A Monotone Slit Mapping with Large Logarithmic Derivative

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1. Introduction

We denote the unit disc in the complex plane, that is $\{z : |z| < 1\}$, by Δ. The class $\mathcal{S}$ is the class of all functions $f(z)$ which are analytic and univalent in Δ and for which $f(0)=0$ and $f'(0)=1$. If $f(z)$ is in $\mathcal{S}$ we set

$$I_2\left(r, \frac{f'}{f}\right) = \int_0^{2\pi} \left| \frac{rf'(re^{i\theta})}{f(re^{i\theta})} \right|^2 d\theta,$$

for each r with $0<r<1$. By a monotone slit mapping we mean a function $f(z)$ in $\mathcal{S}$ whose image domain is the complement of a path $\Gamma(t)$ on $[0,\infty)$ for which $|\Gamma(t_1)| < |\Gamma(t_2)|$ if $t_1 < t_2$. That is, Γ meets each circle centred on the origin at most once.

In this note we prove the following.

THEOREM 1. *There is a monotone slit mapping $\mathfrak{F}(z)$ for which*

$$I_2\left(r, \frac{\mathfrak{F}'}{\mathfrak{F}}\right) \neq o\left(\frac{1}{1-r} \log\log \frac{1}{1-r}\right) \tag{1.1}$$

as $r \to 1$.

A standard way to obtain information on logarithmic coefficients of a function f in $\mathcal{S}$ is to estimate $I_2(r, f'/f)$ (cf. [7]). The logarithmic coefficients play an essential role, for example in the proof of the Bieberbach conjecture by de Branges [5].

Our starting point is an estimate of Biernacki in [4] (see also [11, p. 151]) that if f is a function in $\mathcal{S}$ then, as $r \to 1$,

$$I_2\left(r, \frac{f'}{f}\right) = O\left(\frac{1}{1-r} \log \frac{1}{1-r}\right).$$

It is surprising, but nevertheless true, that this elementary bound is best possible. Hayman produced in [11] an example of a function $f(z)$ in $\mathcal{S}$ for which

$$I_2\left(r, \frac{f'}{f}\right) \neq o\left(\frac{1}{1-r} \log \frac{1}{1-r}\right).$$

Our construction borrows much from the methods he employed there.

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For the Koebe function, $k(z)=z/(1-z)^2$, we find that

$$I_2\left(r, \frac{k'}{k}\right) \sim \frac{1}{1-r}$$

as $r \to 1$. The function $k(z)$ solves the most famous problems in the class $\mathcal{S}$, but the solutions to more general extremal problems are found in the broader class of monotone slit mappings [6, Chap. 9].

Baernstein and Brown [3] introduced, for $0 \leq \lambda < \pi/2$, the class of monotone slit mappings $\mathfrak{M}(\lambda)$. A function $f(z)$ in $\mathcal{S}$ is said to be in the class $\mathfrak{M}(\lambda)$ if the image domain of f is the complement of a path $\Gamma(t)$, parameterised on $[0, \infty)$, such that

$$\limsup_{t \to t_1} \left| \arg \frac{\Gamma(t)-\Gamma(t_1)}{\Gamma(t_1)} \right| \leq \lambda$$

for every t_1 in $[0, \infty)$. If $f \in \mathfrak{M}(\lambda)$ then f is a monotone slit mapping. The classes $\mathfrak{M}(\lambda)$ generalise the class of support points of $\mathcal{S}$ which corresponds to $\mathfrak{M}(\pi/4)$. Extreme points of $\mathcal{S}$ are monotone slit mappings, but it is not known whether or not they belong to $\mathfrak{M}(\lambda)$ for some $\lambda < \pi/2$. Baernstein and Brown note that a special case of their results in [3] is the following.

THEOREM A. *For each λ with $0 \leq \lambda < \pi/2$, there is a constant $C(\lambda)$ depending only on λ such that if $f(z)$ is in $\mathfrak{M}(\lambda)$ then*

$$I_2\left(r, \frac{f'}{f}\right) \leq \frac{C(\lambda)}{1-r}$$

for $0<r<1$.

Theorem 1 shows that Theorem A fails in the limiting case $\lambda = \pi/2$. The existence of the mapping in Theorem 1 was suggested by Hayman (see the remark in [7, p. 38]).

It is interesting to note that the condition

$$I_2\left(r, \frac{f'}{f}\right) = O\left(\frac{1}{1-r}\right) \tag{1.2}$$

is a Tauberian condition. Halász shows in the proof of Theorem 4′ in [8] that if $f(z)=\sum a_n z^n$ is univalent in the disc Δ, if (1.2) holds, and if the necessary conditions that $a_n \to 0$ as $n \to \infty$ and that f has a radial limit at 1 apply, then $\sum a_n$ converges. In a paper [10] published shortly afterwards, Hayman constructs in Theorem 7 a univalent function $f(z)=\sum a_n z^n$ in Δ (which therefore has radial limits a.e.) whose coefficients a_n tend to 0 but for which $\sum a_n e^{in\theta}$ diverges for every θ. Thus, by Halász' result, (1.2) does not hold for this function. I am grateful to the referee for pointing out the relevance of the articles [8] and [10].

I wish to thank Professor W. K. Hayman for suggesting this problem and to thank him and Dr. P. J. Rippon for their valuable assistance in the preparation of this paper. My thanks also go to Professor D. Drasin for carefully reading this manuscript and for his many helpful suggestions. I wish to thank The Open University, England, for financial support.

2. A Key Lemma

Theorem 1 is, as we shall see, a direct consequence of the lemma which is to follow.

LEMMA 1. *There is a univalent function $\mathcal{G}(z)$ in Δ such that*

(A) $\mathcal{G}(z): \Delta \to \mathcal{H}$, *where*

$$\mathcal{H} = \{w = u + iv : \operatorname{Im} \gamma(u) < v < \operatorname{Im} \gamma(u) + 2\pi\} \tag{2.1}$$

and $\gamma(t)$ is some curve on $\mathbf{R}$ *for which* $\operatorname{Re} \gamma(t)$ *is strictly increasing; and*

(B) *there are sequences r_k and r_k' which tend to* 1 *with $r_k < r_k'$ and $(1-r_k)/(1-r_k')$ bounded, such that*

$$A(r_k, r_k') = \int_{r_k}^{r_k'} \int_0^{2\pi} |\mathcal{G}'(re^{i\theta})|^2 r \, d\theta \, dr > C \log\log \frac{1}{1-r_k'}, \tag{2.2}$$

where C is an absolute constant.

Note that $A(r_k, r_k')$ is the Euclidean area of the image under $\mathcal{G}(z)$ of the annulus $\{z : r_k < |z| < r_k'\}$.

