Support Sets and Gleason Parts

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1. Introduction

The function algebra H^∞ is the collection of all bounded holomorphic functions in the unit disc D of the complex plane. Under the supremum norm it is a Banach algebra. The Gelfand theory represents H^∞ as a subalgebra of $C(M)$, the algebra of continuous, complex-valued functions on M, the maximal ideal space of H^∞. With the weak-star topology M is a compact Hausdorff space, and the point evaluations for points in the disc form a dense subset [3].

For points z and w in the disc, the pseudo-hyperbolic distance from z to w is

$$\rho(z,w)=\left|\frac{z-w}{1-\bar{w}z}\right|.$$

Pick's lemma states that, for z and w in D and f a nonconstant H^∞ function with norm not exceeding 1, $\rho(f(w),f(z))\le\rho(z,w)$. Taking points ϕ and ψ in M and extending ρ to $M\times M$ by $\rho(\phi,\psi)=\sup\{\rho(f(\phi),f(\psi)):f\in H^\infty, \|f\|_\infty<1\}$, we can partition M into equivalence classes known as Gleason parts, calling ϕ and ψ equivalent provided $\rho(\phi,\psi)<1$. We denote the Gleason part to which ϕ belongs by P_ϕ.

Hoffman [9] has shown that the Gleason parts of M are either singletons or discs. For the latter case he constructed [11] a one-to-one map L_m of D onto P_m sending 0 to m such that $f\circ L_m$ is holomorphic for all f in H^∞. Such parts and points are called *analytic,* while points whose Gleason parts are singletons are called *trivial.*

Viewing H^∞ functions as continuous over the Shilov boundary of M, which is the maximal ideal space of L^∞ and which we denote by X, one can represent an element ϕ of M as integration against a positive measure μ_ϕ: $f(\phi)=\int f\,d\mu_\phi$. This representation allows us to extend ϕ to L^∞ in such a manner that the Gelfand transforms of L^∞ functions are also continuous on M. The measure μ_ϕ is called a *representing measure,* and its support in X is known as a *support set.* Points in the same Gleason part have the same support set [9]. Support sets may meet, but if they do then one is entirely contained within the other (unpublished work of Hoffman).

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The way one studies points of M outside of the disc is by looking at closures of sequences in D. Two types of Blaschke sequences are of particular interest in such study. An *interpolating* sequence (z_n) is one for which

$$\prod_{n,\, n\neq m}\left|\frac{z_m-z_n}{1-\overline{z_n}z_m}\right|\geq\delta$$

for some positive δ and all m. An interpolating sequence is *thin* if and only if the limit of the above product as m approaches ∞ is 1. If a point lies in the closure of an interpolating sequence, then so too does every point in its Gleason part; these are precisely the analytic points. A point is called thin if it lies in the closure of a thin sequence. It can be shown that any part which contains a thin point consists entirely of thin points (see, e.g., [8]).

A more in-depth introduction to the study of the maximal ideal space of H^∞ can be found in [10].

In this paper the closure of a Gleason part is shown to be a union of parts (Theorem 2.3). For points of M in the closure of a certain broad family of curves approaching the boundary of the disc in a somewhat steady fashion, the support set of a point is shown to be contained in the support set of another point if and only if the Gleason part of the first point lies in the closure of the Gleason part of the second (Theorem 5.7). Finally, it is shown that for two curves in D meeting the boundary at 1, one curve more tangential than the other, then the support set for any point in the closure of the less tangential curve contains a support set of a point in the closure of the more tangential curve (Theorem 5.8). Hence, order of tangency in some sense orders support sets, yet has no relation to the topology of Gleason part closures.

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2. Closures of Gleason Parts

We wish to study the closures of the analytic Gleason parts in M. Our main tool will be an extension of the maps L_m to all of M: $L_m(\phi)(f)=\phi(f\circ L_m)$, for f in H^∞. Then it is easy to see that the extended map is a continuous mapping of M onto the closure of P_m, and that it maps points in the same part to the same part. We would now like to show that the range of L_m is a union of Gleason parts. Recall Hoffman's construction of L_m: for x and y in the disc define the maps $L_x(y)=(y+x)/(1+\bar{x}y)$. If a net $(x(\alpha))$ converges to m, the maps $L_{x(\alpha)}$ converge pointwise to L_m. This will be combined with the following lemmas to obtain our result.

LEMMA 2.1. *For x, y, and z in D,*

$$L_z(L_y(x))=L_{L_z(y)}(x(1+z\bar{y})(1+\bar{z}y)^{-1}).$$

LEMMA 2.2. *Let $m(\alpha)$ be a net in M converging to m, and let $(x(\alpha))$ be a net in D converging to a point x in D. Then $L_{m(\alpha)}(x(\alpha))$ converges to $L_m(x)$.*

The first proof is purely computational, while the second follows from the fact that the holomorphic functions $f\circ L_{m(\alpha)}$ converge pointwise and boundedly on D—and hence, uniformly on compact subsets of D—to $f\circ L_m$.

THEOREM 2.3. *If ζ belongs to $\overline{P_m}$ and η lies in the Gleason part of ζ, then η belongs to $\overline{P_m}$.*

Proof. It will be enough to find ψ in M with $L_m(\psi)=\eta$.

Since ζ is in $\overline{P_m}$, for some ϕ in M we have $L_m(\phi)=\zeta$. Take nets $(z(\alpha))$ and $(y(\beta))$ in the disc converging, respectively, to m and ϕ, and let M_{σ_1} and M_{σ_2} be the respective fibers of m and ϕ. Since η is in P_ζ, for some x in D we have $L_\zeta(x)=\eta$. Setting ψ equal to $L_\phi(x\sigma_2/\sigma_1)$, we claim $L_m(\psi)=\eta$.

