Lindelöf's Theorem for Normal Quasimeromorphic Mappings

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1. Introduction

A classical theorem of Lindelöf states that a bounded analytic function in the upper half plane $\mathbf{H}$, which approaches a limit along a curve terminating at a boundary point of $\mathbf{H}$, approaches the same limit in a Stolz angle about that boundary point. It has been known for some time that Lindelöf's theorem, in its classical formulation, is not true for bounded quasiregular mappings in the upper half-space $\mathbf{H}_n = \{x = (x_1, \ldots, x_n) \in \mathbf{R}^n : x_n > 0\}$ when $n \geq 3$. A counterexample of Rickman [14] shows, in fact, that such a mapping may possess infinitely many distinct asymptotic values at a boundary point. On the other hand, the theorem remains true if the asymptotic path is replaced by an asymptotic $(n-1)$-dimensional surface [14].

In this note we establish a similar positive result for normal quasimeromorphic mappings $f: \mathbf{H}_n \to \bar{\mathbf{R}}^n = \mathbf{R}^n \cup \{\infty\}$, $n \geq 2$, thus extending the classical theorem of Lehto and Virtanen [8]. We shall show that if f is normal and quasimeromorphic in $\mathbf{H}_n$ and if f tends to a limit when x tends to 0 along the set $P = \{x \in \partial \mathbf{H}_n : x_1 > 0\}$ (in the sense of Theorem 2.1 below), then f has the same limit along any Stolz cone with vertex at 0. By a Stolz cone we mean the interior of the closed convex hull of $\{0\} \cup \bar{D}(e_n, M)$, where $\bar{D}(e_n, M)$ is the closed hyperbolic ball centered at $e_n = (0, \ldots, 0, 1)$ with radius $M > 0$. Note that if $n = 2$ then, via conformal mapping, the general situation always can be reduced to the case where f has a limit along the positive real axis; our proof will be fairly elementary even in this classical case.

It was observed by Granlund, Lindqvist, and Martio [4] that for bounded quasiregular mappings Lindelöf's theorem is best stated in terms of a (nonlinear) harmonic measure; Rickman's counterexample is possible because a tangential path in space does not carry enough harmonic measure with respect to points in a Stolz cone. The approach to this problem in both [14] and [4] relies heavily on the boundedness of the mapping and as such cannot be extended to mappings with poles. Our proof is based on an idea of the two-constant theorem for unbounded mappings proved in [8]. As in [14] and

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[4], the methods are purely potential-theoretic, hinging only on the fact that $\log|f|$ is $\mathcal{A}$-subharmonic in the terminology of nonlinear potential theory [2; 5]. Obvious alterations would yield similar results for general $\mathcal{A}$-subharmonic functions.

We shall also indicate that the approach region P can be replaced by a sequence of dyadic cubes in $\partial\mathbf{H}_n$, not converging too rapidly to zero, or by a graph of any continuous function $u: P \to \mathbf{R}_+$. As a second main result, we prove a removable singularity theorem for normal quasimeromorphic mappings.

Angular limits of quasimeromorphic mappings were earlier studied in [10; 17; 19] but, as far as we know, no tangential approach regions have previously been considered in the case of unbounded mappings.

2. Normal Quasimeromorphic Mappings

Our notation is fairly standard. For the definition and properties of quasimeromorphic mappings we refer to [11] and [12].

We say that a quasimeromorphic mapping $f: \mathbf{H}_n \to \bar{\mathbf{R}}^n$ is *normal* if it is uniformly continuous between the metric spaces $(\mathbf{H}_n, \rho)$ and $(\bar{\mathbf{R}}^n, q)$, where ρ is the hyperbolic metric in $\mathbf{H}_n$ defined by

$$\rho(x,y) = \inf_{\gamma} \int_{\gamma} \frac{|dz|}{z_n},$$

γ joins x and y in $\mathbf{H}_n$, and q is the chordal (spherical) metric in $\bar{\mathbf{R}}^n$ defined by