To see how Theorem 1 follows from Lemma 1 we first note that from (A) it follows that $\exp(\mathcal{G}(z))$ maps Δ onto the complement of a path $\Gamma(t) = \exp(\gamma(t))$, where $|\Gamma(t_1)| < |\Gamma(t_2)|$ if $t_1 < t_2$. If we normalise $\exp(\mathcal{G}(z))$ so that it lies in $\mathcal{S}$, then the normalised mapping $\mathcal{F}(z)$ lies in $\mathfrak{M}(\pi/2)$.

To check that this $\mathcal{F}(z)$ satisfies (1.1) we note that, since $I_2(r, \mathcal{G}')$ is an increasing function of r,

$$\begin{aligned} I_2(r_k', \mathcal{G}') &\geq \frac{2}{r_k'^2 - r_k^2} \int_{r_k}^{r_k'} I_2(r, \mathcal{G}') r \, dr \\ &= \frac{2}{r_k'^2 - r_k^2} A(r_k, r_k') \\ &> \frac{2C}{r_k'^2 - r_k^2} \log\log \frac{1}{1-r_k'} \\ &\geq \frac{C_0}{1-r_k'} \log\log \frac{1}{1-r_k'}. \end{aligned} \tag{2.3}$$

The last inequality holds because $(1-r_k)/(1-r_k')$ is bounded. We now note that

$$I_2\left(r_k', \frac{\mathcal{F}'}{\mathcal{F}}\right) \sim I_2(r_k', \mathcal{G}'). \tag{2.4}$$

We obtain (1.1) as a consequence of (2.3) and (2.4).

The proof of Lemma 1 is long and comes in two stages. First of all, an intermediate mapping $G(z)$ from the unit disc to a symmetric domain $\mathfrak{D}$ is obtained which satisfies (2.2). The domain $\mathfrak{D}$ is then changed to a domain $\mathcal{H}$ so that (2.2) remains valid for $\mathcal{G}(z): \Delta \mapsto \mathcal{H}$ and so that, in addition,

(A) holds. It is more convenient to first make the necessary estimates in the symmetric domain $\mathfrak{D}$ and then show that they remain valid in an admissible domain $\mathfrak{H}$ which is similar to $\mathfrak{D}$.

3. Estimates of Hyperbolic Distance

Throughout this note, $\{a_n\}_1^\infty$ is an unbounded, increasing sequence of points on the positive axis.

DEFINITION 1. We define a domain $D=D(\{a_n\})$ by

$D=\{z: |\operatorname{Im} z|<\pi$, unless $\operatorname{Re} z=a_n$ for some n, in which case $|\operatorname{Im} z|<\pi/2\}$.

We write $2d_n=a_{n+1}-a_n$. Lastly, for each n we define

$$B_n^+=\{z: a_n<\operatorname{Re} z<a_{n+1},\ \pi/2<\operatorname{Im} z<\pi\}$$

and

$$B_n^-=\{z: a_n<\operatorname{Re} z<a_{n+1},\ -\pi<\operatorname{Im} z<-\pi/2\}.$$

The domain D resembles Hayman's domain in [11] except that, since a monotone slit mapping is required here, we cannot have boxes but only channels. (See Figure 1.)

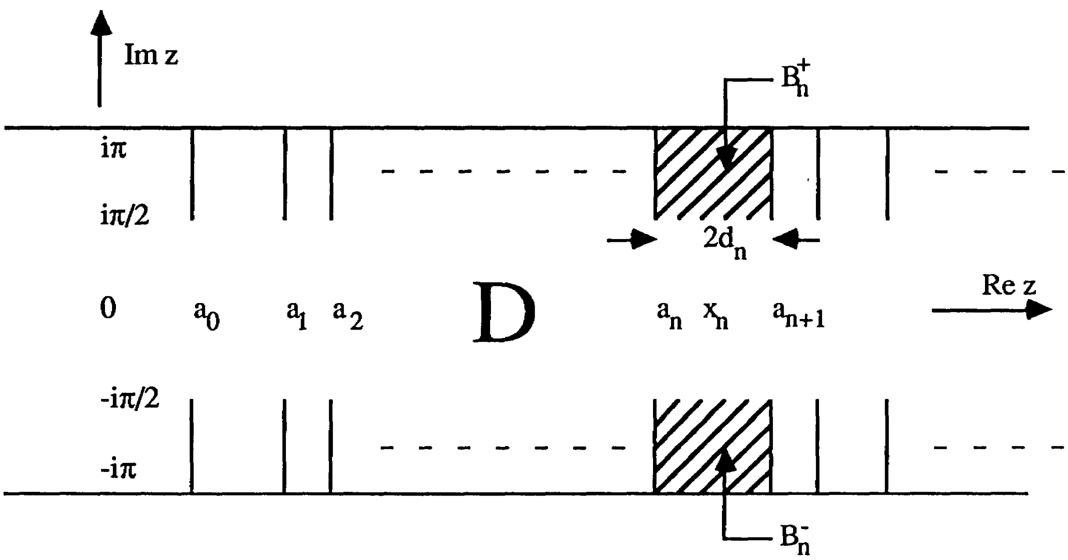

Figure 1

In the next section, a specific choice of the widths d_n is made so that, for this choice, the conformal mapping $G(z)$ from the unit disc to D satisfies (2.2). In this section we obtain the estimates of hyperbolic distance that are needed to accomplish this.

We denote the hyperbolic or Poincaré distance between two points z_1 and z_2 of a simply connected domain Ω by $d(z_1, z_2; \Omega)$ or by $d(z_1, z_2)$ if this is unambiguous. It is given by the formula

$$d(z_1,z_2;\Omega)=\frac{1}{2}\log\frac{1+|f(z_2)|}{1-|f(z_2)|}$$

where $f(z)$ is a conformal mapping, unique up to rotations, which maps Ω to the unit disc Δ and sends z_1 to 0. Hyperbolic distance is a conformally invariant metric on the simply connected domain Ω. We shall also need a relationship between hyperbolic distance and the Green's function. We denote by $g(z_1,z_2;\Omega)$ the value at z_2 of the Green's function for Ω with pole at z_1. Then, for z_1 and z_2 in the simply connected domain Ω, we have

$$d(z_1,z_2;\Omega)=\frac{1}{2}\log\frac{1+\exp(-g(z_1,z_2;\Omega))}{1-\exp(-g(z_1,z_2;\Omega))}. \tag{3.1}$$

This equality may be verified directly when Ω is the unit disc Δ and follows for general Ω by the conformal invariance of both the hyperbolic metric and the Green's function. A useful reference to these matters is [1].

3.1. DISTANCE ALONG THE REAL AXIS. We will need the following lemma which is a special case of a result on Steiner symmetrisation [12, Chap. 5]. In this special case we give an elementary proof which we learned from P. J. Rippon. A domain D is said to be convex with respect to the imaginary axis if whenever $x+iy_1$ and $x+iy_2$ lie in D then so does the line segment joining them.

