On the Coefficients of the Mapping to the Exterior of the Mandelbrot Set

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Introduction

The Mandelbrot set M arises in the dynamics of complex quadratic polynomials $q_w(z) = z^2 + w$. It consists of those parameter values w such that the Julia set obtained from q_w is connected. The complement $\tilde{M}$ of M in the Riemann sphere is known [1] to be simply connected and to have mapping radius equal to 1. Thus we may consider the analytic homeomorphism

(1) $$\psi(z) = z + \sum_{m=0}^{\infty} b_m z^{-m}$$

of $\Delta = \{z : 1 < |z| \leq \infty\}$ onto $\tilde{M}$. It is our purpose to give a useful formula for the coefficients b_m and to show that many of these coefficients are zero. We also determine infinitely many nonzero coefficients, and conclude with a description of the Faber polynomials of the Mandelbrot set. Our work is motivated by an article by Jungreis [2].

Background

Define recursively the polynomials $p_1(w) = w^2 + w$ and

(2) $$p_n(w) = p_{n-1}(w)^2 + w$$

for $n > 1$. Evidently, p_n is a monic polynomial of degree 2^n. It is known [2] that the zeros of p_n all lie in M, and so it is possible to define in $\tilde{M}$ a single-valued branch of $p_n(w)^{1/2^n} = w + O(1)$ as $w \to \infty$. In what follows we shall also have use for $p_n^{m/2^n} \equiv [p_n^{1/2^n}]^m$ for positive integers m.

It turns out [2] that the functions $\phi_n \equiv p_n^{1/2^n}$ converge locally uniformly in $\tilde{M}$ to $\phi = \psi^{-1}$, the inverse of the mapping ψ. Near ∞ the functions ϕ_n are one-to-one, and their inverse functions $\psi_n \equiv \phi_n^{-1}$ evidently satisfy

(3) $$p_n(\psi_n(z)) = z^{2^n}.$$

The functions ψ_n are defined in larger and larger subsets of Δ as $n \to \infty$, and converge locally uniformly in Δ to ψ. In fact, the following lemma shows that this convergence is remarkably strong.

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LEMMA 1. *$\psi(z)=\psi_n(z)+O(1/z^{2^{n+1}-2})$ as $z\to\infty$.*

This lemma is the corollary to Theorem 4 in [2]. For completeness, we include a proof since we do not understand the application of the integral in [2].

Proof of Lemma 1. It is sufficient to show that

$$\psi_{n+1}(z)-\psi_n(z)=O(1/z^{2^{n+1}-2})$$

as $z\to\infty$. Write

$$p_{n+1}(w)-p_{n+1}(\omega)=[w-\omega][w^{2^{n+1}-1}+w^{2^{n+1}-2}\omega+\cdots+\omega^{2^{n+1}-1} + \text{lower powers of } w \text{ and } \omega]$$

to see that

$$p_{n+1}(\psi_{n+1}(z))-p_{n+1}(\psi_n(z))=[\psi_{n+1}(z)-\psi_n(z)] \times[2^{n+1}z^{2^{n+1}-1}+\text{lower powers of } z]. \tag{4}$$

On the other hand, (2) and (3) imply that

$$\begin{aligned} p_{n+1}(\psi_{n+1}(z))-p_{n+1}(\psi_n(z)) &= p_{n+1}(\psi_{n+1}(z))-p_n(\psi_n(z))^2-\psi_n(z) \\ &= z^{2^{n+1}}-z^{2^{n+1}}-\psi_n(z)=-z+O(1) \end{aligned} \tag{5}$$

as $z\to\infty$. The result follows from comparing (4) and (5). □

Coefficient Formula

A useful formula for the coefficients in (1) is contained in the following.

THEOREM 1. *If $1\le m\le 2^{n+1}-3$ and R is sufficiently large, then*

$$-mb_m=\frac{1}{2\pi i}\int_{|w|=R} p_n(w)^{m/2^n}\,dw. \tag{6}$$

REMARKS. To use formula (6) for a particular coefficient b_m, one may choose any n sufficiently large that $m\le 2^{n+1}-3$. Of course, the circle $|w|=R$ can be replaced by any simple closed rectifiable curve that surrounds the Mandelbrot set once in the positive sense. Furthermore, the right side is just the coefficient of $1/w$ in the Laurent expansion of $p_n(w)^{m/2^n}$ about ∞. As a result, this formula is particularly well-suited to precise computation by any of the symbolic manipulation programs now available on computers.

Proof of Theorem 1. If $1\le m\le 2^{n+1}-3$ and R is sufficiently large, then Lemma 1 and a change of variables imply that

$$\begin{aligned} -mb_m &= \frac{1}{2\pi i}\int_{|z|=R} z^m\psi_n'(z)\,dz=\frac{1}{2\pi i}\int_{|w|=R}\phi_n(w)^m\,dw \\ &= \frac{1}{2\pi i}\int_{|w|=R} p_n(w)^{m/2^n}\,dw. \end{aligned}$$ □

Zero Coefficients

If $m=2^n$, then $2^n \le 2^{n+1}-3$ is valid for $n \ge 2$. In this case, the integrand in (6) is a polynomial and so the integral is zero. This conclusion was first observed with a different proof in [2].