By Lemma 2.1,

$$L_{z(\alpha)}\left(L_{y(\beta)}\frac{x\sigma_2}{\sigma_1}\right)=L_{L_{z(\alpha)}(y(\beta))}\left(\frac{x\sigma_2(1+z(\alpha)\overline{y(\beta)})}{\sigma_1(1+\overline{z(\alpha)}y(\beta))}\right). \tag{2.1}$$

Since $L_{z(\alpha)}(y(\beta))$ converges to $L_m(y(\beta))$ as $\alpha\to\infty$, Lemma 2.2 applied to both sides of (2.1) gives

$$L_m\left(L_{y(\beta)}\frac{x\sigma_2}{\sigma_1}\right)=L_{L_m(y(\beta))}\left(\frac{x\sigma_2(1+\sigma_1\overline{y(\beta)})}{\sigma_1(1+\overline{\sigma_1}y(\beta))}\right). \tag{2.2}$$

Since $L_{y(\beta)}(x\sigma_2/\sigma_1)$ converges to ψ as $\beta\to\infty$, and since L_m is continuous, the left side of (2.2) converges to $L_m(\psi)$. Again by Lemma 2.2, the right side of (2.2) converges to $L_{L_m(\phi)}(x)$. Thus, $L_m(\psi)=L_\zeta(x)=\eta$. □

COROLLARY 2.4. *If m is in the fiber of M over σ_1 and ϕ is in the fiber over σ_2, and if S is the rotation defined by $S(x)=x\sigma_2/\sigma_1$, then $L_m\circ L_\phi\circ S(x)=L_{L_m(\phi)}(x)$.*

COROLLARY 2.5. *If x is in $\overline{P_y}$ and y is in $\overline{P_z}$ then x is in $\overline{P_z}$.*

The first corollary is simply a restatement of the main calculation in the theorem, while the second is an immediate consequence of the theorem.

Corollary 2.4 is a statement about factorization; this is emphasized in the following reformulation.

COROLLARY 2.6. *Suppose that P_ψ is contained in $\overline{P_m}$, and let ϕ satisfy $L_m(\phi)=\psi$. Then $L_\psi=L_m\circ L_\phi\circ S$, where S is a rotation of the disc extended continuously to M.*

COROLLARY 2.7. *If $L_m(\phi)=\psi$ and ψ is an analytic point, then the restriction of L_m to P_ϕ is a one-to-one map onto P_ψ.*

Proof. By Corollary 2.6, $L_\psi=L_m\circ L_\phi\circ S$. This shows that when restricted to P_ϕ, the image of L_m is precisely P_ψ. That the map is injective follows from the fact that L_ψ is one-to-one. □

As a final consequence of this result, we see that there is an abundance of trivial points in M, for if ϕ is any trivial point then so too is $L_m(\phi)$. In particular, we have the following.

COROLLARY 2.8. *For every point m in M there is a trivial point in $\overline{P_m}$.*

Whether every trivial point, excluding those in the Shilov boundary, lies in the closure of an analytic part is unknown.

At this point it should be noted that Abram and Weiss [1] have also looked at extensions of analytic maps from D to analytic parts. Using the theory of transformation groups, they too have shown that the closure of a Gleason part is a union of parts, and that any such closure contains trivial points.

Mortini [12] has conjectured that any closed, primary ideal of H^∞ contained in a maximal ideal determined by a trivial point is itself maximal. (An ideal is primary if it is contained in a unique maximal ideal.) He has used the existence of these trivial points in the closure of analytic parts to show that there exist nonmaximal, closed prime ideals (ideals I for which if $f \in I$ and $f = gh$ then either g or $h \in I$) which lie in a maximal ideal determined by a trivial part. Indeed, if ψ is an analytic point, then $I = \{f \in H^\infty : f$ vanishes identically on $P_\psi\}$ is such an example. In contrast to this, he has shown that any closed, prime ideal contained in a maximal ideal determined by a point in X is itself maximal.

For a thin point m, Hoffman [11] has noted that L_m is a homeomorphism of the disc. In fact, if (z_n) is a thin sequence containing m in its closure, and if $B(z)$ is the Blaschke product with zero sequence (z_n), then $B \circ L_m(z) = \lambda z$ for some constant λ of unit modulus. Likewise, in case m is an oricycular point and L_m a homeomorphism, Gerber and Weiss [5] have shown that L_m is inverted by a multiple of a Blaschke product. The question of whether L_m or its extension to M is a homeomorphism is equivalent to the question of whether it is one-to-one. From Corollary 2.7, we see that a sufficient condition for L_m to be one-to-one is that it map distinct parts to distinct parts. Hoffman [11] has constructed a point m in M for which L_m maps some points in $M - D$ into P_m, and so L_m is not a homeomorphism. His example actually produces two distinct Gleason parts whose closures are identical. This cannot happen if L_m is a homeomorphism, as the next result shows.

PROPOSITION 2.9. *For some point ψ of M suppose L_ψ is a homeomorphism. If $\overline{P_\phi} = \overline{P_\psi}$ then ϕ and ψ belong to the same Gleason part.*

Proof. Take points m_1 and m_2 in M such that $L_\phi(m_1) = \psi$ and $L_\psi(m_2) = \phi$. Then

$$L_\psi(0) = \psi = L_{L_\psi(m_2)}(m_1) = L_\psi \circ L_{m_2} \circ S(m_1)$$

for some rotation S. Since L_ψ is one-to-one, we conclude that m_2 is a point of the disc, and so ϕ is in P_ψ. □

For a thin point m, not only is L_m a homeomorphism but P_m is also maximal; that is, P_m does not lie in the closure of another Gleason part aside (of course) from the disc itself. To prove this we will use the following lemma.

LEMMA 2.10. *Let (z_n) be an interpolating sequence and let $\phi \in \overline{(z_n)}$. If $\phi \in \overline{P_\psi}$, then ϕ also lies in the closure of those points in P_ψ which $\overline{(z_n)}$ meets.*

Proof. Suppose $\phi \in \overline{P_\psi}$ and $L_\psi(m) = \phi$. Let K be the closure of $P_\psi \cap \overline{(z_k)}$. Let us assume that $\phi \notin K$. Take disjoint, open neighborhoods U and V of ϕ and K, respectively. Let (w_k) be the subsequence of (z_n) which lies in U, and let its associated Blaschke product be $A(z)$. Then $A(\phi) = 0$. Let $g(z) = A \circ L_\psi(z)$. Note that $g(z)$ does not vanish on D, yet $g(m) = 0$. It follows that g vanishes identically on P_m [11], and so A does the same on P_ϕ. But because A is interpolating, it must instead be true that $\phi \in K$. □

PROPOSITION 2.11. *A thin part is maximal.*

Proof. Suppose ϕ is a thin point in a Gleason part P. Then there is a thin sequence (z_n) and an associated Blaschke product $B(z)$ such that $B(\phi) = 0$. If $\phi \in \bar{Q}$, where $P \neq Q$, then by the preceding lemma ϕ belongs to the closure of the zeros of B that lie in Q. But since B is thin, it has at most one zero in any Gleason part. Hence, $\phi \notin \bar{Q}$. □

If ϕ and ψ are thin points and $m = L_\phi(\psi)$, then L_m is a homeomorphism, yet m lies in $\overline{P_\phi}$. Proposition 2.9 shows that P_m is properly contained in $\overline{P_\phi}$; thus, there are many points whose Hoffman maps are homeomorphisms, yet their Gleason parts are not maximal.