LEMMA 2. *Let D be a simply connected domain containing 0 which is symmetric about the real axis and convex with respect to the imaginary axis. Denote by $g(0,z;D)$ the Green's function for D with pole at 0. Then for each real x_0 in D,*

$$g(0,x_0;D)=\max\{g(0,x_0+iy;D):x_0+iy\in D\}.$$

Proof. Suppose that x_0 and x_0+iy_0, where y_0 is positive, are in D. We wish to show that $g(0,x_0;D)\geq g(0,x_0+iy_0;D)$. Write Ω for that component of $D\cap\{z:\operatorname{Im}z>y_0/2\}$ containing x_0+iy_0. Define the function $u(t+i\tau)$ on Ω by

$$u(t+i\tau)=g(0,t+i(y_0-\tau);D)-g(0,t+i\tau;D).$$

Then u is well defined because of the assumptions on D and, moreover, is superharmonic in Ω and nonnegative on the finite boundary of Ω. Because $u(t+i\tau)\to 0$ as $|t+i\tau|\to\infty$ if Ω is unbounded, the maximum principle yields that $u(t+i\tau)$ is positive throughout Ω. In particular, $u(x_0+iy_0)>0$ which gives

$$g(0,x_0;D)\geq g(0,x_0+iy_0;D).$$

The case of $y_0<0$ follows since $g(0,z;D)=g(0,\bar{z};D)$ for z in D, and this completes the proof of Lemma 2. □

We also need a theorem from [9] which we now state.

Suppose that ω_n $(n=0,1,2,\dots)$ is a sequence of complex numbers for which $|\omega_n|=r_n$ is strictly increasing and unbounded and for which

$$\omega_0=0, \quad \omega_1=-1.$$

We write δ_n for $\log(r_{n+1}/r_n)$ if $n\geq 1$, and write ϵ_n for $\min(\delta_n,\delta_n^2)$. Then Theorem 1 of [9] runs as follows.

THEOREM B. *If $f(\zeta)$ is regular in Δ, if $|f(0)|\leq 1$, and if f never assumes the values ω_n, then for $r_n<M(\rho,f)\leq r_{n+1}$ we have*

$$\log M(\rho,f)<2\log\frac{1+\rho}{1-\rho}+\sum_1^n\epsilon_i+30.$$

Here $M(\rho,f)=\max\{|f(\rho e^{i\theta})|: 0\leq\theta<2\pi\}$.

LEMMA 3. *Suppose that a and b are real with $0\leq a<b$. We let n and m be the positive integers for which $a_n\leq a<a_{n+1}$ and $a_m<b\leq a_{m+1}$. Suppose, in addition, that $a_{n+1}-a\leq 1$. Then*

$$d(a,b;D)>\frac{b-a}{2}-\frac{1}{4}\sum_n^m e_i-8, \tag{3.2}$$

where $e_i=\min(4d_i,16d_i^2)$ and D is the domain in Definition 1.

Proof. We first estimate $d(0,x;D)$, where x is positive. Let $h(\zeta)$ be the conformal mapping from Δ to D for which $h(0)=0$ and $h'(0)$ is positive. We choose ρ, with $0<\rho<1$, so that $h(\rho)=x$. Suppose that ζ is in Δ and that $|\zeta|=\rho$. Then we have that $\operatorname{Re}h(\zeta)\leq x$. For otherwise we would have, by Lemma 2 and the fact that the Green's function for D is strictly decreasing on $[0,\infty)$, that

$$g(0,h(\zeta);D)\leq g(0,\operatorname{Re}h(\zeta);D)<g(0,x;D)$$

which contradicts the assumption that ρ and ζ lie on a level line for the Green's function. If we now define

$$f(\zeta)=e^{2(h(\zeta)-a_1)}$$

then

$$M(\rho,f)=e^{2(x-a_1)}.$$

Thus, on setting $r_n=\exp(2(a_n-a_1))$ for $n>1$, we have that $r_n<M(\rho,f)\leq r_{n+1}$ if and only if $a_n<x\leq a_{n+1}$.

Next we note that $h(\zeta)$ certainly omits all points $a_n+i(\pi/2+k\pi)$ with $n=1,2,\dots$ and k in $\mathbf{Z}$. Hence $f(\zeta)$ omits ω_n $(n=0,1,\dots)$, where $\omega_0=0$, $\omega_1=-1$, and $\omega_n=-e^{2(a_n-a_1)}$ for $n>1$. Moreover, $|f(0)|=e^{-2a_1}\leq 1$ since $a_1\geq 0$. Thus $f(\zeta)$ satisfies the hypotheses of Theorem B.

Also, for $n\geq 1$,

$$\delta_n=\log\left|\frac{\omega_{n+1}}{\omega_n}\right|=2(a_{n+1}-a_n)=4d_n.$$

Theorem B yields

$$2x-2a_1<2\log\frac{1+\rho}{1-\rho}+\sum_1^n e_i+30,$$

and so, if $a_1\le 1$,

$$d(0,x;D)=\frac{1}{2}\log\frac{1+\rho}{1-\rho}>\frac{x}{2}-\frac{1}{4}\sum_1^n e_i-8. \tag{3.3}$$

To obtain the more general (3.2) we need only note that

$$d(a,b;D)=d(0,b-a;D-a),$$

where $D-a$ is the domain D translated by a to the left, and apply (3.3). □

The upper bound is easier.

LEMMA 4. *If* $0\le a<b$, *then*

$$d(a,b;D)\le\frac{b-a}{2}. \tag{3.4}$$

Proof. Since D contains the strip S,

$$S=\{z:|\operatorname{Im} z|<\pi/2\},$$

it follows that

$$d(a,b;D)\le d(a,b;S)=\frac{b-a}{2}.$$

Here we have used the basic fact that if D_1 and D_2 are simply connected domains for which $D_1\subset D_2$, and if z_1 and z_2 are in D_1, then $d(z_1,z_2;D_2)\le d(z_1,z_2;D_1)$. This is easily proved using Schwarz's lemma. □

3.2. DISTANCE IN A CHANNEL. The following lemma, which is Lemma 6 in [11], will be useful.

LEMMA A. *Suppose that* D_0 *is a simply connected domain containing the rectangle*

$$R_0=\{s:|\operatorname{Re} s|<d,\ -\tau_0-d<\operatorname{Im} s<\tau_0+d\},$$

where d *and* τ_0 *are positive. Then*

$$d(-i\tau_0,i\tau_0;D_0)\le\frac{\pi}{2d}\tau_0+\frac{\pi}{2}. \tag{3.5}$$

Suppose further that E *is a subset of length* l *of the interval* $[-\tau_0,\tau_0]$ *such that if* $\tau\notin E$ *then both* $d+i\tau$ *and* $-d+i\tau$ *are in the complement of* D_0. *If* $|\sigma_1|<d$ *and* $|\sigma_2|<d$ *then*

$$d(\sigma_1-i\tau_0,\sigma_2+i\tau_0;D_0)\ge\frac{\pi}{2d}\left(\tau_0-\frac{l}{2}\right)-\frac{\pi}{2}. \tag{3.6}$$

We write $x_n=(a_{n+1}+a_n)/2$. We then have the following lemma.

