Crossed Product Criteria and Skew Linear Groups II

B. A. F. WEHRFRITZ

This paper concerns the class $\langle P, L\rangle(\mathfrak{A}\mathfrak{F})$ of groups built from the classes $\mathfrak{A}$ of abelian groups and $\mathfrak{F}$ of finite groups by repeated use of the local and poly operators L and P, transfinitely if necessary. The nature of the paper makes the use of Hall's calculus of group classes (see the opening pages of [3]) more of a necessity than a convenience. Our previous paper [7] with this title concerned the smaller class $\langle P, L\rangle\mathfrak{A}$.

Throughout this paper F denotes a (commutative) field, D a division F-algebra and n a positive integer. For any group G, $\tau(G)$ is the unique maximal locally finite normal subgroup of G, $\eta(G)$ the Hirsch–Plotkin radical of G, $\zeta_1(G)$ the centre and $\zeta_2(G)$ the second centre of G, and $\alpha(G)$ and $\beta(G)$ are defined by

$$\beta(G)/\tau(G)=\eta(G/\tau(G)) \quad \text{and} \quad \alpha(G)/\tau(G)=\zeta_1(\beta(G)/\tau(G)).$$

The core of our main result here, of which there are many corollaries, can be summarized as follows.

Let G be a primitive subgroup of $GL(n,D)$ with $G\in\langle P, L\rangle(\mathfrak{A}\mathfrak{F})$. Then the F-subalgebra $F[G]$ of the full matrix ring $D^{n\times n}$ generated by G is a crossed product over the locally-finite by abelian normal subgroup $\alpha(G)$ of G; that is, $F[G]$ is free as a left and right $F[\alpha(G)]$-module on any transversal of $\alpha(G)$ to G.

The proofs below depend heavily on the results and proofs of [7]. One difference perhaps we should specifically mention at the outset. Unlike [7], the major results here depend ultimately on the classification of the finite simple groups (in the weak form, that there exist only a finite number of sporadic groups). In this sense [7] and the present paper operate at different levels. When [7] was written I did not believe that comparable results for $\langle P, L\rangle(\mathfrak{A}\mathfrak{F})$ existed. In one way this is true: Unlike $\langle P, L\rangle\mathfrak{A}$-groups, $\langle P, L\rangle(\mathfrak{A}\mathfrak{F})$-groups in general do not have a "Zalesskiĭ" subgroup; specifically, our F-algebras need not be crossed products over normal FC-groups, canonical or otherwise. However the partial results obtained by working directly with $\alpha(G)$ do hold and are almost as useful.

Received July 19, 1989.
Michigan Math. J. 37 (1990).

We now prepare the ground for the statement of the main theorem. Our hypotheses are weaker and our conclusions stronger than the above suggests. First consider the class $\mathfrak{S}\mathfrak{F}$ of all soluble-by-finite groups. An $\mathfrak{S}\mathfrak{F}$-group has a characteristic soluble subgroup of finite index. It follows that $\mathfrak{S}\mathfrak{F}$ is poly closed and hence that

$$(*) \qquad \mathfrak{S}\mathfrak{F} = P(\mathfrak{S} \cup \mathfrak{F}).$$

There is a simple extension to $\mathfrak{S}\mathfrak{F}$-groups of the notion of the Zalesskiĭ subgroup of a soluble group as expounded in [2, p. 364] or [4, p. 188]. Thus, if G is a soluble-by-finite group then there is a particular characteristic FC-subgroup of G to be defined below, which we denote by Zal G and call the *Zalesskiĭ subgroup* of G. If G is actually soluble, Zal G coincides with the Zalesskiĭ subgroup of G as defined in [2] and [4].

If G is a primitive subgroup of $GL(n, D)$ and if N is a normal subgroup of G, then N is homogeneous by Clifford's theorem and so $F[N] \leq D^{n\times n}$ is prime (see [4, 1.1.6 and 1.1.14b]). Suppose G is a group with $\tau(G) \in \mathfrak{S}\mathfrak{F}$. Then $\alpha(G)$ also lies in $\mathfrak{S}\mathfrak{F}$ by $(*)$ above and consequently $\mathrm{Zal}(\alpha(G))$ is defined. We can now state the main theorem.