These extensions $L_m: M \to \overline{P_m}$ have been used to construct counterexamples of conjectures about ideals in subalgebras of L^∞. For example, it is known that for every closed ideal I in H^∞ whose zero set (the set of points in M on which every function in the ideal vanishes) lies in X, there is an ideal J in L^∞ such that $I = J \cap H^\infty$ [6]. For certain types of Douglas algebras B (closed algebras between H^∞ and L^∞), Gorkin and Mortini [7] have characterized when closed ideals in $QA_B = \bar{B} \cap H^\infty$ "lift" to $QB = B \cap \bar{B}$. (Here, QB is the largest C^* algebra contained in B, and QA_B is its analytic subalgebra.) They have found that such ideals lift if and only if they have the property that for every function in the ideal, its outer factor is also in the ideal. Given a Douglas algebra B, one might ask whether every closed ideal I in QA_B whose zero set lies in the Shilov boundary of $M(QA_B)$ has this property. Gorkin and Mortini [7], however, have provided a counterexample using these extensions of L_m.

In another application, Gorkin, Hedenmalm, and Mortini [6] have shown that if I is a closed ideal in H^∞ such that $\{m \in M : f(m) = 0 \text{ for all } f \in I\} \subset M(L^\infty)$, then there is a unique, continuous homomorphism from L^∞ onto H^∞/I which is canonical on H^∞. One might ask whether the same is true when L^∞ is replaced by a Douglas algebra B. They answer this question affirmatively for $B = H^\infty + C$, under the added condition that the weak-star closure of I is H^∞. In general, however, this is not true. They use the extension of L_m for a thin point m to construct a counterexample, where B is the smallest Douglas algebra containing the complex conjugates of all thin Blaschke products. (See Hedenmalm [8] for a study of this Douglas algebra.)

3. Closures of *K*-Curves in the Disc

Several authors have attempted to classify points in M according to how they can be approached from the disc. Most notable among this work has been Hoffman's discovery [9] that points with nontrivial Gleason parts are precisely those that can be approached along interpolating sequences. Wortman [15] has shown that the closures of convex curves contain only analytic points. Weiss [14] has demonstrated that convexity is not crucial: as long as the curve can be parametrized as $r(t)e^{i\theta(t)}$ $(0 \le t \le 1)$, where $r(t)$ is strictly increasing and $\theta(t)$ strictly decreasing, the same conclusion holds. Such curves are known as M-curves. It is not too difficult to see, however, that not every analytic point lies in the closure of such a curve. (For a classification of curves which meet only analytic points, and which meet every analytic point in the relative interior of a fiber, see Gerber and Weiss [4].)

In this section we want to examine the relationships of the support sets and Gleason parts of points in the closure of the curves in a certain family. Symmetry allows us to restrict our attention to M_1^+, the points in the fiber M_1 which can be approached from the set of points in D with nonnegative imaginary parts. M-curves are too general for our analysis, and so we will work with a certain subclass called K-curves, which were first studied by Weiss and Satyanarayana ([13], [14]).

Letting Γ be an M-curve defined by $f(\theta) = 1 - r$ for some real-valued continuous function f which is defined near 0, vanishes at 0, and is strictly increasing, we say Γ is a K-curve if $f(\theta + kf(\theta))/f(\theta) \to 1$ as θ approaches 0^+ for all real numbers k. Weiss [14] has characterized these curves extensively. One particularly useful characterization is that Γ is a K-curve if and only if the slope of the tangent to Γ approaches $-\infty$ as Γ tends to 1. Of course, there is no *a priori* reason why the tangent should exist at all points along Γ. Curves are defined to be equivalent if their closures are identical on $M - D$. It is easy to see that for any K-curve there is an equivalent curve with a well-defined tangent, and so we may assume all K-curves have well-defined tangents everywhere. Clearly, tangential convex curves are K-curves.

Although Stolz curves are not K-curves, both classes behave similarly. If Γ is a Stolz curve, define $A(\Gamma)$ to be the union of all Stolz points. Then $A(\Gamma)$ is an open union of analytic parts, each of which intersects each Stolz curve. In case Γ is a K-curve, similar results hold ([13], [14]). Suppose Γ is defined by $f(\theta) = 1 - r$. For any positive number λ let Γ_λ be the curve given by $f(\theta) = \lambda(1 - r)$. Define $A(\Gamma)$ to be the union of $\overline{\Gamma_\lambda} - \Gamma_\lambda$ over all possible λ. Then

$A(\Gamma)$ is an open union of analytic parts, each of which intersects each Γ. Furthermore, $A(\Gamma)$ separates M_1 topologically. Thus, relative to a K-curve Γ there are three types of points in M_1^+: points with the same order of tangency (those in $A(\Gamma)$), points more tangential, and points less tangential.

This leads to several definitions. For a K-curve Γ approaching 1 and ϕ in M_1^+, we say ϕ is more tangential than Γ if every net $(z(\alpha))$ approaching ϕ eventually lies between Γ and the upper half of the boundary of D and if $\rho(z(\alpha),\Gamma)$ converges to 1 as $\alpha \to \infty$. We say ϕ is less tangential than Γ if every net $(w(\alpha))$ approaching ϕ eventually lies between Γ and the positive real axis and if $\rho(w(\alpha),\Gamma)$ converges to 1 as $\alpha \to \infty$. These definitions correspond to the topological picture obtained when separating M_1^+ by $A(\Gamma)$.

The crucial question in what follows will be whether points can be separated by a K-curve: Given two points in M_1^+, can we find a K-curve Γ such that one point is more tangential than Γ and the other either lies in $A(\Gamma)$ or is less tangential than Γ? If not, does this mean that both points belong to $A(\Gamma)$ for some K-curve Γ? There are pairs of points in M for which both of these questions have negative answers. One such example is a pair of analytic points in $\overline{A(\Gamma)}-A(\Gamma)$, for Γ the positive real axis. Unfortunately, pairs of this type are not amenable to our techniques.