$$q(x,y) = \frac{|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}, \quad x \neq \infty \neq y;$$

$$q(x,\infty) = \frac{1}{\sqrt{1+|x|^2}}.$$

Thus f is normal if and only if there is a strictly increasing continuous function $h_f: (0,\infty) \to (0,1]$, with $h_f(s) \to 0$ as $s \to 0$, such that

$$q(f(x), f(y)) \leq h_f(\rho(x,y))$$

for all $x, y \in \mathbf{H}_n$. Likewise, f is normal if and only if the family $\{f \circ \varphi : \varphi$ is a conformal automorphism of $\mathbf{H}_n\}$ is a normal family; this follows from Ascoli's theorem and [20, Thm. 13.6]. (Vuorinen [20, Cor. 13.5] has also shown that one can always choose $h_f(s) = cs^{\alpha}$, where $\alpha = K^{1/(1-n)}$ and K is the maximal dilatation of f.)

Our work was partly motivated by the result of Rickman [15, Thm. 2.4] which implies that a K-quasimeromorphic mapping $f: \mathbf{H}_n \to \bar{\mathbf{R}}^n$ is normal if it omits a finite number $p = p(n,K)$ of points. It is well known that $p(2,K) = 3$ (while $p(n,K) \to \infty$ as $K \to \infty$, at least when $n = 3$ [16]) and in this case the uniformization theorem, untenable in higher dimensions, can be used (see [18, p. 307]). As a further example we mention that if $n \geq 3$, then f is normal if it is locally homeomorphic and omits two points [12, Thm. 2.9].

Our main theorem reads as follows.

2.1. THEOREM. *Suppose that f is a normal quasimeromorphic mapping in $\mathbf{H}_n$ and that*

$$\lim_{y\to 0}\left(\limsup_{x\to y} q(f(x),b)\right)=0$$

for $y\in P=\{x\in\partial\mathbf{H}_n : x_1>0\}$ and $b\in\bar{\mathbf{R}}^n$. Then $f(x)\to b$ when $x\to 0$ along any Stolz cone with vertex at 0.

Generalizations of Theorem 2.1 are briefly discussed in Section 5.

Normal functions on general planar domains are usually defined by using the hyperbolic (Poincaré) metric obtained from the covering map. The Poincaré density in the punctured disk $\mathbf{B}_2^*=\{z:|z|<1, z\neq 0\}$ is asymptotic to $(|z|\log(1/|z|))^{-1}$ as $z\to 0$ [1, pp. 17–18], and a classical result asserts that no normal meromorphic function in $\mathbf{B}_2^*$ can have 0 as an essential singularity [8]. In a subsequent paper [9] Lehto and Virtanen further improved that result by a nice argument which can be used in higher dimensions as well. For completeness we state and prove the following quasiregular analogue of their theorem.

Let δ be any positive continuous function in $\mathbf{B}_n^*=\{x\in\mathbf{R}^n : |x|<1, x\neq 0\}$, with $\delta(x)=o(|x|^{-1})$ as $x\to 0$. Define the metric

$$k_\delta(x,y)=\inf\int_\gamma \delta(z)\,|dz|, \tag{2.2}$$

where the infimum is taken over all curves γ joining x and y in $\mathbf{B}_n^*$.

2.3. THEOREM. *If f is quasimeromorphic on $\mathbf{B}_n^*$ and uniformly continuous between the spaces $(\mathbf{B}_n^*, k_\delta)$ and $(\bar{\mathbf{R}}^n, q)$, then* 0 *is a removable singularity for f.*

2.4. REMARK. The proof of Theorem 2.3 actually shows that is is enough to assume that

$$q(f(x,),f(y))<\tfrac{1}{4}-\epsilon,\quad \epsilon>0,$$

whenever $|x|=|y|$ is small enough; see (4.1) below. For functions meromorphic in $\mathbf{B}_2^*$, Lehto proved (see [6], [7]) that one can replace $\frac{1}{4}$ by 1. It would be nice to know if this is true in higher dimensions as well.