MAIN THEOREM. *Let G be a $\langle P, L\rangle(\mathfrak{K}\mathfrak{F})$-subgroup of $GL(n, D)$ such that the subalgebra $F[N]$ of $D^{n\times n}$ is a prime ring for every characteristic subgroup N of G.*

(a) *$F[G] \leq D^{n\times n}$ is a crossed product over $\alpha(G)$.*
(b) *If $\tau(G) \in \mathfrak{S}\mathfrak{F}$ then $F[G]$ is a crossed product over the characteristic FC-subgroup* $\mathrm{Zal}(\alpha(G))$ *of G.*
(c) *If* char $F = 0$ *or $n = 1$ then $F[G]$ is a crossed product over* $\mathrm{Zal}(\alpha(G))$.
(d) *If $\tau(G) \leq \zeta_2\alpha(G)$, then $F[G]$ is a crossed product over an abelian characteristic subgroup of G that depends only on G as a group.*
(e) *Suppose $\tau(G) \in \mathfrak{S}\mathfrak{F}$. Then there are characteristic subgroups $A \leq G_0$ of G, depending only on the group structure of G, such that A is abelian, $F[G_0]$ is a crossed product over A, and $(G : G_0)$ is finite.*
(f) *There is an integer-valued function $f(n)$ of n only, such that if* char $F =$ *0 or if $n = 1$ then A and G_0 exist as in* (e) *with $(G : G_0) \leq f(n)$.*

Unlike [7, 1.1a], here $F[G]$ need not be a crossed product over any normal FC-subgroup of G. Counterexamples are easy. By (c) of the theorem, they exist only for char $F = p > 0$ and $n > 1$. Let F be any infinite locally finite field (e.g., the algebraic closure of the field of p elements), and set $D = F$ and $G = SL(2, F)$. If Z is a normal FC-subgroup of G then Z is central, $(G : Z)$ is infinite, and $\dim_F F[G] = 4$ is finite. Thus $F[G]$ is not a crossed product over Z. The same example shows that G need not have a subgroup G_0 of finite index such that $F[G_0]$ is a crossed product over a normal FC-subgroup or over a normal abelian (or even a $\langle P, L\rangle\mathfrak{K}$) subgroup of G_0. Part (a) of the theorem seems to be the strongest general conclusion of this kind.

The theorem has a substantial number of corollaries. The first three below are more or less immediate from the theorem (cf. [7]). They depend upon the notion of control of ideals in group algebra (see [2, p. 8] etc. or, just for the definition, [7]).

COROLLARY 1. *Let F be a field, let G be a $\langle P,L\rangle(\mathfrak{RF})$-group, and let $\mathfrak{p}$ be an ideal of the group algebra FG such that $G\cap(1+\mathfrak{p})=\langle 1\rangle$, $FG/\mathfrak{p}$ is either left or right Goldie, and $\mathfrak{p}\cap FN$ is a prime ideal of FN for every characteristic subgroup N of G.*

(a) *$\mathfrak{p}$ is controlled by $\alpha(G)$.*
(b) *If $\tau(G)\in\mathfrak{SF}$ then $\mathfrak{p}$ is controlled by* $\mathrm{Zal}(\alpha(G))$.
(c) *If* $\operatorname{char} F=0$ *or if $FG/\mathfrak{p}$ is a domain then $\mathfrak{p}$ is controlled by* $\mathrm{Zal}(\alpha(G))$.
(d) *If $\tau(G)\le\zeta_2\alpha(G)$ then there is an abelian characteristic subgroup of G controlling all such $\mathfrak{p}$.*
(e) *Suppose $\tau(G)\in\mathfrak{SF}$. Then there are characteristic subgroups $A\le G_0$ of G such that A is abelian, the ideal $\mathfrak{p}\cap FG_0$ of FG_0 is controlled by A for all such $\mathfrak{p}$, and $(G:G_0)$ is finite.*

COROLLARY 2. *Let F be a field and G a $\langle P,L\rangle(\mathfrak{RF})$-group. Suppose $\mathfrak{p}$ is an ideal of FG such that $G\cap(1+\mathfrak{p})=\langle 1\rangle$, $FG/\mathfrak{p}$ is simple Artinian, and G does not permute any nontrivial set of orthogonal idempotents of $FG/\mathfrak{p}$ under conjugation. Then the hypotheses and hence the conclusions of Corollary 1 hold.*

COROLLARY 3. *Let F be a field and G a nonperiodic simple $\langle P,L\rangle(\mathfrak{RF})$-group. Then the only nonzero prime ideal $\mathfrak{p}$ of the group algebra FG with $FG/\mathfrak{p}$ left or right Goldie is the augmentation ideal of G in FG. If $R=F[G]$ is a prime F-subalgebra of some one-sided Artinian F-algebra, then R is the group algebra FG.*

We can use crossed product techniques to compute the normalizers of certain subgroups of $GL(n,D)$. The following is a corollary of the proof of the theorem, rather than of the theorem itself, in that it is immediate from 7.5 (and 3.1) of [7] and from a relativised version of the theorem (viz., Theorem 7 below).