4. Behavior of Blaschke Products with Zeros along K-Curves

We would like to estimate the behavior of certain Blaschke products which have their zeros on K-curves. As these calculations are more easily done in the right half-plane (RHP), we will transfer everything to that setting via the map

$$w=\tau(z)=\frac{1+z}{1-z}, \quad z\in D.$$

For $z=x+iy$ and $w=u+iv$ in the RHP, the pseudo-hyperbolic metric (which we again denote by $\rho(\cdot,\cdot)$) becomes

$$\rho^2(z,w)=\left|\frac{z-w}{z+w}\right|^2=\frac{(x-u)^2+(y-v)^2}{(x+u)^2+(y-v)^2},$$

and so we have

(4.1) $$1-\rho^2(z,w)=\frac{4xu}{(x+u)^2+(y-v)^2};$$

(4.2) For $0<\delta<1$ the set $\{z\in \text{RHP}: \rho(z,w)=\delta\}$ is a circle with center $(u(1+\delta^2)/(1-\delta^2), v)$ and radius $2\delta u/(1-\delta^2)$.

A Blaschke product in the RHP has the form

$$B(z)=\prod_n c_n\frac{z-z_n}{z+\overline{z_n}},$$

where the c_n are constants of unit modulus, chosen to make the product converge. Under the map τ the image of a K-curve approaching 1 with positive imaginary part has a particularly nice form: it approaches ∞ through the first quadrant in such a manner that the absolute value of the slope of the tangent to the curve also tends to ∞. Curves in the RHP that are images of K-curves in the disc will also be called K-curves.

We are interested in those Gleason parts which meet the closure of a particular Stolz or K-curve. Since the closure of every Stolz curve meets precisely the same Gleason parts, for this case we may assume that the curve is the positive real axis. In either case, for such a curve Γ we select a sequence of distinct points (z_n) along Γ which have the property that $\rho(z_n, z_{n+1})$ is constant. Every part which $\bar{\Gamma}$ meets will also be met by $\overline{(z_n)}$, so it is sufficient to study the closure of this sequence; we shall do so by examining the behavior of the Blaschke product with zero sequence (z_n). (Such a sequence is interpolating [14], so the Blaschke product certainly exists.)

It is known that a Blaschke product with zeros ρ-equally spaced along a radial arc has unit modulus on every Gleason part except the Stolz parts. In particular, such a Blaschke product has unit modulus at every point of M which is more tangential than some Stolz curve. Additionally, a sequence of points (z_n) on a radial arc is thin precisely when $\rho(z_n, z_{n+1})$ tends to 1 as n approaches ∞. Both of these results have analogs valid for K-curves, as we now intend to demonstrate.

THEOREM 4.1. *Let Γ be a K-curve approaching* 1 *from above and let (z_n) be a sequence of distinct points ρ-equally spaced along Γ. Let $B(z)$ be the Blaschke product with zero sequence (z_n). If ϕ is more tangential and ψ less tangential than Γ, then $|B(\phi)|=1$ and $|B(\psi)|<1$.*

Proof. We will work entirely in the RHP, and so we assume that (z_n) and Γ are the images under the map $w=\tau(z)$ of the sequence and curve mentioned in the theorem.

Let $z_n = x_n + iy_n$. Since $\rho(z_n, z_{n+1}) = \delta$, by (4.2) we see that

$$x_{n+1} = \frac{(1+\delta^2)x_n}{(1-\delta^2)} + \frac{2\delta x_n}{(1-\delta^2)}\cos\theta, \qquad y_{n+1} = y_n + \frac{2\delta x_n}{(1-\delta^2)}\sin\theta,$$

where θ is the angle with vertex at the Euclidean center of the circle, one ray in the direction of the positive real axis, and the other towards (x_{n+1}, y_{n+1}). This immediately leads to

$$\frac{1-\delta}{1+\delta} \le \frac{x_{n+1}}{x_n} \le \frac{1+\delta}{1-\delta}. \tag{4.3}$$

Using the fact that the slope of the tangent to Γ tends, in absolute value, to ∞, we see that $\theta \to \operatorname{Arccos}(-\delta)$, and so for sufficiently large n we have the following:

$$\frac{2\delta}{1-\delta^2}\sin\left(\operatorname{Arccos}\left(-\frac{1+\delta}{2}\right)\right) \le \frac{y_{n+1}-y_n}{x_n} \le \frac{2\delta}{1-\delta^2}. \tag{4.4}$$

Any positive constant whose value depends only on δ will be denoted by C in the estimates that follow. Different uses of C may represent different constants.

Suppose (w_n) is a sequence containing ϕ in its closure, where $\rho(w_n, \Gamma)\to 1$ as $n\to\infty$. Fix n and let $w=w_n=u+iv$. Let $a_k=\rho(w,z_k)$ and $d=\inf_k a_k$ (so that $d\to 1$ as $w\to\phi$). Using the estimate

$$1-x\le -\log x\le \frac{1-x}{d} \quad \text{for } d\le x\le 1,$$

we find that

$$\exp\left(-\frac{1}{d}\sum_k(1-a_k)\right)\le |B(w)|\le \exp\left(-\sum_k(1-a_k)\right).$$

To see that $|B(w)|\to 1$ it will suffice to show $\sum_k(1-a_k^2)\to 0$.

Choose j so that $y_j\le v<y_{j+1}$. Then by (4.1), (4.3), and (4.4),

$$\begin{aligned}
\sum_k(1-a_k^2) &= \sum_k \frac{4x_k u}{(x_k+u)^2+(y_k-v)^2}\\
&\le \sum_1^{j-2}+\sum_{j+3}^{\infty} \frac{4x_k u}{(x_k+u)^2+(y_k-v)^2}+4(1-d^2)\\
&\le C\cdot\sum_1^{j-2}\frac{u(y_{k+1}-y_k)}{(x_k+u)^2+(y_k-v)^2}\\
&\quad +C\cdot\sum_{j+3}^{\infty}\frac{u(y_k-y_{k-1})}{(x_k+u)^2+(y_k-v)^2}+4(1-d^2)\\
&\le C\left(\int_{y_1}^{y_{j-1}}\frac{u}{u^2+(y-v)^2}\,dy+\int_{y_{j+2}}^{\infty}\frac{u}{u^2+(y-v)^2}\,dy\right)+4(1-d^2)\\
&\le C\left[\operatorname{Arctan}\frac{u}{v-y_{j-1}}+\operatorname{Arctan}\frac{u}{y_{j+2}-v}\right]+4(1-d^2)\\
&\le C\left[\operatorname{Arctan}\frac{u}{y_j-y_{j-1}}+\operatorname{Arctan}\frac{u}{y_{j+2}-y_{j+1}}\right]+4(1-d^2)\\
&\le 2C\cdot\operatorname{Arctan}\frac{Cu}{x_j}+4(1-d^2).
\end{aligned}$$

It will be enough to show that $u/x_j\to 0$ as $w\to\phi$.