3. Proof of Theorem 2.1

Suppose that f and P are as in Theorem 2.1. We are clearly free to assume that $b=0$.

Our proof will be based on an analysis of the $\mathfrak{A}$-harmonic measure whose precise definition and basic properties can be found in [3] and [4]. For convenience, however, we recall that if f is K-quasimeromorphic in an open set $\Omega\subset\mathbf{R}^n$ and E is a subset of $\partial\Omega$, then there is a unique function $\omega=\omega(E,\Omega;\mathfrak{A})$, the $\mathcal{A}$-*harmonic measure* of E in Ω, which is a continuous weak solution to the quasilinear elliptic equation

$$\operatorname{div} \mathcal{A}(x, \nabla u) = 0 \tag{3.1}$$

in Ω. The mapping $\mathcal{A}: \Omega \times \mathbf{R}^n \to \mathbf{R}^n$ in (3.1) is defined by

$$\mathcal{A}(x, h) = J_f(x) f'(x)^{-1} |f'(x)^{-1^*} h|^{n-2} f'(x)^{-1^*} h$$

if $x \in \Omega$ such that $J_f(x) \neq 0$, and by

$$\mathcal{A}(x, h) = |h|^{n-2} h$$

if $J_f(x)$ does not exist or $J_f(x) = 0$.

Above, $f'(x)$ is the formal derivative of f at x, $f'(x)^{-1}$ its inverse, and T^* is the transpose of a linear map $T: \mathbf{R}^n \to \mathbf{R}^n$. The ellipticity and boundedness constants of $\mathcal{A}$ depend only on n and K. It is a fundamental fact in the theory of quasiregular mappings that the function $u = \log|f|$ is a solution to (3.1) outside the zeros and poles (see e.g. [2]).

The function ω $(0 \leq \omega \leq 1)$ is defined via the Perron method with respect to supersolutions of (3.1), and its behavior resembles that of the classical harmonic measure. However, the set function $E \mapsto \omega(E, \Omega; \mathcal{A})(x)$ never defines a measure unless $n = 2$.

For the next two lemmas suppose that $\omega = \omega(E, \Omega; \mathcal{A})$ as above.

3.2. TWO-CONSTANT THEOREM ([3, Thm. 5.8]; cf. [13, Thm. 4.22]). *Suppose $|f| \leq M < \infty$ in Ω and that*

$$\limsup_{x \to E} |f(x)| \leq m.$$

Then $|f(x)| \leq M(m/M)^{\omega(x)}$ for all $x \in \Omega$.

The second lemma follows from [4, Lemma 4.12].

3.3. LEMMA. *Suppose that $E \subset \partial\Omega$ is connected and that $B_n(y, r) \cap \partial\Omega \subset E$ for some $y \in \partial\Omega$; here $B_n(y, r) = \{x \in \mathbf{R}^n : |x - y| < r\}$. Then there is $\eta = \eta(n, K, \kappa) > 0$ such that $\omega(x) \geq \eta$ for all $x \in \Omega$ with $|x - y| \leq \kappa r < r$.*

We now turn to the proof of Theorem 2.1. Let Q denote the $(n-1)$-dimensional cube $Q = \{x \in P: 0 < x_i < 1, i = 1, \ldots, n-1\}$. By rescaling, if necessary, we may assume that

$$\limsup_{x \to Q} |f(x)| < b_0 < 1, \tag{3.4}$$

where b_0 is a fixed constant (depending on n, K, and h_f only) which will be determined later on. Fix $\lambda \in (\frac{1}{2}, 1)$ and let Q_λ denote the cube concentric with Q and with side length λ. For each $t > 0$ denote by Ω_t the interior of the closed convex hull of $Q_\lambda \cup \{z_0 + te_n\}$, where $z_0 = (\frac{1}{2}, \ldots, \frac{1}{2}, 0)$ is the midpoint of Q_λ.