LEMMA 5. *Suppose that* $d_n < \pi/4$ *and that* $s = x_n + i(\pi/2 + \rho)$, *where* $d_n \le \rho \le \pi/2 - d_n$. *Then*

$$d(x_n, s; D) \le \frac{\pi}{4}\left(\frac{\rho}{d_n}\right) + \frac{1}{2}\log\frac{1}{d_n} + K_1, \tag{3.7}$$

where $K_1 = \frac{1}{2}\log 2\pi + \pi/4 + \log 7$, *and*

$$d(x_n, s; D) \ge \frac{\pi}{4}\left(\frac{\rho}{d_n}\right) - \frac{3\pi}{4}. \tag{3.8}$$

Furthermore, if x is the point on the real axis closest to s in the hyperbolic metric on D, we have

$$d(x_n, s; D) - d(x, s; D) \le \frac{1}{2}\log\frac{1}{d_n} + K_2, \tag{3.9}$$

where $K_2 = \frac{1}{2}\log 2\pi + \pi + \log 7$.

Proof. Let s_n^+, s_n^- be the points $x_n + i(\pi/2 + d_n)$ and $x_n + i(\pi - d_n)/2$, respectively. By the triangle inequality,

$$d(x_n, s; D) \le d(x_n, s_n^-; D) + d(s_n^-, s_n^+; D) + d(s_n^+, s; D). \tag{3.10}$$

Let B_1 be the disc $|z - x_n| < \pi/2$. Then

$$\begin{aligned} d(x_n, s_n^-; D) &\le d(x_n, s_n^-; B_1) \\ &= \frac{1}{2}\log\frac{2\pi - d_n}{d_n} \\ &< \frac{1}{2}\log\frac{1}{d_n} + \frac{1}{2}\log 2\pi. \end{aligned} \tag{3.11}$$

Here we have again used the fact that hyperbolic distance is larger in a smaller domain. Let B_2 be the disc $|z - x_n - i(\pi/2 + d_n/4)| < d_n$. Then

$$\begin{aligned} d(s_n^-, s_n^+; D) &\le d(s_n^-, s_n^+; B_2) \\ &= \log 7. \end{aligned} \tag{3.12}$$

Since D contains the rectangle

$$R = \{z : a_n < \operatorname{Re} z < a_{n+1} \text{ and } \pi/2 < \operatorname{Im} z < \pi/2 + \rho + d_n\},$$

it follows from Lemma A that

$$d(s_n^+, s; D) \le \frac{\pi}{4}\left(\frac{\rho}{d_n}\right) + \frac{\pi}{4}. \tag{3.13}$$

The inequality (3.10) and the estimates (3.11), (3.12), and (3.13) together yield (3.7) with the stated value of K_1.

The vertical sides of R are both part of the boundary of D so that, in this case, E is the empty set in Lemma A and (3.6) gives that

$$d(s_n^+ + t, s; D) \ge \frac{\pi}{4}\left(\frac{\rho}{d_n}\right) - \frac{3\pi}{4} \tag{3.14}$$

for $-d_n < t < d_n$.

We now consider (3.9). Let γ_n be the geodesic from s to that point x of the real axis closest to s. Let Q_n be the point where γ_n meets the line $\operatorname{Im} z = \pi/2 + d_n$ on its way from s. Because γ_n is a geodesic, $d(x,s;D) = l(\gamma_n) > d(s,Q_n;D)$ where $l(\gamma_n)$ is the hyperbolic length of γ_n.

By (3.14),

$$d(Q_n,s;D) \geq \frac{\pi}{4}\left(\frac{\rho}{d_n}\right) - \frac{3\pi}{4}. \tag{3.15}$$

So we see by (3.7) and (3.15) that (3.9) holds.

Moreover, since $d(x_n, s; D) \geq l(\gamma_n)$, (3.8) follows from (3.15) and the proof of Lemma 5 is complete. □

4. The Symmetric Intermediate Domain

The next task is to make a specific choice of the numbers a_n in Definition 1, or equivalently of the widths d_n, in order to construct the symmetric intermediate domain.

We suppose that $a_1 = 0$ and that $2d_1 = 1$. For each $k > 1$, we define

$$2d_n = \frac{1}{2\sqrt{2^k - n}} \tag{4.1}$$

when $2^{k-1} < n < 2^k$ and, for $k \geq 1$, we define

$$2d_{2^k} = k+1. \tag{4.2}$$

This defines the intermediate domain $\mathfrak{D}$ in accordance with Definition 1. (See Figure 2.)

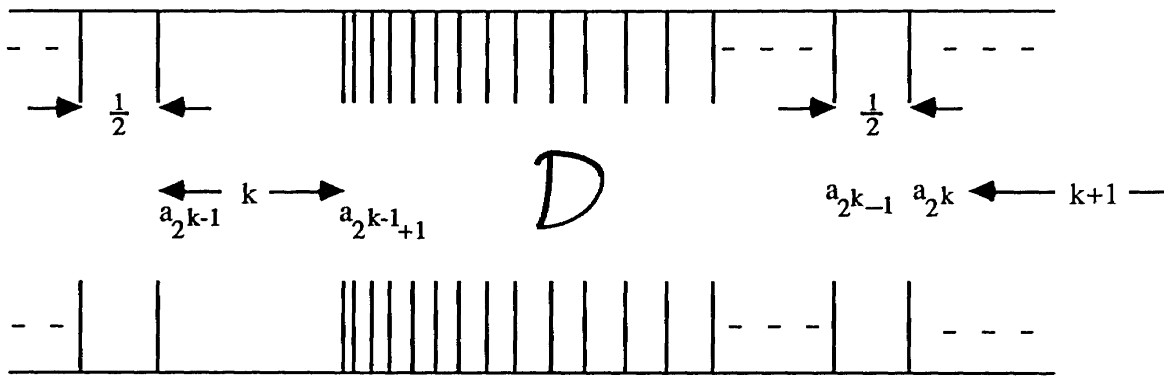

Figure 2

We need the following lemma [11, Lemma 4], which yields an estimate for the error in the triangle inequality for hyperbolic distance in $\mathfrak{D}$.

LEMMA B. *Suppose that D is a simply connected domain with a line of symmetry L. Suppose that w_1 is a point of D not in L and that w_2 and w_3 are points of D on L. We let δ be the hyperbolic distance of w_1 from L with respect to D and we put*

$$\alpha = d(w_1, w_2) - \delta.$$

Then

$$d(w_1, w_3) \geq d(w_1, w_2) + d(w_2, w_3) - 2\alpha - \log 2.$$

We now prove the following.