Using (4.1), (4.3), and (4.4) again, we see that

$$\rho^2(z_j,w)\le\frac{(1-u/x_j)^2+C}{(1+u/x_j)^2+C}\le 1.$$

But $\rho(z_j,w)\to 1$ as $w\to\phi$, and since ϕ is more tangential than Γ, u/x_j is bounded above by $(1+\delta^2)/(1-\delta^2)$. It follows that $u/x_j\to 0$, so $|B(w)|\to 1$.

Next, take ψ less tangential than Γ. For w on Γ, $|B(w)|$ is clearly bounded away from 1. It will be enough to show $|B(w)|$ is bounded from 1 along the positive real axis, and we shall do so by showing $\sum(1-\rho^2(w,z_k))$ is bounded away from 0.

Let $w=u$ and let $y_*=\sup_{x_k\le u} y_k$. Then

$$\sum_k (1-\rho^2(w,z_k)) \ge \sum_{k,\,x_k\le u} \frac{4x_k u}{(x_k+u)^2+y_k^2}$$
$$\ge C\cdot \sum_{k,\,x_k\le u} \frac{u(y_{k+1}-y_k)}{4u^2+y_k^2}$$
$$\ge C\cdot \int_{y_1}^{y_*} \frac{u}{4u^2+y^2}\,dy$$
$$=\frac{C}{2}\cdot\left[\operatorname{Arctan}\frac{y_*}{2u}-\operatorname{Arctan}\frac{y_1}{2u}\right]$$

for some positive constant C. As $w\to\psi$, y_*/u is bounded away from 0 while y_1/u approaches 0. It follows that $\sum(1-\rho^2(w,z_k))$ is also bounded away from 0. This proves the second statement. □

THEOREM 4.2. *Let (z_n) be a sequence of points ordered along a K-curve Γ. This sequence is thin if and only if $\rho(z_n,z_{n+1})$ tends to 1 as n approaches ∞.*

Proof. Clearly, it is necessary that $\rho(z_n,z_{n+1})$ tends to 1 in order that (z_n) be thin. To see that this is sufficient will require an estimate similar to the one used in the last theorem. Again, we assume (z_n) and Γ are the corresponding sequence and arc in the RHP. Let

$$z_n=x_n+iy_n \quad\text{and}\quad d_n=\inf_{k\ne n}\rho(z_k,z_n).$$

In order to show $\prod_{k,\,k\ne n}\rho(z_k,z_n)\to 1$ as $n\to\infty$, it will be enough to check that $\sum_{k,\,k\ne n}(1-\rho^2(z_k,z_n))\to 0$. This follows from the fact that

$$\text{(4.5)}\qquad \frac{x_n}{y_n-y_{n-1}}\to 0 \quad\text{and}\quad \frac{x_n}{y_{n+1}-y_n}\to 0 \quad\text{as } n\to\infty;$$

indeed,

$$\sum_{k,\,k\ne n}(1-\rho^2(z_k,z_n)) \le \sum_1^{n-2}+\sum_{n+2}^{\infty}\frac{4x_kx_n}{(x_k+x_n)^2+(y_k-y_n)^2}+2(1-d_n^2)$$
$$\le C\cdot\sum_1^{n-2}\frac{x_n(y_{k+1}-y_k)}{x_n^2+(y_k-y_n)^2}$$
$$+C\cdot\sum_{n+2}^{\infty}\frac{x_n(y_k-y_{k-1})}{x_n^2+(y_k-y_n)^2}+2(1-d_n^2)$$
$$\le C\cdot\int_0^{y_{n-1}}\frac{x_n}{x_n^2+(y-y_n)^2}\,dy$$
$$+C\cdot\int_{y_{n+1}}^{\infty}\frac{x_n}{x_n^2+(y-y_n)^2}\,dy+2(1-d_n^2)$$
$$\le C\cdot\operatorname{Arctan}\frac{x_n}{y_n-y_{n-1}}+C\cdot\operatorname{Arctan}\frac{x_n}{y_{n+1}-y_n}+2(1-d_n^2)$$

for some positive constant C. As n increases this last quantity tends to 0, proving our result. It remains for us to show that (4.5) holds.

From (4.1), note that

$$\rho(z_n, z_{n+1}) = \frac{\left(\dfrac{x_{n+1}-x_n}{y_{n+1}-y_n}\right)^2+1}{\left(\dfrac{x_{n+1}+x_n}{y_{n+1}-y_n}\right)^2+1} \leq 1.$$

As n increases, $\rho(z_n, z_{n+1})$ converges to 1 and $(x_{n+1}-x_n)/(y_{n+1}-y_n)$ tends to 0, since these points lie on a K-curve; hence, the quantity $(x_{n+1}+x_n)/(y_{n+1}-y_n)$ also tends to 0. This proves (4.5). □

Notice that we actually proved a stronger result; it is stated as a corollary.

COROLLARY 4.3. *If Γ is a K-curve, $B(z)$ an interpolating Blaschke product with zero sequence (z_n) along Γ, and (w_k) a sequence of points on Γ for which $\rho(w_k, (z_n))$ tends to 1 as k increases, then $|B(w_k)|$ converges to 1.*

5. Separation of Support Sets

Our major concern now is to recognize when support sets are separated. Hoffman's work (unpublished) shows that either two support sets are disjoint or one is contained in the other. A sufficient condition for the latter possibility to occur concerns Gleason parts.

PROPOSITION 5.1. *Let ϕ and ψ be points in M. If ϕ is in $\overline{P_\psi}$ then the support of μ_ϕ is contained in the support of μ_ψ.*

Proof. Suppose x is a point in supp μ_ϕ, yet not in supp μ_ψ. Take a nonnegative L^∞ function f which vanishes on the support of μ_ψ and is positive at x. Then f vanishes identically on P_ψ, yet $f(\phi)>0$. This shows that ϕ is not in $\overline{P_\psi}$. □

This condition is certainly not necessary: some points of the Shilov boundary lie in analytic support sets, yet Shilov points are never in the closure of an analytic part.