COROLLARY 4. *Let H be a $\langle P,L\rangle(\mathfrak{RF})$-subgroup of $GL(n,D)$ such that $F[N]\le D^{n\times n}$ is prime for every characteristic subgroup N of H, and suppose that $D^{n\times n}$ is the (classical) ring $F(H)$ of quotients of its subalgebra $F[H]$. Assume $\tau(H)\le\zeta_1(H)$ and set $Z=\zeta_1\alpha(H)$. Then*

$$N_{GL(n,D)}(H)=H\cdot N_{F(Z)^*}(H).$$

Here the quotient field $F(Z)$ of $F[Z]$ is naturally embedded in $D^{n\times n}$ (e.g., by 3.1 and 3.3 of [7]), so the conclusion of the corollary is meaningful. Corollary 4 enables one to exploit the theorem to study absolutely irreducible, skew linear groups. (For basic definitions and results concerning skew linear groups, see [4].) Most of the proof of [7, 1.5] concerns the case where $H \in \langle P, L\rangle \mathfrak{A}$. If one repeats this proof with $H \in \langle P, L\rangle(\mathfrak{A}\mathfrak{F})$, using part (a) of the main theorem and Corollary 4 in the place of [7, 1.1a and 7.6], one obtains the following.

COROLLARY 5. *Let G be an absolutely irreducible subgroup of $GL(n, D)$ and H a normal $\langle P, L\rangle(\mathfrak{A}\mathfrak{F})$-subgroup of G. Then H is abelian-by-locally finite and $G/C_G(H)$ is abelian-by-periodic.*

This corollary stands at the end of a very long road, stretching back in some sense to Jordan (see [4, Chap. 5], [7, 1.5], and further references given in [7]). I think this is probably the end of this particular development. Corollary 5 contains all the results of this type that are known to me, and the class $\langle P, L\rangle(\mathfrak{A}\mathfrak{F})$ seems much more natural than some of its subclasses considered earlier (e.g., the class $PL(\langle P, L\rangle \mathfrak{A} \cup \mathfrak{F})$ of [7]).

What scope is there in Corollary 5 for widening further the class to which H is confined? For linear groups G we need only insist that H has no free subgroup of rank 2, for then Tits' theorem [5, 10.17] ensures that H lies in $\langle P, L\rangle(\mathfrak{A}\mathfrak{F})$ and indeed in $\mathfrak{S}L\mathfrak{F}$. Even if it is possible in Corollary 5 to weaken $H \in \langle P, L\rangle(\mathfrak{A}\mathfrak{F})$ to "H has no free subgroup of rank 2," we are still, I think, a very long way from proving it. Is there some intermediate class that is more accessible? One might as well assume that the class is $\langle S, Q, P, L\rangle$-closed initially, since otherwise the same questions arise about its $\langle S, Q, P, L\rangle$-closure. One possible candidate is the class $\langle P, L\rangle((\hat{P}\mathfrak{A})^{QS}\mathfrak{F})$; this class contains $\langle P, L\rangle(\mathfrak{A}\mathfrak{F})$, although whether it is strictly larger I do not know. Indeed I do not know of any group in $(\hat{P}\mathfrak{A})^{QS}$ $(=\overline{\overline{SN}}$ in the Kuroš notational scheme) that is not in $\langle P, L\rangle\mathfrak{A}$. Nonetheless, my feeling (based though on very little evidence) is that such groups must exist.

Corollary 5 can be translated into the language of group algebras as follows.

COROLLARY 6. *Let F be a field, G a group, $\mathfrak{m}$ an ideal of FG such that $FG/\mathfrak{m}$ is simple Artinian, and H a normal $\langle P, L\rangle(\mathfrak{A}\mathfrak{F})$-subgroup of G. Then H modulo $\mathfrak{m}$ and $G/C_G(H$ modulo $\mathfrak{m})$ are both abelian-by-periodic. If $G \in \langle P, L\rangle(\mathfrak{A}\mathfrak{F})$ then G modulo $\mathfrak{m}$ is abelian-by-locally finite.*

For a given field F let $\mathfrak{Y}_F$ (resp. $\mathfrak{Z}_F$) denote the class of groups G such that every primitive image of the group algebra FG satisfies a polynomial identity (resp. is Artinian). See [4, Chap. 6] for alternative definitions and basic results concerning these classes. Now $\mathfrak{Y}_F \subseteq \mathfrak{Z}_F$ for every field F. The following is immediate from Corollary 5 and [7, 9.1].

COROLLARY 7. *Let $\mathfrak{G}$ denote the class of finitely generated groups.*

(a) $\mathfrak{Z}_F \cap \mathfrak{G} \cap \langle P, L\rangle(\mathfrak{A}\mathfrak{F}) \subseteq \mathfrak{Y}_F$.

(b) *Suppose F is not locally finite. Then* $\mathfrak{Z}_F \cap \langle P, L\rangle(\mathfrak{A}\mathfrak{F}) \subseteq \mathfrak{Y}_F$.

We must now prove the main theorem. In view of the above remarks, the deduction of the corollaries can be left to the reader. We begin the main proofs with two lemmas concerning periodic skew linear groups.