LEMMA 6. *There are absolute constants C_1 and C_2 and a sequence of open subsets $\{\Omega_k\}_1^\infty$ of $\mathfrak{D}$, together with an unbounded, increasing sequence $\{\lambda_k\}$, such that*

(i) *$\lambda_k - C_1 < d(0, z; \mathfrak{D}) < \lambda_k + C_1$ whenever z is in Ω_k if k is sufficiently large; and*
(ii) *the Euclidean area of Ω_k exceeds $C_2 \log \lambda_k$.*

Proof. We shall show that there is a positive integer K_0 such that for each fixed, large k, it is possible to choose a point s_n from each channel B_n^+ so that the points s_n in the range $2^{k-1} < n < 2^k - K_0$ all lie at the same hyperbolic distance λ_k from the origin. If $s_n = x_n + i(\pi/2 + \rho_n)$ then the ρ_n are monotonically decreasing for n in this range from about $\pi/2$ to about 0. Then Ω_k consists of small discs (centred on s_n) in the channels B_n^+, and the estimate for the area of Ω_n follows easily.

Let s be a point $x_n + i(\pi/2 + \rho)$ in the nth channel B_n^+, where $d_n < \rho < \pi/2 - d_n$ and $2^{k-1} < n < 2^k$. By the triangle inequality,

$$\begin{aligned} d(0,s) &\leq d(0,x_n) + d(x_n, s) \\ &= d(0,x_n) + d(x_n, x_{2^k-1}) + d(x_n, s) - d(x_n, x_{2^k-1}) \\ &= d(0, x_{2^k-1}) + d(x_n, s) - d(x_n, x_{2^k-1}). \end{aligned}$$

Or,

$$d(0,s) - d(0, x_{2^k-1}) \leq d(x_n, s) - d(x_n, x_{2^k-1}). \tag{4.3}$$

In the other direction, Lemma B gives that

$$d(0,s) \geq d(0,x_n) + d(x_n,s) - 2\alpha - \log 2,$$

where α is the difference between the hyperbolic distance from s to the point x_n and the minimum distance from s to the real axis. It follows from (3.9) that

$$\alpha \leq \frac{1}{2} \log \frac{1}{d_n} + K_2.$$

Thus,

$$\begin{aligned} d(0,s) &\geq d(0,x_n) + d(x_n,s) - \log \frac{1}{d_n} - 2K_2 - \log 2 \\ &= d(0, x_{2^k-1}) + d(x_n, s) - d(x_n, x_{2^k-1}) - \log \frac{1}{d_n} - K_3, \end{aligned}$$

where $K_3 = 2K_2 + \log 2$. Thus,

$$d(0,s) - d(0, x_{2^k-1}) \geq d(x_n, s) - d(x_n, x_{2^k-1}) - \log \frac{1}{d_n} - K_3. \tag{4.4}$$

We define $\lambda(\rho)$, for $d_n < \rho < \pi/2 - d_n$, by

$$\lambda(\rho) = d(0, s) - d(0, x_{2^k-1}) = d(0, x_n + i(\pi/2 + \rho)) - d(0, x_{2^k-1}).$$

If $\lambda(d_n)$ is negative and $\lambda(\pi/2 - d_n)$ is positive then, by continuity, $\lambda(\rho_n) = 0$ for some ρ_n in $(d_n, \pi/2 - d_n)$.

It follows from (4.3) that $\lambda(d_n)$ is negative if

$$d(x_n, s) - d(x_n, x_{2^k-1}) < 0 \tag{4.5}$$

when $s = x_n + i(\pi/2 + d_n)$, and from (4.4) that $\lambda(\pi/2 - d_n)$ is positive if

$$d(x_n, s) - d(x_n, x_{2^k-1}) - \log \frac{1}{d_n} - K_3 > 0 \tag{4.6}$$

when $s = x_n + i(\pi - d_n)$. The estimates (3.2) and (3.7) show that (4.5) holds if

$$\left[\frac{\pi}{4}\left(\frac{\rho}{d_n}\right) + \frac{1}{2}\log\frac{1}{d_n} + K_1 - \left\{\frac{1}{2}(x_{2^k-1} - x_n) - 4\sum_{n+1}^{2^k-1} d_i^2 - 8\right\}\right]_{\rho = d_n} < 0,$$

that is, if

$$\frac{1}{2}\log\frac{1}{d_n} - \frac{1}{2}(x_{2^k-1} - x_n) + 4\sum_{n+1}^{2^k-1} d_i^2 + K_4 < 0, \tag{4.7}$$

where $K_4 = \pi/4 + K_1 + 8$. Likewise, the estimates (3.4) and (3.8) show that (4.6) holds if

$$\left[\frac{\pi}{4}\left(\frac{\rho}{d_n}\right) - \frac{3\pi}{4} - \frac{1}{2}(x_{2^k-1} - x_n) - \log\frac{1}{d_n} - K_3\right]_{\rho = \pi/2 - d_n} > 0,$$

that is, if

$$\frac{\pi^2}{8}\left(\frac{1}{d_n}\right) - \frac{1}{2}(x_{2^k-1} - x_n) - \log\frac{1}{d_n} - K_5 > 0, \tag{4.8}$$

where $K_5 = \pi + K_3$. We write $n = 2^k - N$, so that $d_n = \frac{1}{4}N^{-1/2}$, and obtain from (4.1) that

$$x_{2^k-1} - x_n = \frac{1}{2}\sum_1^N \frac{1}{\sqrt{i}} - \frac{1}{4}\left(1 + \frac{1}{\sqrt{N}}\right)$$

and

$$4\sum_{n+1}^{2^k-1} d_i^2 = \frac{1}{4}\sum_1^{N-1}\frac{1}{i}.$$

Since $\sum_1^N(1/\sqrt{i})$ and $\sum_1^N(1/i)$ are within additive constants of $2\sqrt{N}$ and $\log N$ respectively, (4.7) holds if

$$\frac{1}{2}\log(4\sqrt{N}) - \frac{1}{2}\left(\sqrt{N} - \frac{3}{2}\right) + \frac{1}{4}(\log N + 1) + K_4 < 0,$$

that is, if

$$\sqrt{N} \ge \log N + K_6.$$

Similarly, (4.8) holds if

$$\left(\frac{\pi^2}{2}-\frac{1}{2}\right)\sqrt{N}\geq\frac{1}{2}\log N+K_7,$$

that is, if

$$\sqrt{N}\geq\frac{1}{(\pi^2-1)}\log N+K_8.$$

Thus (4.7) and (4.8) both hold if $N>K_0$, where K_0 is a suitable absolute constant.

Hence (4.5) and (4.6) hold if $N>K_0$, and so, for large fixed k, there are points $s_n=x_n+i(\pi/2+\rho_n)$ in each of the channels B_n^+, B_n^- for n in the range $2^{k-1}<n<2^k-K_0$ for which $\lambda(\rho_n)=0$. In other words, the points s_n lie at the same hyperbolic distance $d(0,x_{2^k-1})$ from the origin. We write $\lambda_k=d(0,x_{2^k-1})$. Then

$$\begin{aligned}\lambda_k&<\frac{1}{2}x_{2^k-1}\\&<\frac{1}{2}\sum_{j=1}^{k}\left(\sum_{n=1}^{2^{j-1}}\frac{1}{2\sqrt{n}}+j+1\right)\end{aligned}$$

so that $\log\lambda_k\leq K_9\log 2^k$ for large k.