There are two simple techniques we shall use to separate support sets; they are described in the next two lemmas.

LEMMA 5.2. *If $B(z)$ is a Blaschke product, ϕ and ψ belong to M, and $|B(\psi)|<1$, $|B(\phi)|=1$, then the support of μ_ψ is not contained in the support of μ_ϕ.*

One can see this by viewing points in M as integrations against representing measures over support sets.

If E is a measurable subset of the boundary of D, denote its characteristic function by χ_E.

LEMMA 5.3. *For ϕ and ψ in M:*

(a) *the support sets of ϕ and ψ are disjoint if and only if, for some measurable subset E of the unit circle, $\chi_E(\phi)=1$ and $\chi_E(\psi)=0$;*
(b) *the support of μ_ψ is contained in the support of μ_ϕ if and only if, for all measurable subsets E with $\chi_E(\phi)=1$, we have $\chi_E(\psi)=1$.*

These results hold because $L^\infty=C(X)$ and the sets $\{\phi\in M:\chi_E(\phi)=1\}$ form a basis for the topology of X.

Of course, the support sets of points in different fibers can always be separated. Upper- and lower-tangential support sets of M_1 can be separated as in Lemma 5.3(a) by taking E to be the upper half of the boundary of D. Thus, we are safe in restricting our attention to the Stolz and upper-tangential points of M_1. Combining Lemma 5.2 with Theorem 4.1, we immediately deduce the following theorem.

THEOREM 5.4. *Let Γ be a K-curve approaching 1 from above and let ϕ and ψ be points in M with ϕ more tangential than Γ, and ψ either in $A(\Gamma)$ or less tangential than Γ. Then the support of μ_ψ is not contained in the support of μ_ϕ.*

In Proposition 2.11 we showed that thin parts are maximal. We say that a support set is maximal if it is not properly contained in any other support set. Hoffman (unpublished) has shown that thin points also have maximal support sets. This follows rather easily from the next result, which is presented for completeness but can also be found in [8]. Note that both versions of the proof rely in an essential manner on an estimate of Hoffman of where an interpolating Blaschke product can be small.

PROPOSITION 5.5. *Let $B(z)$ be a Blaschke product whose zero sequence (z_n) is thin. Then $|B(\phi)|<1$ if and only if B vanishes at some point in P_ϕ.*

Proof. If $B(\psi)=0$ and ψ is in P_ϕ, then certainly $|B(\phi)|<1$.

Suppose that $|B(\phi)|=d<1$. Discarding a finite number of zeros of B, we may assume

$$\prod_{n,\, n\neq k}\left|\frac{z_n-z_k}{1-\overline{z_k}z_n}\right|\geq\sigma$$

for σ as near to 1 as we like. Note that this does not change the value of $|B|$ on $M-D$.

Choose δ strictly between d and 1. For some sequence (w_k) in D we have $\phi\in\overline{(w_k)}$ and $|B(w_k)|<\delta$. By a result of Hoffman [11], for some $\xi<1$ the set of points in D where $|B(z)|<\delta$ is contained in the union of pseudo-hyperbolic discs about the z_n with radius ξ. (To get this for δ arbitrarily near 1 we need to be able to choose σ arbitrarily near 1.) Thus, the sequence (w_k) lies entirely in the union of these discs. Using the lower semi-continuity of ρ [11], we see that $\overline{(z_n)}$ meets P_ϕ. It follows that B vanishes somewhere on P_ϕ. □

COROLLARY 5.6 (originally due to Hoffman). *If ϕ lies in the closure of a thin sequence then the support of μ_ϕ is maximal.*

Proof. Let (z_n) be a thin sequence containing ϕ in its closure. Take ψ in $M-P_\phi$. The set $P_\psi \cap \overline{(z_n)}$ consists of at most one point, so by adjusting (z_n) we may in fact assume $P_\psi \cap \overline{(z_n)}$ is empty. Using Proposition 5.5 and Lemma 5.2 we can conclude that the support of μ_ϕ is not contained in the support of μ_ψ. Since ψ was arbitrary, the support of μ_ϕ must be maximal. □

Now we would like to describe the situation in the closure of a K-curve. For this case the converse of Proposition 5.1 is true.

THEOREM 5.7. *Let Γ be a Stolz curve or a K-curve approaching* 1 *from above, and let $\phi, \psi \in \bar{\Gamma}-\Gamma$. Then* $\operatorname{supp}\mu_\phi \subset \operatorname{supp}\mu_\psi$ *if and only if $\phi \in \overline{P_\psi}$.*

Proof. The easy half being contained in Proposition 5.1, we may assume ϕ is not in $\overline{P_\psi}$. Working in the RHP, we shall construct a subset E of the boundary of the RHP such that $\chi_E(\psi)=1$ and $\chi_E(\phi)<1$. Combining this with Lemma 5.3(b) will give the desired conclusion.

First consider the case where Γ is a Stolz curve. Since our result concerns Gleason parts rather than points, and because the closure of the sequence $(2^n)_{n=1}^{\infty}$ meets every Stolz Gleason part, we may assume $\phi, \psi \in \overline{(2^n)}$. Take disjoint open sets U and V with ϕ in U and $\overline{P_\psi}$ contained in V. Take nets $2^{m(\alpha)} \to \psi$ and $2^{n(\beta)} \to \phi$. We assume that the latter net lies entirely in U. For any integer k, $2^{m(\alpha)+k}$ will converge to a point in P_ψ; hence, for any positive integer K there exists α_K in the directed set such that $(2^{m(\alpha)+k})_{k=-K}^{K} \subset V$ for all $\alpha \geq \alpha_K$.