1. LEMMA. *Let G be a subgroup of $GL(n,D)$, where* char $F=p>0$, *and set $T=\tau(G)$. Suppose $F[T]\leq D^{n\times n}$ is prime. Then $G/T\cdot C_G(T)$ is an extension of a finite group by an abelian group. (In fact, $G/T\cdot C_G(T)$ is an extension of a finite group of n-bounded order by a super-residually cyclic group.)*

We could instead make use of [4, 5.4.1], but Lemma 1 makes better use of the special situation we have below and thus avoids certain complications.

Proof. Since $F[T]$ is prime and $O_p(T)$ is unitriangular, $O_p(T)=\langle 1\rangle$. Let k be the prime subfield of D and set $S=k[T]\leq D^{n\times n}$. Then S is a simple Artinian ring (e.g., by [7, 2.1] and [4, 1.1.14(b) and (c)]), so S is a matrix ring, say of degree m, over some locally finite field K (all locally finite division rings are fields).

The group G induces automorphisms on S by conjugation. By the Skolem–Noether theorem [1, p. 364],

$$\operatorname{Aut} S\cong \operatorname{Aut} K^{m\times m}\cong P\Gamma L(m,K)=\operatorname{Aut} K\cdot PGL(m,K)$$

in an obvious way. Let U denote the group of units of S. Then G has a normal subgroup $N\geq T\cdot C_G(T)$ such that $N\leq U\cdot C_{GL(n,D)}(T)$ and G/N is isomorphic to a subgroup of Aut K. Since K is a locally finite field, Aut K is procyclic and G/N is abelian (even super-residually cyclic). Set $V=N_U(T)$. Clearly $N\leq V\cdot C_{GL(n,D)}(T)$. Also $U\cong GL(m,K)$, so by [4, 5.1.6] the group $V/T\cdot C_V(T)$ is finite (even of n-bounded order). But

$$(N\colon T\cdot C_G(T))\leq (V\colon T\cdot C_V(T)).$$

The proof is complete. □

Let $\mathfrak{X}$ denote the class of all groups of finite exponent. Curiously, this class plays an important role in the proof of the main theorem.

2. LEMMA. *Let G be a locally finite subgroup of $GL(n,D)$ with $G\in\mathfrak{S}\mathfrak{X}$. Then $G\in\mathfrak{S}\mathfrak{F}$.*

Proof. If char $D=0$ then $G\in\mathfrak{S}\mathfrak{F}$ by [4, 2.5.14]. Assume char $D=p>0$. Then $O_p(G)$ is unitriangularizable and hence nilpotent, and $G/O_p(G)$ is isomorphic to a linear group of degree n and characteristic p (by [4, 2.3.1]). Thus assume G is linear. We can now factor out by the maximal soluble normal subgroup of G (see [5, 5.9, 5.11 and 6.4]) and assume that G has finite exponent. Then $G/O_p(G)$ is finite by Burnside's theorem [5, 1.23], and the proof is complete. (Alternatively, G clearly cannot generate the variety of all groups, and Platonov's theorem [5, 10.15] then implies that G is soluble-by-finite.) □

3. *The Zalesskiĭ subgroup of a soluble-by-finite group.* Let G be a soluble-by-finite group. Then G has a unique maximal soluble normal subgroup S. There is a canonical hyper FC-central subgroup $E(S)$ of S such that, by definition, the Zalesskiĭ subgroup Zal S of S is $\Delta_S(E(S)) = \Delta(E(S))$ (see [2, p. 364] or [4, p. 188]). Set $E = E(S) \cdot \Delta_G(E(S))$. Then E is a characteristic subgroup of G. We claim that E is hyper FC-central. For

$$E \cap S = E(S) \cdot \Delta_S(E(S)) = E(S)$$

and $(E : E(S))$ is finite. Thus, for example,

$$\Delta(E(S)) = S \cap \Delta(E).$$

A simple argument shows that $E(S)$ is hyper FC-central in E; that is, $E(S)$ lies in the hyper FC-centre of E. Consequently E is hyper FC-central. Furthermore, $E(S) \leq E$, so

$$\Delta_G(E) \leq \Delta_G(E(S)) \leq E.$$

Define $E(G)$ and the Zalesskiĭ subgroup Zal G of G by $E(G) = E$ and Zal $G = \Delta_G(E) = \Delta(E)$. Clearly, any nilpotent-by-finite group is hyper FC-central. These remarks applied to [4, 5.3.10] yield the following.