Around each point s_n lies a disc of radius $d_n/2$ whose points have hyperbolic distance no more than $\frac{1}{2}\log 3$ from s_n. We define Ω_k to be the union of these discs and C_1 to be $\frac{1}{2}\log 3$. To complete the proof of Lemma 6, it remains to show that the area of Ω_k exceeds $C_2\log\lambda_k$ for a fixed constant C_2. We have

$$\begin{aligned}\text{Area of }\Omega_k&=\frac{\pi}{64}\sum_{K_0}^{2^{k-1}-1}\frac{1}{N}\\&>\frac{\pi}{64}\log(2^{k-1}-1)-\frac{\pi}{64}(1+\log K_0)\\&>\frac{\pi}{128}\log 2^k\end{aligned}$$

for all sufficiently large k. Thus,

$$\text{Area of }\Omega_k>\frac{\pi}{128K_9}\log\lambda_k$$

for all sufficiently large k, and we take $C_2=\pi/(128K_9)$. This proves Lemma 6. □

5. Proof of Lemma 1

We are now in a position to produce a domain $\mathcal{H}$ so that the normalised mapping function $\mathcal{G}(z)$ from the unit disc Δ onto $\mathcal{H}$ satisfies conditions (A) and (B) of Lemma 1. At present we have a domain $\mathfrak{D}$, defined by (4.1), (4.2), and Definition 1, for which it follows from Lemma 6 that the mapping function $G(z):\Delta\to\mathfrak{D}$ satisfies (2.2). However, certainly $G(z)$ does not satisfy condition (A).

The domain $\mathcal{H}$ is obtained from $\mathfrak{D}$ by replacing the inward-pointing slits of $\mathfrak{D}$ by rapid oscillations in such a way that $\mathcal{H}$ is of the form (2.1). These modifications to the domain $\mathfrak{D}$ are carried out making sure that the estimates obtained in Lemma 6 continue to hold (cf. Lemma 7). To make this precise we need some more notation.

For $-\infty<a<\infty$ and δ positive, we let $I_{a,\delta}=(a-\delta, a+\delta)$ and let $\gamma_{a,\delta}$ be the function on $I_{a,\delta}$ whose graph is the polygonal path having successive vertices

$$a-\delta,\ a-\delta/2+i\pi/2,\ a+\delta/2-i\pi/2,\ a+\delta.$$

Suppose that $\{\alpha_i\}_1^\infty$ and $\{\beta_i\}_1^\infty$ are two sequences of positive integers for which $\alpha_i<\alpha_{i+1}$ and β_i is "large" compared with α_i. For $n>1$, we let E_n be the set of integers k with $2^{\alpha_{n-1}}<k\le 2^{\alpha_n}$ and we let $E_1=\{k: 1\le k\le 2^{\alpha_1}\}$. If $k\in E_n$, where $n\ge 1$, we put

$$I_k=I_{a_k, 2^{-\beta_n}} \quad\text{and}\quad \gamma_k=\gamma_{a_k, 2^{-\beta_n}}.$$

We let

$$F_n=\bigcup_{i=1}^{n}\bigcup_{k\in E_i} I_k=\bigcup_{k=1}^{2^{\alpha_n}} I_k.$$

We now define a sequence of domains $\{\mathfrak{D}_n\}_0^\infty$ depending on the sequences $\{\alpha_i\}$ and $\{\beta_i\}$. Firstly, we set $\mathfrak{D}_0=\mathfrak{D}$.

DEFINITION 2. For $n\ge 1$ we define $\mathfrak{D}_n$ to be the domain

$$\mathfrak{D}_n=\{z: \operatorname{Re} z\in I_k,\ k\le 2^{\alpha_n},\ \gamma_k-i\pi<\operatorname{Im} z<\gamma_k+i\pi \text{ or } \operatorname{Re} z\notin F_n,\ z\in\mathfrak{D}\}.$$

(See Figure 3.)

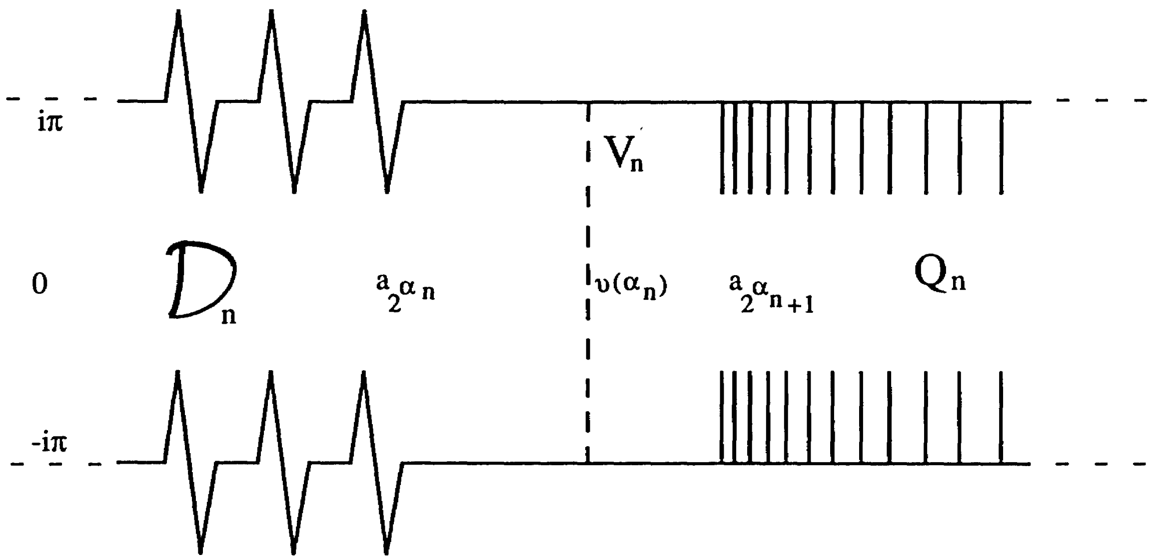

Figure 3

Thus $\mathfrak{D}_n$ differs from $\mathfrak{D}_{n-1}$ only in that the next set of slits, those corresponding to a_k with $k\in E_n$, have been changed to oscillations and these oscillations are probably much faster than those gone before. We also note that the domain $\mathfrak{D}_n$ depends only on α_k and β_k, $k=1,2,\dots,n$.

LEMMA 7. *The numbers α_i and β_i may be chosen so that the sequence of domains $\{\mathfrak{D}_n\}_0^\infty$ has the property that if z is in Ω_k for some k, then z is in $\mathfrak{D}_n$ for each n and*

$$|d(0,z;\mathfrak{D}_n)-d(0,z;\mathfrak{D}_{n+1})|<2^{-n} \tag{5.1}$$

for each $n\geq 0$.