We define a certain subset of the boundary of the RHP. Let

$$E^* = \bigcup_{K=1}^{\infty} \bigcup_{\alpha \geq \alpha_K} [i2^{m(\alpha)-K}, i2^{m(\alpha)+K}],$$

$$E = E^* \cup \overline{E^*} \quad \text{(here “}\overline{}\text{” denotes complex conjugate).}$$

Extending χ_E by its Poisson integral, we see that

$$\chi_E(2^{m(\alpha)}) \geq 2\chi_{[i2^{m(\alpha)-K}, i2^{m(\alpha)+K}]}(2^{m(\alpha)}) \geq \frac{2}{\pi}(\operatorname{Arctan} 2^K - \operatorname{Arctan} 2^{-K})$$

and

$$\chi_E(2^{n(\beta)}) \leq 1 - \frac{2}{\pi}\left(\operatorname{Arctan} 2 - \operatorname{Arctan} \frac{1}{2}\right)$$

for all $\alpha \geq \alpha_K$ and all β. Thus, $\chi_E(\psi)=1$ and $\chi_E(\phi)<1$, which completes the proof of the theorem. □

Now suppose Γ is a K-curve, (z_n) is a sequence of points along Γ equally spaced with respect to pseudo-hyperbolic distance, and ϕ, ψ lie in $\overline{(z_n)}$. We assume $(z_{n(\alpha)})$ and $(w_{n(\beta)})$ are subnets of (z_n) converging, respectively, to ψ and ϕ, and that $\phi \notin \overline{P_\psi}$. Separate ϕ and $\overline{P_\psi}$ with disjoint open sets U and V, say ϕ in U and $\overline{P_\psi}$ contained in V. Again, we may assume the net $(w_{n(\beta)})$ lies entirely in U. As in the Stolz case, for each positive integer K there is an α_K such that for $\alpha \geq \alpha_K$, $(z_{n(\alpha)+k})_{k=-K}^{K} \subset V$.

In order to simplify the notation for the remainder of this argument, we will write α for $n(\alpha)$ and $\alpha+k$ for $n(\alpha)+k$.

Let $z_\alpha = x_\alpha + iy_\alpha$ and $w_\beta = u_\beta + iv_\beta$, and define

$$E = \bigcup_{K=1}^{\infty} \bigcup_{\alpha \geq \alpha_K} [iy_{\alpha-K}, iy_{\alpha+K}].$$

Extending χ_E by its Poisson integral and taking $\alpha \geq \alpha_K$ for some large K, we have

$$\begin{aligned}\chi_E(z_\alpha) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \chi_E(t)\frac{x_\alpha}{x_\alpha^2+(y_\alpha-t)^2}\,dt \\ &\geq \frac{1}{\pi}\int_{y_{\alpha-K}}^{y_{\alpha+K}} \frac{x_\alpha}{x_\alpha^2+(y_\alpha-t)^2}\,dt \\ &= \frac{1}{\pi}\left[\operatorname{Arctan}\frac{y_{\alpha+K}-y_\alpha}{x_\alpha} - \operatorname{Arctan}\frac{y_{\alpha-K}-y_\alpha}{x_\alpha}\right]\end{aligned}$$

and

$$\begin{aligned}\chi_E(w_\beta) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \chi_E(t)\frac{u_\beta}{u_\beta^2+(v_\beta-t)^2}\,dt \\ &\leq 1-\frac{1}{\pi}\int_{v_{\beta-1}}^{v_{\beta+1}} \frac{u_\beta}{u_\beta^2+(v_\beta-t)^2}\,dt \\ &= 1-\frac{1}{\pi}\left[\operatorname{Arctan}\frac{v_{\beta+1}-v_\beta}{u_\beta} - \operatorname{Arctan}\frac{v_{\beta-1}-v_\beta}{u_\beta}\right].\end{aligned}$$

For each α let $K(\alpha)$ denote the largest integer K for which $\alpha_K \leq \alpha$. Note that $K(\alpha) \to \infty$ as $\alpha \to \infty$. It will be enough to show

(i) $(y_{\alpha+K(\alpha)} - y_\alpha)/x_\alpha \to \infty$ as $\alpha \to \infty$,
(ii) $(y_\alpha - y_{\alpha-K(\alpha)})/x_\alpha \to \infty$ as $\alpha \to \infty$,
(iii) $(v_{\beta+1} - v_\beta)/u_\beta \geq C$, and
(iv) $(v_\beta - v_{\beta-1})/u_\beta \geq C$

for some positive constant C. Statements (iii) and (iv) follow immediately from (4.3) and (4.4). Writing

$$\rho(z_\alpha, z_{\alpha+K(\alpha)})^2 = \frac{\left(\dfrac{x_{\alpha+K(\alpha)}-x_\alpha}{y_{\alpha+K(\alpha)}-y_\alpha}\right)^2+1}{\left(\dfrac{x_{\alpha+K(\alpha)}+x_\alpha}{y_{\alpha+K(\alpha)}-y_\alpha}\right)^2+1}$$

and noting that

$$\frac{x_{\alpha+K(\alpha)}-x_\alpha}{y_{\alpha+K(\alpha)}-y_\alpha} \to 0 \quad \text{and} \quad \rho(z_\alpha, z_{\alpha+K(\alpha)}) \to 1 \quad \text{as } z \to \infty,$$

we see that (i) holds. A similar argument is used to show that (ii) is true. Hence, $\chi_E(\psi)=1$ while $\chi_E(\phi)<1$.

For a K-curve or a Stolz curve Γ, recall that $A(\Gamma)$ is an open union of parts. This means that if Γ_1 and Γ_2 are two K-curves approaching 1, one more tangential than the other, and if ϕ and ψ are points of M_1 in $\overline{\Gamma_1}$ and $\overline{\Gamma_2}$, then $\overline{P_\phi}$ and $\overline{P_\psi}$ do not meet. Support sets of points in $\overline{\Gamma_1}$ and $\overline{\Gamma_2}$, however, are not separated this way; hence, the converse to Theorem 5.7 fails badly—even for analytic points.

THEOREM 5.8. *Let Γ_1 and Γ_2 be two curves in D tending to 1 with Γ_2 more tangential than Γ_1. Let ϕ belong to $\overline{\Gamma_1}-\Gamma_1$. Then for some ψ in $\overline{\Gamma_2}$, the support of μ_ψ is contained in the support of μ_ϕ.*

Proof. Suppose at first that ϕ is not a Stolz point. Continuing to work in the RHP, let Ω denote the collection of measurable subsets of E of the imaginary axis with the property that $\chi_E(\phi)=1$. For each E in Ω let A_E denote the set of points in $\overline{\Gamma_2}$ where χ_E takes the value 1. Since L^∞ functions are continuous on M, A_E is closed and hence compact. Let $A=\bigcap_{E\in\Omega} A_E$. By Lemma 5.3(b) it will be enough to show that A is nonempty. The intersection of a finite collection of sets in Ω is again in Ω, and $\bigcap_{k=1}^n A_{E_k}=A_{\bigcap E_k}$. Together with the compactness of the sets A_E, this implies it is enough to know that A_E is never empty.

