bounded from H_s to H_k for any s and k. Therefore, the kernel of the operator in (5.7) must be smooth. Note that

$$\psi_1(z',t)\partial_t(\psi_2(z',t)S(z',t;w',s))\psi_1(w',s) + \psi_1(z',t)\partial_s S(z',t;w',s)\psi_1(w',s)$$

is the kernel. This completes the proof. □

LEMMA 5.6. *The operator T given in* (5.4) *has a kernel which is smooth off the diagonal; that is, K given in* (5.5) *is smooth off the diagonal.*

Proof. Let $Z_1 = (z_1^1, z_2^1, t^1)$ and $Z_2 = (z_1^2, z_2^2, t^2)$ be two different points in U_1. We first assume that $(z_1^1, t^1) \neq (z_1^2, t^2)$. Let π be the projection from $\mathbf{C}^2 \times \mathbf{R} \to \mathbf{C}^1 \times \mathbf{R}$ defined by $\pi(z_1, z_2, t) = (z_1, t)$. Choose functions η, η', η'', and ψ in $C_0^\infty(U_1)$ so that $\{\eta, \eta', \eta''\}$ is nested and $\pi(\mathrm{supp}(\eta'')) \cap \pi(\mathrm{supp}(\psi)) = \emptyset$. We will prove that for any integer s there exists a constant C_s such that

$$\|\eta T\psi f\|_s \le C_s\|f\|.$$

Then the rest follows from the duality and interpolations as before. Since $\|\partial_{z_2} f\| = \|\bar{\partial}_{z_2} f\|$ if f is compactly supported,

$$\|\eta T f\|_s^2 \le C_s \sum_{j=0}^{s} \int_{z_2} \|\bar{\partial}_{z_2}^j \eta P_1 \zeta P_2 \psi f(\cdot, z_2, \cdot)\|_{s-j}^2 \, dm(z_2).$$

And, since $\bar{\partial}_{z_2}^j P_1 = P_1 \bar{\partial}_{z_2}^j$,

$$\begin{aligned}\|\eta T f\|_s^2 &\le C_s \sum_{j=0}^{s} \int_{z_2} \|\eta' P_1 \bar{\partial}_{z_2}^j \zeta P_2 \psi f(\cdot, z_2, \cdot)\|_{s-j}^2 \, dm(z_2)\\ &\le C_s \sum_{j=0}^{s} \left(\int_{z_2} \|\eta' P_1 \zeta \bar{\partial}_{z_2}^j P_2 \psi f(\cdot, z_2, \cdot)\|_{s-j}^2 \, dm(z_2) + \|f\|^2\right),\end{aligned}$$

since $\zeta \equiv 1$ on U_1.

Note that $\bar{\partial}_{z_2} = \bar{L}_2 + i(\bar{\partial}_{z_2} p_2)(z_2)\partial_t$ and $\partial_t \bar{L}_2 = \bar{L}_2 \partial_t$. Therefore, because $\bar{L}_2 P_2 = 0$, we have

$$\|\eta T f\|_s^2 \le C_s \sum_{j=0}^{s} \left(\int_{z_2} \|\eta' P_1 \zeta \phi_j \partial_t^j P_2 \psi f(\cdot, z_2, \cdot)\|_{s-j}^2 \, dm(z_2) + \|f\|^2\right)$$

for some smooth functions ϕ_j. It then follows from Corollary 3.2 that

$$\begin{aligned}\|\eta T f\|_s^2 &\le C_s\Big(\sum_{j=0}^{s}\int_{z_2}\|\eta''\partial_t^j P_2\psi f(\cdot,z_2,\cdot)\|_{s-j+1}^2\,dm(z_2)\\ &\qquad+\sum_{j=0}^{s}\int_{z_2}\|\zeta\partial_t^j P_2\psi f(\cdot,z_2,\cdot)\|_{-j}^2\,dm(z_2)+\|f\|^2\Big)\\ &\le C_s\|f\|^2\end{aligned}$$

since the kernel of $\eta''\partial_t^j P_2\psi$ is smooth.

If $(z_2^1, t^1)\neq(z_2^2, t^2)$, then we note that $T^*=\phi P_2\zeta P_1\phi$ and apply the same argument to T^*. This completes the proof. □

In order to prove Theorem 5.1, it is now enough to show that the operator T has an almost reproducing property.

LEMMA 5.7. *Let T be the operator defined in* (5.4). *If ψ is a smooth function supported in U_1, then there exists a constant C such that*

$$\|\psi(I-T)f\|\le C\|\bar\partial_b(\phi f)\|$$

for any $f\in\mathbf{H}_p$.

Proof. Note that

$$\psi(I-P)f=\psi(I-P_1)\phi f+\psi P_1\zeta(I-P_2)\phi f.$$

Therefore, it follows from Lemma 4.1 that

$$\begin{aligned}\|\psi(I-P)f\| &\le \|\psi(I-P_1)\phi f\|+\|\psi P_1\zeta(I-P_2)\phi f\|\\ &\le C(\|\bar L_1(\phi f)\|+\|\zeta(I-P_2)\phi f\|)\\ &\le C(\|\bar L_1(\phi f)\|+\|\bar L_2(\phi f)\|).\end{aligned}$$

The proof is completed. □

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