4. PROPOSITION. *Let J be a ring and G a soluble-by-finite group. Set $K = [G, E(G)]$ and $N =$ Zal G. Alternatively, if G is actually nilpotent-by-finite then set $K = G'$ and $N = \Delta(G)$. Suppose $\mathfrak{a}$ is an ideal of the group ring JG that is left annihilator-free over K. Set $\mathfrak{b} = \mathfrak{a} \cap JN$. Then $JG/\mathfrak{a}$ is a crossed product of $JN/\mathfrak{b}$ by G/N via the natural maps.*

5. PROPOSITION. *Let G, H, and M be subgroups of $GL(n, D)$ with H soluble-by-finite, $M' \leq H \leq M \leq G$, and H normal in G. Suppose $F[G] \leq D^{n \times n}$ is prime and $F[A]$ is prime for every abelian characteristic subgroup A of* Zal H. *Then there is a normal FC-subgroup Z_M of M such that $F[M]$ is a crossed product over Z_M. Moreover, $\zeta_1(M) \leq Z_M$, $Z_M \cap H \leq$ Zal H, Z_M is normalized by any automorphism of M normalizing H, and if $H = M$ then $Z_M =$ Zal H. Further, $F[M]$ is a crossed product over $\beta(M)$.*

To prove Proposition 5, we copy the proof of [7, 6.1]. Set

$$Z_M = \Delta_M(E(H) \cdot \Delta_M(E(H)))$$

as in [7, 6.1] and repeat the proof verbatim, except for the proof of the hyper FC-centrality of $\Delta = \Delta_M(E(M))$ and the final line, where Proposition 4 is used in place of [4, 5.3.10]. Note that, unlike [7, 6.1], if $H = M'$ then it is no longer clear whether $Z_M =$ Zal M.

To see that Δ is hyper FC-central, note that Δ has a soluble normal subgroup S of finite index and $S' \leq H$, so $S' \leq \Delta_H(E(H)) \leq E(H)$ and $S = \Delta_S(S')$. Then S is hyper FC-central ([2, p. 363] or [4, p. 188]) and consequently Δ is also.

6. DEFINITION. For each ordinal α, define the class $\mathfrak{Y}_\alpha$ of groups as follows. Let $\mathfrak{Y}_0$ denote the class of trivial groups. If $\mathfrak{Y}_\beta$ is defined for all ordinals $\beta<\alpha$, set $\mathfrak{Y}_\alpha=\bigcup_{\beta<\alpha}\mathfrak{Y}_\beta$ if α is a limit ordinal and $\mathfrak{Y}_\alpha=(L\mathfrak{Y}_{\alpha-1})\mathfrak{S}\mathfrak{F}$ otherwise. Then $\mathfrak{Y}_1=\mathfrak{S}\mathfrak{F}$, each $\mathfrak{Y}_\alpha$ is QS-closed, and $\langle P,L\rangle(\mathfrak{A}\mathfrak{F})=\bigcup_\alpha\mathfrak{Y}_\alpha$. The classes $\mathfrak{Y}_\alpha$ will occupy the role of the $\mathfrak{X}_\alpha$ of [7].

An elementary induction on α, using $\mathfrak{S}\mathfrak{F}=P(\mathfrak{S}\cup\mathfrak{F})$ for the case $\alpha=1$, yields the following.

Let H be a group and A a soluble-by-finite normal subgroup of H. Then, for all $\alpha\geq 1$:

(a) *$H\in\mathfrak{Y}_\alpha$ if and only if $H/A\in\mathfrak{Y}_\alpha$; and*

(b) *$H\in L\mathfrak{Y}_\alpha$ if and only if $H/A\in L\mathfrak{Y}_\alpha$.*

7. THEOREM. *Let G, H, and $\bar{G}$ be subgroups of $GL(n,D)$ with $H\in\langle P,L\rangle(\mathfrak{A}\mathfrak{F})$, $G'\leq H\leq G\leq\bar{G}$, and H normal in $\bar{G}$. Suppose $F[\bar{G}]$ is prime and $F[N]$ is prime for every characteristic subgroup N of H. Then $F[G]$ is a crossed product over $\beta(G)$. Moreover, $F[G]$ is a crossed product over a normal subgroup J of G that is (locally finite) by (nilpotent of class at most 2) such that J is normalized by every automorphism of G normalizing H. If $H=G$ or G', then J is characteristic in G.*

This theorem is a substitute for 8.1 of [7]. Apart from the weaker conclusion we have had to introduce the extra group $\bar{G}$ to allow the induction to go through. (In [7, 8.1] we have, in effect, $\bar{G}=G$.)

Proof. By the assertion concluding Definition 6, there is a least α with $H\in\mathfrak{Y}_\alpha$; we induct on this α. If $\alpha\leq 1$ then the result is immediate from Proposition 5 with G, H, and $\bar{G}$ for M, H, and G. Also, α cannot be a limit ordinal. Thus we assume the following.