We can now prove Lemma 1 assuming Lemma 7.

Proof of Lemma 1. The domains $\mathfrak{D}_n$ clearly converge to a simply connected domain $\mathcal{H}$ in the sense of kernel convergence, and $\mathcal{H}$ is of the form (2.1) so that Lemma 1(A) holds for the conformal mapping $\mathcal{G}(z):\Delta\to\mathcal{H}$.

Now each of the sets Ω_k lies in $\mathcal{H}$ and for z in Ω_k we have, by Lemmas 6 and 7, that

$$\lambda_k-C_1-\sum_1^\infty\frac{1}{2^n}<d(0,z;\mathcal{H})<\lambda_k+C_1+\sum_1^\infty\frac{1}{2^n}.$$

Thus, for z in Ω_k,

$$\lambda_k-C_3<d(0,z;\mathcal{H})<\lambda_k+C_3,$$

and, as before, the area of Ω_k exceeds $C_2\log\lambda_k$.

We define r_k and r_k', for $k\geq 1$, by

$$\frac{1}{2}\log\frac{1+r_k}{1-r_k}=\lambda_k-C_3\quad\text{and}\quad\frac{1}{2}\log\frac{1+r_k'}{1-r_k'}=\lambda_k+C_3.$$

Hence, $(1-r_k)/(1-r_k')$ is bounded and

$$\begin{aligned}A(r_k,r_k')&\geq\text{Area of }\Omega_k\\&\geq C_0\log\log\frac{1}{1-r_k'},\end{aligned}$$

which is (2.2) and completes the proof of Lemma 1 and hence of our theorem. □

It remains only to prove Lemma 7. In order to do so, we need one further result.

LEMMA 8. *Let R_0 be the rectangle*

$$R_0=\{s:-L<\operatorname{Re}s<L,\ |\operatorname{Im}s|<\pi\}.$$

Suppose that $u_1(s)$ and $u_2(s)$ are two positive harmonic functions in R_0 which are continuous on the boundary of R_0 and which vanish on $|\operatorname{Im}s|=\pi$. Suppose also that C is a constant for which

$$\frac{u_1(0)}{u_2(0)}<C.$$

Then, if $-\pi<r<\pi$ and $L>2\log(24/\pi)$, we have

$$\frac{u_1(ir)}{u_2(ir)}<C(1+16e^{-L/2}).$$

Proof. The proof of Lemma 8 involves some standard calculations which we omit. We let $V=\{s: \operatorname{Re} s=-L, -\pi<\operatorname{Im} s<\pi\}$ on the boundary of R_0. Then $\omega(0, V; R_0)$ denotes the harmonic measure of V with respect to R_0 and evaluated at the origin.

Write S^+ for the half-strip $\{s: \operatorname{Re} s>-L, |\operatorname{Im} s|<\pi\}$. It follows from the reflection principle that

$$\begin{aligned}\omega(0, V; R_0) &< \omega(0, V; S^+)\\ &= 2\omega(0, H; S),\end{aligned}$$

where S is the strip $\{s: |\operatorname{Im} s|<\pi\}$ and $H=\{s: \operatorname{Re} s<-L \text{ and } |\operatorname{Im} s|=\pi\}$ is the boundary of S to the left of V.

The Poisson integral formula for S gives

$$\begin{aligned}2\omega(0, H; S) &= \frac{4e^{L/2}}{\pi}\int_{-\infty}^{0}\frac{e^{\xi}}{e^{2\xi}+e^{L}}\,d\xi\\ &< \frac{4e^{L/2}}{\pi}e^{-L}\left[\int_{-\infty}^{0}e^{\xi}\,d\xi\right]\\ &= \frac{4}{\pi}e^{-L/2}.\end{aligned}$$

We conformally map R_0 to Δ, fixing the origin and sending the real axis onto the interval $(-1, 1)$. Then $u_1(s)$ and $u_2(s)$ in R_0 give rise to harmonic functions $u_1(z)$ and $u_2(z)$ in Δ. Moreover, by conformal invariance of harmonic measure, $u_1(e^{it})$ and $u_2(e^{it})$ vanish when t lies outside $(-\delta, \delta)$ or $(\pi-\delta, \pi+\delta)$ where $\delta=4e^{-L/2}$.

Hence, for $0<r<1$,

$$\begin{aligned}u_1(\pm ir) &= \frac{1}{2\pi}\int_0^{2\pi}u_1(e^{it})\mathcal{P}\left(r, t\mp\frac{\pi}{2}\right)dt\\ &< \frac{1}{2\pi}\left(\int_0^{2\pi}u_1(e^{it})\,dt\right)\frac{1-r^2}{1-2r\sin\delta+r^2}\\ &= u_1(0)\frac{1-r^2}{1-2r\sin\delta+r^2},\end{aligned}$$

where $\mathcal{P}(r, \theta)$ is the Poisson kernel in the unit disc. Similarly,

$$u_2(\pm ir) > u_2(0)\frac{1-r^2}{1+2r\sin\delta+r^2}. \tag{5.2}$$

Thus, for $-1<r<1$,

$$\begin{aligned}\frac{u_1(ir)}{u_2(ir)} &< \frac{u_1(0)}{u_2(0)}\left(\frac{1+2|r|\sin\delta+r^2}{1-2|r|\sin\delta+r^2}\right)\\ &< C\left(\frac{1+\sin\delta}{1-\sin\delta}\right)\\ &< C(1+4\delta)\end{aligned}$$

if $\delta<\pi/6$. So, under the correspondence between R_0 and Δ,

$$\frac{u_1(ir)}{u_2(ir)}<C(1+16e^{-L/2})$$

for $-\pi<r<\pi$ if $e^{L/2}>24/\pi$. This proves Lemma 8. □

Proof of Lemma 7. The sequences α_i and β_i are chosen inductively. We suppose therefore that α_i and β_i $(i=1,2,\ldots,N)$ have been chosen, and we recall that $\mathfrak{D}_N$ denotes the corresponding domain according to Definition 2. What follows works equally well when $N=0$, $\mathfrak{D}_N=\mathfrak{D}$ and we wish to choose α_1 and β_1. To begin with, we recall from (3.1) that if Ω is simply connected then

$$d(0,z;\Omega)=\frac{1}{2}\log\frac{1+e^{-g(0,z;\Omega)}}{1-e^{-g(0,z;\Omega)}}. \tag{5.3}$$

As h decreases to 0, $h(1+e^{-h})/(1-e^{-h})$ decreases to 2. Therefore, if ϵ is positive then there exists a positive δ such that, for $0<h<\delta$,

$$\log\frac{2}{h}<\log\left(\frac{1+e^{-h}}{1-e^{-h}}\right)<\log\frac{2}{h}+\log(1+\epsilon). \tag{5.4}$$

We choose ϵ so that $\log(1+\epsilon)=2^{-N}$ and choose a positive δ so that (5.4) holds.