THEOREM 2 (Jungreis). $b_{2^n}=0$ *for* $n\ge 2$.

We shall now show that many more coefficients are zero.

THEOREM 3. *For any integers* k *and* ν *satisfying* $k\ge 1$ *and* $2^\nu \ge k+3$, *let* $m=(2k+1)2^\nu$. *Then* $b_m=0$.

Proof. We shall use the (generalized) binomial coefficients defined for real α and $|x|<1$ by $(1+x)^\alpha=\sum_{j=0}^{\infty} C_j(\alpha)x^j$. Then for $|w|$ sufficiently large the recursion (2) leads to the expansions

$$
\begin{aligned}
p_n(w)^{m/2^n} &= [p_{n-1}(w)^2+w]^{m/2^n} = p_{n-1}(w)^{m/2^{n-1}}\left[1+\frac{w}{p_{n-1}(w)^2}\right]^{m/2^n} \\
&= \sum_{j_1=0}^{\infty} C_{j_1}(m/2^n)w^{j_1}p_{n-1}(w)^{m/2^{n-1}-2j_1} \\
&= \sum_{j_1=0}^{\infty} C_{j_1}(m/2^n)w^{j_1}p_{n-2}(w)^{m/2^{n-2}-2^2 j_1}\left[1+\frac{w}{p_{n-2}(w)^2}\right]^{m/2^{n-1}-2j_1} \\
&= \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty} C_{j_1}(m/2^n)C_{j_2}(m/2^{n-1}-2j_1) \\
&\qquad \times w^{j_1+j_2}p_{n-2}(w)^{m/2^{n-2}-2^2j_1-2j_2} = \cdots \\
&= \sum_{j_1=0}^{\infty}\cdots\sum_{j_N=0}^{\infty} C_{j_1}(m/2^n)\cdots C_{j_N}(m/2^{n-N+1}-2^{N-1}j_1-\cdots-2j_{N-1}) \\
&\qquad \times w^{j_1+\cdots+j_N}p_{n-N}(w)^{m/2^{n-N}-2^Nj_1-\cdots-2j_N}.
\end{aligned} \tag{7}
$$

Now let $m=(2k+1)2^\nu$, where k and ν are any fixed integers satisfying $k\ge 1$ and $2^\nu\ge k+3$. Choose n larger than ν and sufficiently large that $m\le 2^{n+1}-3$. Let $N=n-\nu$ so that $n-N=\nu$. Then the final factors in (7) reduce to

$$w^{j_1+\cdots+j_N}p_\nu(w)^{2k+1-2^Nj_1-\cdots-2j_N}. \tag{8}$$

To show that

$$-mb_m=\frac{1}{2\pi i}\int_{|w|=R} p_n(w)^{m/2^n}\,dw=0,$$

we divide the indices $j_1,\dots,j_N$ into two sets. Let

$$u=2k+1-2^Nj_1-\cdots-2j_N. \tag{9}$$

If $u\ge 0$, then we are integrating a polynomial in (8) and the integral is zero. So it remains to consider the case $u\le -1$. In this case the factors (8) are of the form

$$w^{j_1+\cdots+j_N+2^\nu u}[1+O(1/w)] \tag{10}$$

as $w \to \infty$. We shall show that the exponent satisfies $j_1 + \cdots + j_N + 2^\nu u \leq -2$, so that the integrals of these terms are also zero.

It follows from (9) that

$$j_N = \tfrac{1}{2}[2k+1-u] - 2^{N-1}j_1 - \cdots - 2j_{N-1}$$

and

$$j_1 + \cdots + j_N = \tfrac{1}{2}[2k+1-u] - (2^{N-1}-1)j_1 - \cdots - j_{N-1} \leq \tfrac{1}{2}[2k+1-u]. \tag{11}$$

Therefore we have

$$\begin{aligned} j_1 + \cdots + j_N + 2^\nu u &\leq \tfrac{1}{2}[2k+1-u] + 2^\nu u = \tfrac{1}{2}[2k+1+(2^{\nu+1}-1)u] \\ &\leq \tfrac{1}{2}[2k+1-(2^{\nu+1}-1)] = k+1-2^\nu. \end{aligned} \tag{12}$$

Since $2^\nu \geq k+3$, this is at most -2. □

Some Nonzero Coefficients

On the basis of Theorems 2 and 3, the coefficients b_m are zero with indices $m = 4, 8, 16, 32, \ldots, 12, 24, 48, 96, \ldots, 40, 80, 160, 320, \ldots, 56, 112, 224, 448, \ldots$, etc. So far, computations have not produced a zero coefficient besides those indicated in Theorems 2 and 3. The following theorem shows that the condition $2^\nu \geq k+3$ in Theorem 3 is in some sense necessary and, more importantly, provides explicit values for infinitely many nonzero coefficients.