(a) *Suppose G, H, and $\bar{G}$ are as in Theorem 7, with $H\in\mathfrak{Y}_\alpha$ for some $\alpha>1$ for which $\beta=\alpha-1$ exists. Assume inductively that $F[G]$ is a crossed product over $\beta(G)$ whenever $H\in\mathfrak{Y}_\beta$.*

As in the proof of [7, 8.1], we need a digression before we can analyse the structure of $F[G]$. For β as in (a) suppose K is a $L\mathfrak{Y}_\beta$-subgroup of $GL(n,D)$ such that, for some finitely generated subgroup X of K, if N is any normal subgroup of any finitely generated subgroup of K containing X then $F[N]$ is homogeneously faithful. Then $F[N]$ is prime by [7, 2.1] for all such N, and a simple argument shows that $F[N]$ is prime whenever N is a normal subgroup of any subgroup of K containing X.

Statements (b), (c), and (d) below are proved exactly as their counterparts in the proof of [7, 8.1], but using the classes $\mathfrak{Y}_\alpha$ instead of the $\mathfrak{X}_\alpha$ of [7].

(b) *$F[K]$ is a crossed product over $\alpha(K)$.*

(c) *Suppose $\tau(K)\leq\zeta_1\alpha(K)$. Then $F[K]$ is a crossed product over $\zeta_1\alpha(K)$.*

(d) *Suppose $\tau(K)\leq\zeta_1(K)$ and let W be the group of units of the ring $F(K)$ of quotients of $F[K]\leq D^{n\times n}$. Then for $Z=\zeta_1\alpha(K)$ we have $N_W(K)=K\cdot N_{F(Z)^*}(K)$.*

Note that K is irreducible in $F(K)$ by [7, 3.1], and so $F(Z)$ is naturally embedded in $F(K)$ by [7, 3.3]. Thus the conclusion of (d) is meaningful.

This completes our digression, and we return to the consideration of the groups G, H, and $\bar{G}$ as in (a). We wish to express $F[G]$ as a crossed product. Since $F[\bar{G}]$ is prime, $F[\bar{G}]$ acts faithfully on at least one D-$\bar{G}$ composition factor of row n-space over D. Thus we may assume the following.

(e) *$\bar{G}$ is an irreducible subgroup of $GL(n,D)$.*

Set $\bar{L}=\bigcap_A C_{\bar{G}}(A)$, where A ranges over all the abelian characteristic subgroups of H. By [7, 4.1] the subalgebra $F[\bar{G}]$ of $D^{n\times n}$ is a crossed product over $\bar{L}$. Hence we conclude as follows.

(f) *$F[G]$ is a crossed product over $L=G\cap\bar{L}$.*

Clearly $H\cap L$ $(=\lambda_c H)$ is characteristic in H. Every abelian characteristic subgroup of $H\cap L$ is characteristic in H and hence lies in the centre Z of L. Consider $P=\tau(H\cap L)$. Suppose char $F=0$. Then P has a metabelian characteristic subgroup Q of finite index whose Hirsch–Plotkin radical Q_1 is abelian [4, 2.5.14]. Then $Q_1\leq Z$ is central in Q. Thus $Q=Q_1$ is abelian and $Q\leq Z$. Trivially, $L/C_L(P/Q)$ is finite. By stability theory, $C_L(P/Q)/C_L(P)$ embeds into $\mathrm{Hom}(P/Q,Q)$, and the latter group is finite since P/Q is finite and Q has finite rank [4, 2.5.1]. Therefore $L/C_L(P)$ is finite. If char $F>0$ then $L/P\cdot C_L(P)$ is finite-by-abelian by Lemma 1. Thus $L/C_L(P)\in(L\mathfrak{F})\mathfrak{A}$ in all cases. Clearly $C_{H\cap L}(P)$ is characteristic in $H\cap L$, and $H\in(L\mathfrak{Y}_\beta)\mathfrak{S}\mathfrak{F}$ by hypothesis. Hence there is a characteristic subgroup K_1 of $H\cap L$ such that $K_1\leq C_{H\cap L}(P)$, $K_1\in L\mathfrak{Y}_\beta$, and $(H\cap L)/K_1\in(L\mathfrak{F})\mathfrak{S}\mathfrak{T}$.

Since $\bar{G}$ is now irreducible by (e), every characteristic subgroup of H is homogeneously faithful by [7, 2.2]. Apply [7, 2.6] to K_1. Thus there is a characteristic subgroup K of K_1 with $K_1/K\in\mathfrak{T}\cap L\mathfrak{F}$ and a finitely generated subgroup X of K such that N is homogeneously faithful whenever N is a normal subgroup of any finitely generated subgroup of K containing X. Clearly

$$\tau(K)=K\cap P\leq\zeta_1(K)=K\cap Z\quad\text{and}\quad\zeta_1\alpha(K)=K\cap Z,$$

since every abelian characteristic subgroup of $H\cap L$ lies in Z. By (c), the algebra $F[K]$ is a crossed product over $K\cap Z$. Set $M=C_L(K)$. Clearly $Z\leq M$. Let $S=F(K)$ and note that S is naturally a subring of $D^{n\times n}$ by (e) and [7, 3.3]. Since $F[K]$ is prime, S is simple Artinian. Denote the group of units of S by W. Just as in the proof of [7, 8.1] (see paragraphs (g) and (h)), we obtain the following two results.