Since $d_{2^k}=k+1$ for each k, we have that a_{2^k} is separated from a_{2^k+1} by a rectangle of width 2π and length $k+1$. We set

$$\nu(k)=\tfrac{1}{2}(a_{2^k}+a_{2^k+1})$$

so that certainly $\nu(k)$ increases to infinity with k. Thus, since $g(0,z;\mathfrak{D}_N)$ approaches zero as $\operatorname{Re} z$ tends to infinity, we may choose α_{N+1} greater than α_N so that, if $\operatorname{Re} z>\nu(\alpha_{N+1})$, then

$$g(0,z;\mathfrak{D}_N)<e^{-2}\delta \tag{5.5}$$

and so that

$$\log(1+16e^{-\alpha_{N+1}/4})<2^{-N-1}. \tag{5.6}$$

Set $Q_N=\{z \text{ in } \mathfrak{D}_N\colon \operatorname{Re} z>\nu(\alpha_{N+1})\}$. It follows from (5.3), (5.4), and (5.5) that for z in Q_N,

$$\frac{1}{2}\log\frac{2}{g(0,z;\mathfrak{D}_N)}<d(0,z;\mathfrak{D}_N)<\frac{1}{2}\log\frac{2}{g(0,z;\mathfrak{D}_N)}+\frac{1}{2^{N+1}}. \tag{5.7}$$

Thus, α_{N+1} has been chosen so that $\frac{1}{2}\log(2/g)$ gives a good approximation to hyperbolic distance in Q_N. It remains to choose β_{N+1}.

For a positive integer n, we let D^n denote the domain which corresponds to $\mathfrak{D}_{N+1}$ in Definition 2 with the choice of $\beta_{N+1}=n$. By allowing n to vary over the positive integers, we produce a sequence of domains $\{D^n\}$. Our objective is to show that if $n=n_0$ is chosen sufficiently large then (5.1) holds, in which case we put $\beta_{N+1}=n_0$. The sequence of domains $\{D^n\}$ converges to $\mathfrak{D}_N$ in the sense of kernel convergence and, for large n,

$$\bigcup_{k=1}^{\infty} \Omega_k \subset D^n.$$

Suppose that $f_n(z)$ maps D^n conformally onto Δ with $f_n(0)=0$, $f_n'(0)$ positive, and that $f(z)$ maps $\mathfrak{D}_N$ to Δ normalised in the same way. The Carathéodory kernel theorem (cf. [6, Chap. 3]) yields that, for z in Ω_k ($k=1,2,\ldots,\alpha_{N+1}$), we have

$$|d(0,z;\mathfrak{D}_N)-d(0,z;D^n)|<\frac{1}{2^N} \tag{5.8}$$

if $n \geq n_1$, say.

It remains to show that, by choosing n sufficiently large, we can also arrange that (5.8) holds for z in Ω_k when $k>\alpha_{N+1}$.

We choose a positive ϵ' so that $\log(1-\epsilon')>-2^{-N-1}$. Again by kernel convergence of $f_n(z)$ to $f(z)$, it follows that for $n \geq n_2$,

$$1-\epsilon'<\frac{g(0,\nu(\alpha_{N+1});D^n)}{g(0,\nu(\alpha_{N+1});\mathfrak{D}_N)}<1+\epsilon'. \tag{5.9}$$

Since the rectangle

$$R_0=\left\{z: \nu(\alpha_{N+1})-\frac{\alpha_{N+1}}{2}<\operatorname{Re} z<\nu(\alpha_{N+1})+\frac{\alpha_{N+1}}{2} \text{ and } |\operatorname{Im} z|<\pi\right\}$$

is contained in $\mathfrak{D}_N$, and since both Green's functions are harmonic there and vanish on the sides of R_0 where $|\operatorname{Im} z|=\pi$, it follows from Lemma 8 and (5.9) that

$$c_1=\frac{1-\epsilon'}{1+16e^{-\alpha_{N+1}/4}}<\frac{g(0,z;D^n)}{g(0,z;\mathfrak{D}_N)}<(1+\epsilon')(1+16e^{-\alpha_{N+1}/4})=c_2$$

for $\operatorname{Re} z=\nu(\alpha_{N+1})$ and $|\operatorname{Im} z|<\pi$. Because of (5.6) and the choice of ϵ', both $\log(1/c_1)$ and $\log c_2$ are less than 2^{-N}.

Both Green's functions $g(0,z;D^n)$ and $g(0,z;\mathfrak{D}_N)$ are defined in Q_N and vanish on the boundary of Q_N where $\operatorname{Re} z>\nu(\alpha_{N+1})$ strictly. On the remaining boundary, that is on $\{z: \operatorname{Re} z=\nu(\alpha_{N+1})$ and $|\operatorname{Im} z|<\pi\}$, we have

$$c_1 g(0,z;\mathfrak{D}_N) \leq g(0,z;D^n) \leq c_2 g(0,z;\mathfrak{D}_N). \tag{5.10}$$

Hence (5.10) holds on the boundary of Q_N and by the maximum principle throughout Q_N.

Since certainly $\log c_2<2$ and $g(0,z;\mathfrak{D}_N)<e^{-2}\delta$ in Q_N by (5.5), we have that $g(0,z;D^n)<\delta$ throughout Q_N. Therefore (5.7) holds for the Green's function for D^n as well as for $\mathfrak{D}_N$. That is,

$$\frac{1}{2}\log\frac{2}{g(0,z;D^n)}<d(0,z;D^n)<\frac{1}{2}\log\frac{2}{g(0,z;D^n)}+\frac{1}{2^{N+1}}. \tag{5.11}$$

From the inequalities (5.7), (5.10), and (5.11), we obtain that for z in Q_N (in particular for z in Ω_k, where $k>\alpha_{N+1}$),

$$\begin{aligned} d(0,z;D^n) &< \frac{1}{2}\log\frac{2}{g(0,z;D^n)}+\frac{1}{2^{N+1}} \\ &< \frac{1}{2}\log\frac{2}{g(0,z;\mathfrak{D}_N)}+\frac{1}{2}\log\frac{1}{c_1}+\frac{1}{2^{N+1}} \\ &< d(0,z;\mathfrak{D}_N)+\frac{1}{2^N}; \\ d(0,z;D^n) &> \frac{1}{2}\log\frac{2}{g(0,z;D^n)} \\ &> \frac{1}{2}\log\frac{2}{g(0,z;\mathfrak{D}_N)}-\frac{1}{2}\log c_2 \\ &> d(0,z;\mathfrak{D}_N)-\frac{1}{2^N}. \end{aligned}$$

Thus, if $n \geq n_2$, we have

$$|d(0,z;\mathfrak{D}_N)-d(0,z;D^n)| < \frac{1}{2^N} \tag{5.12}$$

whenever z is in Ω_k for $k \geq \alpha_{N+1}$. We choose $\beta_{N+1}=\max\{n_1, n_2\}$. It then follows from (5.8) and (5.12) that (5.1) holds. □

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