THEOREM 4. *For $\nu \geq 1$, let $m = (2^{\nu+1}-3)2^\nu$. Then*

$$b_m = -\frac{(2^{\nu+1}-4)!}{2^{2^{\nu+1}-3}2^\nu!\,(2^\nu-2)!}.$$

Proof. We proceed as in the proof of Theorem 3. Write $m = (2k+1)2^\nu$, where $k = 2^\nu - 2$, so that $2^\nu = k+2$. Fix n and N as before. For the same reasons, the indices that satisfy $u \geq 0$ contribute zero to the integral of (7). Therefore assume $u \leq -1$.

In this case (12) implies that the exponent in (10) satisfies

$$j_1 + \cdots + j_N + 2^\nu u \leq -1. \tag{13}$$

Because of the estimate (11), equality in (13) occurs precisely when $j_1 = \cdots = j_{N-1} = 0$, $u = -1$, and $j_N = k+1$. Thus the integral of (7) becomes $-mb_m = C_{k+1}(m/2^{\nu+1})$. In other words, we have

$$\begin{aligned} b_m &= \frac{-1}{m} C_{k+1}\left(k+\frac{1}{2}\right) \\ &= \frac{-1}{m}\frac{(2k+1)!}{2^{2k+1}k!\,(k+1)!} = -\frac{(2^{\nu+1}-4)!}{2^{2^{\nu+1}-3}2^\nu!\,(2^\nu-2)!}. \end{aligned}$$ □

REMARKS. One hopes that formula (6) can be used to provide estimates for all nonzero coefficients b_m. If one could show that the series (1) converges uniformly on $\bar{\Delta}$, then the mapping ψ would extend continuously to $|z|=1$. In this case it would follow that the boundary of the Mandelbrot set is locally connected—a well-known open problem.

On the basis of Theorem 4 we have some information about the growth of the coefficients b_m. It leaves open the possibility that $|b_m| \le A/m^{1+\epsilon}$ is true for some ϵ, $0<\epsilon\le \frac{1}{4}$.

COROLLARY. $\limsup_{m\to\infty} m^{5/4}|b_m| \ge 2^{1/4}/\sqrt{\pi}$.

Proof. A lengthy (but straightforward) application of Stirling's formula to the coefficients in Theorem 4 leads to the asymptotics $-b_m \sim 1/(2^{1+5\nu/2}\sqrt{\pi})$ as $\nu\to\infty$. Since $2^\nu \sim \sqrt{m/2}$ as $\nu\to\infty$, we have $m^{5/4}|b_m| \sim 2^{1/4}/\sqrt{\pi}$ as $m\to\infty$ through the given sequence. □

Faber Polynomials

Expand $p_n(w)^{m/2^n} = w^m + \sum_{k=-\infty}^{m-1} B_k w^k$ in a neighborhood of ∞. Because of (3), each B_k is a polynomial in the coefficients b_j of ψ_n. Furthermore, B_k depends only on $b_0, \ldots, b_{m-k-1}$. Therefore, if we define $\mathfrak{F}_m$ to be the polynomial part of $p_n^{m/2^n}$ at ∞, namely,

$$\mathfrak{F}_m(w) = w^m + \sum_{k=0}^{m-1} B_k w^k, \tag{14}$$

then $\mathfrak{F}_m$ depends only on $b_0, \ldots, b_{m-1}$. It is a consequence of Lemma 1 that $\mathfrak{F}_m$ is independent of n whenever $m \le 2^{n+1}-2$. In this case, we shall see that $\mathfrak{F}_m$ is the mth Faber polynomial of the Mandelbrot set M.

We shall show that $\mathfrak{F}_m(\psi(z)) = z^m + o(1)$ as $z\to\infty$, which is a characterization of the mth Faber polynomial of M (cf. [3, p. 131]). Since $\mathfrak{F}_m(w) - p_n(w)^{m/2^n} = o(1)$ as $w\to\infty$, it is clear that

$$\mathfrak{F}_m(\psi(z)) - p_n(\psi(z))^{m/2^n} = o(1) \tag{15}$$

as $z\to\infty$. Next, observe that

$$p_n(w)^{m/2^n} - p_n(\omega)^{m/2^n} = [w-\omega][w^{m-1} + w^{m-2}\omega + \cdots + \omega^{m-1} + \text{lower powers of } w \text{ and } \omega],$$

so that

$$p_n(\psi(z))^{m/2^n} - p_n(\psi_n(z))^{m/2^n} = [\psi(z) - \psi_n(z)][mz^{m-1} + \text{lower powers of } z].$$

For $m \le 2^{n+1}-2$, the latter is $o(1)$ as $z\to\infty$, by Lemma 1. Combining this with (15) and (3), we see that

$$\mathfrak{F}_m(\psi(z)) = p_n(\psi_n(z))^{m/2^n} + o(1) = z^m + o(1)$$

as $z\to\infty$. This shows that (14) defines the Faber polynomials of the Mandelbrot set whenever $m \le 2^{n+1}-2$. Note that they are easily constructed by symbolic manipulation programs on computers.

References

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