(g) *$F[L]$ is a crossed product over KM.*

(h) *Suppose A is a normal subgroup of G with $K\cap Z\leq A\leq M$ such that $F[M]$ is a crossed product over A. Then $F[KM]$ is also a crossed product over A.*

Putting (f), (g), and (h) together, we have proved the following.

(i) *If A is as in* (h) *then $F[G]$ is a crossed product over A.*

Thus by (i) it suffices to express $F[M]$ as a suitable crossed product. The requirement that $K\cap Z\leq A$ we can ignore, for $K\cap Z$ is central in M and characteristic in H, so we can always replace A by $A(K\cap Z)$. Certainly $M'\leq H\cap M\leq M\leq \bar{G}$, $H\cap M$ is normal in $\bar{G}$, $F[\bar{G}]$ is prime, and $F[N]$ is prime for every characteristic subgroup N of $H\cap M$. Any automorphism of G normalizing H also normalizes L, K, and M and hence induces an automorphism of M normalizing $H\cap M$. Thus we may replace G by M. Define T by $T/(K\cap Z)=\tau((H\cap M)/(K\cap Z))$. Since Z is central in M, Schur's theorem yields that T' is locally finite. Also, $K\cap M=K\cap Z$, so $(H\cap M)/(K\cap Z)$ is isomorphic to a subgroup of the $(L\mathfrak{F})\mathfrak{S}\mathfrak{T}$-group $(H\cap L)/K$. Hence

$$H\cap M\in (L\mathfrak{F})\mathfrak{A}\cdot\mathfrak{S}\mathfrak{T}=(L\mathfrak{F})\mathfrak{S}\mathfrak{T},$$

and we have proved the following.

(j) *We may assume that* $H\in(L\mathfrak{F})\mathfrak{S}\mathfrak{T}$.

(k) *The completion of the proof of Theorem 7.*

We now make another choice for K_1 and hence for K. Set $K_1=C_{H\cap L}(P)$. Since here $H\in(L\mathfrak{F})\mathfrak{S}\mathfrak{T}$, by (j) we have $K_1\in\mathfrak{S}\mathfrak{T}$. It is easy to see that

$$\langle P,L\rangle(\mathfrak{A}\mathfrak{F})\cap\mathfrak{T}\subseteq L\mathfrak{F}.$$

Hence $K_1\in\mathfrak{S}\cdot L\mathfrak{F}\subseteq L(\mathfrak{S}\mathfrak{F})\subseteq L\mathfrak{Y}_\beta$, since $\beta\geq 1$; therefore this choice for K_1 is legitimate. We choose K as before and set $M=C_L(K)$. Again, by (i) we need only consider $F[M]$.

Now $L'\leq H\cap L$ and $L/C_L(P)\in(L\mathfrak{F})\mathfrak{A}$. Consequently L/K lies in $(L\mathfrak{F})\mathfrak{A}$ and therefore so does $M/(K\cap M)\cong KM/K\leq L/K$. But $K\cap M$ is central in M. It follows that $[M',M]\leq\tau(M)$ since $M''\leq\tau(M)$ by Schur's theorem, and if $x\in M'$ and $y\in M$ then $x^r\in K\cap M$ for some positive integer r and

$$[x,y]^r\in[x^r,y]M''=M''\leq\tau(M).$$

We have now shown that $M/\tau(M)$ is nilpotent of class at most 2. Trivially, $F[M]$ is a crossed product over M. Consequently $F[G]$ is also by (i), and the proof of the theorem is complete. □

8. PROOF OF THE MAIN THEOREM.

(a) This follows from Theorem 7 and [7, 5.4].

(b) Here $\alpha(G)$ is soluble-by-finite. Hence apply Proposition 5 to $\alpha(G)$; that is, take G, H, and M in Proposition 5 all to be $\alpha(G)$.

(c) If char $F=0$ or if $n=1$ then every locally finite subgroup of $GL(n,D)$ is soluble-by-finite [4, 2.1.1 and 2.5.5]. Now apply (b).

(d) Here, $\alpha(G)$ is clearly soluble. Now apply [7, 1.1c] to $\alpha(G)$.