Fix a measurable subset E of the imaginary axis. For z in the RHP and $0<\theta<\pi/2$ we define

$$I_\theta(z)=\{(0,y)\in\partial\text{RHP}: |y-\operatorname{Im} z|\le(\operatorname{Re} z)(\tan\theta)\};$$

$$R_\theta(z)=\{(x,y)\in\text{RHP}: |y-\operatorname{Im} z|\le|x-\operatorname{Re} z|(\tan\theta)\}.$$

Let $m(\theta)$ denote the Poisson integral of $\chi_{I_\theta(z)}$ evaluated at z; note that $m(\theta)$ is independent of z, and $m(\theta)\to1$ as $\theta\to\pi/2$.

Assume A_E is empty. For some $a>0$ and all w in Γ_2 with sufficiently large imaginary part, we have $\chi_E(w)\le1-a$. We now restrict our attention to such w and choose θ near enough to $\pi/2$ that $m(\theta)=1-a/2$. Denote the Lebesgue measure of a set S by $|S|$. We need to estimate $|I_\theta(w)\cap E|$. Note that $\chi_{I_\theta(w)-E}(w)\ge a/2$. Choose θ_1 so that $m(\theta_1)=a/2$. Then $I_{\theta_1}(w)$ is the subset of $I_\theta(w)$ with the smallest Lebesgue measure whose characteristic function at w is at least $a/2$; hence, $|I_\theta(w)-E|\ge|I_{\theta_1}(w)|=b|I_\theta(w)|$, where b is a constant depending only on a $(0<b<1)$. Thus,

$$\frac{|I_\theta(w)\cap E|}{|I_\theta(w)|}\le1-b.$$

Choose a point z of Γ_1 so large that if $w\in R_\theta(z)\cap\Gamma_2$, then $\chi_E(w)\le1-a$. Select a finite sequence of points $w_1,\ldots,w_n$ on Γ_2 so that

(i) the $I_\theta(w_k)$ are pairwise disjoint (aside from endpoints),
(ii) $I_\theta(z)\subset\bigcup_1^n I_\theta(w_k)$, and
(iii) $\sum_{I_\theta(w_k)\subset I_\theta(z)}|I_\theta(w_k)|\ge|I_\theta(z)|/2$.

Now,

$$\begin{aligned}|I_\theta(z)\cap E| &= \sum_1^n |I_\theta(z)\cap I_\theta(w_k)\cap E| \\ &= \sum_{I_\theta(w_k)\subset I_\theta(z)} |I_\theta(w_k)\cap E| + \sum_{I_\theta(w_k)\not\subset I_\theta(z)} |I_\theta(z)\cap I_\theta(w_k)\cap E| \\ &\le (1-b)\sum_{I_\theta(w_k)\subset I_\theta(z)} |I_\theta(w_k)| + ((1-b)+b)\sum_{I_\theta(w_k)\not\subset I_\theta(z)} |I_\theta(z)\cap I_\theta(w_k)| \\ &\le (1-b)|I_\theta(z)| + b\sum_{I_\theta(w_k)\not\subset I_\theta(z)} |I_\theta(z)\cap I_\theta(w_k)| \\ &\le (1-b)|I_\theta(z)| + b\frac{|I_\theta(z)|}{2} \\ &\le \left(1-\frac{b}{2}\right)|I_\theta(z)|.\end{aligned}$$

So for some $\delta > 0$, $\chi_E(z) \le 1-\delta$. Since this is true for all large z, we conclude that $\chi_E(\phi) \le 1-\delta$, so that $E \notin \Omega$; hence, for all E in Ω, A_E is nonempty. It follows that for some ψ in $\overline{\Gamma_2}$ we have $\operatorname{supp}\mu_\psi \subset \operatorname{supp}\mu_\phi$.

In case ϕ is a Stolz point, the only place the preceding argument might fail is in our ability to choose z so large that those w's in $R_\theta(z)$ are also sufficiently large. At this point we choose a ray Γ from 0 to ∞ making an angle of $(\pi-\theta)/4$ with the positive imaginary axis. As z approaches ∞ along Γ, $I_\theta(z)$ is a set whose values also tend to ∞. Choose a point $\tilde{\phi}$ of $\bar{\Gamma}$ which is in the same Gleason part as ϕ. Note that this does not change the set Ω. Continuing our argument with $\tilde{\phi}$ in place of ϕ, we complete the proof. □

In the above proof we never really used the fact that ϕ lies in the closure of a curve. Indeed, we actually proved the following.

COROLLARY 5.9. *Let ϕ be a point in M_1 lying in the closure of some subset S of the upper semi-disc. Suppose Γ is an upper tangential curve in D tending to* 1 *strictly between S and the upper semi-circle. Then, for some ψ in $\bar{\Gamma}$, the support of μ_ψ is contained in the support of μ_ϕ.*

In particular, the support set of any trivial point lying in the closure of such a set S (of which there are many, by Corollary 2.8) properly contains support sets of analytic points.

Another way to see this uses a result of Axler and Gorkin [2]. They have shown that if b is a Blaschke product, $\varphi \in M$, and $|b(\varphi)| < 1$, then for some $\psi \in M$ for which $\operatorname{supp}\mu_\psi \subset \operatorname{supp}\mu_\varphi$, $b(\psi) = 0$. Combining this with Ziskind's result [16] that for every point φ not in the Shilov boundary there is an interpolating Blaschke product b such that $|b(\varphi)| < 1$, we see again that the support sets for trivial points outside of the Shilov boundary contain support sets of analytic points. What Corollary 5.9 adds to this is a relationship between the orders of tangency of the trivial point and the analytic part.

A final consequence of Theorem 5.8 is that there are analytic parts which do not lie in the closure of a thin part. We can see this as follows. Let ϕ be a thin Stolz point in M_1. Let Γ be an oricycle approaching 1. By Theorem 5.8 there exists $\psi \in \bar{\Gamma}$ such that $\operatorname{supp} \mu_\psi \subset \operatorname{supp} \mu_\phi$. If $\psi \in \overline{P_\eta}$ for some thin point η, then one can show P_η also meets Γ. Furthermore, the supports of μ_ϕ and μ_η both contain $\operatorname{supp} \mu_\psi$, and both are maximal. It follows that they are identical, yet Theorem 5.4 shows that this cannot happen. Hence, ψ does not lie in the closure of a thin part. It is not known whether every part lies in the closure of a maximal part, but if this is true, it is then clear that there must be maximal parts which are not thin.

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