(e) $\tau(G)$ has a characteristic subgroup Q of finite index such that either Q is abelian or Q is metabelian with all its Sylow subgroups abelian (see [4, 2.5.14 and 2.3.1]). Set $G_0=C_G(\tau(G)/Q)$. Now $F[G_0]$ is a crossed product over $L=\lambda_c G_0$ by [7, 4.1]. Then $\eta(L\cap Q)$ is abelian and central in L. Consequently $L\cap Q=\eta(L\cap Q)$ is central in L and $\tau(L)=L\cap\tau(G)\leq\zeta_2(L)$. Now apply (d) to L.

(f) In the proof of (e) we can choose Q so that $(\tau(G):Q)$ is bounded by a function of n only [4, 2.5.14]. Then $(G:G_0)$ is also so bounded and the proof is complete. □

We conclude this paper by stating analogues of certain other results from [7]. The proofs, which we omit, are not significantly different from those in [7]. Corresponding to [7, 8.2], we have the following.

9. PROPOSITION. *Let G be a $\langle P, L\rangle(\mathfrak{R}\mathfrak{F})$-subgroup of $GL(n,D)$ such that $F[G]$, $F[\beta(G)]$, and $F[N]$ are prime for every characteristic subgroup N of G'. Then $F[G]$ is a crossed product over $\alpha(G)$.*

Next we state an analogue of [7, 6.3]. It is a consequence of Proposition 5 and [7, 5.4 and 5.5].

10. PROPOSITION. *With the hypotheses of Proposition 5, assume also either than $F[\beta(M)]$ is prime, or that $F[H]$ and $F[\eta(H)]$ are prime and that $\beta(H)\in(L\mathfrak{N})(L\mathfrak{F})$ and $D^{n\times n}$ contains a ring of quotients of $F[H]$ containing M. Then $F[M]$ is a crossed product over $\alpha(M)$.*

The following should be compared with [7, 6.2]. It looks less attractive since we do not have a short symbol for what is called Z_G below.

11. PROPOSITION. *Let G be a soluble-by-finite subgroup of $GL(n,D)$ with $F[G]$ and $F[A]$ prime for every abelian characteristic subgroup A of $G'\cap Z_G$ for*

$$Z_G=\Delta(E(G')\cdot\Delta_G(E(G'))).$$

Then $F[G]$ is a crossed product over Z_G and hence also over $\beta(G)$.

If we repeat the proof of [7, 7.4], using Proposition 5 in place of [7, 6.1], we obtain the following.

12. PROPOSITION. *Let H be a soluble-by-finite subgroup of $GL(n,D)$ such that $D^{n\times n}$ is the ring of quotients of its subalgebra $F[H]$. Suppose $A=$ Zal H is abelian and $F[A]$ is a domain. Then*

$$N_{GL(N,D)}(H)=H\cdot N_{F(A)^*}(H).$$

13. *The case $n=1$.* Suppose G is a $\langle P, L\rangle(\mathfrak{R}\mathfrak{F})$-subgroup of D^*. In [6] we analysed in detail the crossed product structure of $F[G]\leq D$ for $G\in\langle P, L\rangle\mathfrak{R}$. If char $F>0$ then necessarily $\alpha(G)\in\langle P, L\rangle\mathfrak{R}$ and [6] applies, so assume that char $F=0$. If $\tau(G)$ is soluble then so is $\alpha(G)$, and we can apply the theorem of [6] to $\alpha(G)$. If $\tau(G)$ is not soluble then it must be the binary icosahedral group of order 120 (see [4, 2.5.9]). Thus $F[G]$ is a crossed product over a normal subgroup B of G, where B is abelian or B contains a finite normal subgroup T of G (with B/T abelian and T quaternion of order 8, or binary tetrahedral of order 24, or binary icosahedral of order 120). If G has a normal quaternion subgroup of order 8, then $\alpha(G)$ is soluble and part (c) of the theorem of [6] applies. Finally, if $f(n)$ is as in part (f) of the proof of the main theorem, then the best value for $f(1)$ is 120, which is quite a bit larger than the corresponding value (namely, 6) of part (d) of the theorem of [6].

References

1. P. M. Cohn, *Algebra,* v. 2, John Wiley & Sons, London, 1977.
2. D. S. Passman, *The Algebraic Structure of Group Rings,* John Wiley & Sons, New York, 1977.
3. D. J. S. Robinson, *Finiteness Conditions and Generalized Soluble Groups,* v. 1, Springer, Berlin, 1972.
4. M. Shirvani and B. A. F. Wehrfritz, *Skew Linear Groups,* Cambridge Univ. Press, Cambridge, 1986.
5. B. A. F. Wehrfritz, *Infinite Linear Groups,* Springer, Berlin, 1973.
6. ———, *Crossed products, control in group algebras and absolutely irreducible groups,* J. London Math. Soc. (2), to appear.
7. ———, *Crossed product criteria and skew linear groups,* J. Algebra, to appear.

School of Mathematical Sciences
Queen Mary & Westfield College
London E1 4NS
United Kingdom