Next, for $y \in \partial\Omega_t \cap \mathbf{H}_n$ we define x_y to be the point that lies on the perpendicular line segment from y to Q_λ and satisfies $\operatorname{dist}(x_y, Q_\lambda) = \tau \operatorname{dist}(y, Q_\lambda)$, where $\tau < 1$ is determined by the condition $h_f(\log 1/\tau) = \frac{1}{4}$. In particular, by normality

$$q(f(y), f(x_y)) \leq \tfrac{1}{4}. \tag{3.5}$$

If y is as above and $y^* \in Q_\lambda$ such that $|y - y^*| = \operatorname{dist}(y, Q_\lambda) \equiv d$, it then follows by elementary geometry that

$$B_n\left(y^*, \frac{1+\tau}{2} d\right) \cap \partial\Omega_t \subset Q_\lambda \tag{3.6}$$

as soon as t is less than some constant $t_0 = t_0(\tau) > 0$. From now on we suppose that $t \leq t_0$. Write $\omega_t = \omega(Q_\lambda, \Omega_t; \mathfrak{A})$. Since $x_y \in B_n(y^*, ((1+\tau)/2)d)$ and $|x_y - y^*| = \tau d$, it follows from (3.6) and Lemma 3.3, with $\kappa = 2\tau/(1+\tau) < 1$, that

$$\omega_t(x_y) \geq \eta > 0, \tag{3.7}$$

where η depends only on n, K, and h_f.

By continuity, f is bounded by b_0 in a neighborhood of Q_λ; in particular, f is bounded by b_0 in Ω_t for some $t > 0$. Note that $\bar{Q}_\lambda \subset Q$. If $|f| > 1$ somewhere in Ω_{t_0}, then there is the largest $t_1 < t_0$ such that $|f| \leq 1$ in Ω_{t_1}. Let $y \in \partial\Omega_{t_1} \cap \mathbf{H}_n$ be a point such that $|f(y)| = 1$. It follows from (3.5) that

$$\frac{1}{4} \geq q(f(y), f(x_y)) \geq \frac{1 - |f(x_y)|}{\sqrt{2}\sqrt{1 + |f(x_y)|}} \geq \frac{1 - |f(x_y)|}{2}$$

and hence that

$$|f(x_y)| \geq \tfrac{1}{2}. \tag{3.8}$$

On the other hand, the two-constant Theorem 3.2 together with (3.4) and (3.7) implies that

$$|f(x_y)| \leq b_0^\eta.$$

This contradicts (3.8) as soon as $b_0 < 2^{-1/\eta}$. We thus have that $|f|$ is bounded by 1 in Ω_{t_0}. Furthermore, by letting $\lambda \to 1$, it follows from the continuity that $|f|$ is bounded by 1 in the "pyramid" Ω_0, where Ω_0 is the interior of the closed convex hull of $Q \cup \{z_0 + t_0 e_n\}$.

To complete the proof of Theorem 2.1, we shall show that $f(x) \to 0$ as $x \to 0$ along the line segment L that joins 0 and the midpoint $x_0 = z_0 + \frac{1}{2} t_0 e_n$ of the pyramid Ω_0. Because f is bounded in Ω_0, this follows from [4, Thm. 4.21], provided that the condition

$$\lim_{r \to 0} \liminf_{\substack{x \to 0 \\ x \in L}} \omega(B_n(0, r) \cap Q, \Omega_0; \mathfrak{A})(x) > 0 \tag{3.9}$$

is met; but (3.9) is an easy consequence of Lemma 3.3 so that $f(x) \to 0$ as $x \to 0$ along L, as desired. The proof can now be completed by the following (well-known) reasoning: If C is a Stolz cone with vertex at 0, then for each x in C there is a subsegment L_x of L, such that the hyperbolic diameter of L_x is (say) 1 and such that the hyperbolic diameter of $L_x \cup \{x\}$ is bounded by a constant $c_0 = c_0(t_0, C)$. Suppose then, for some sequence $\{x_j\} \subset C$, that

$x_j \to 0$ but $|f(x_j)| \geq \epsilon > 0$. Let $L_j = L_{x_j}$ be a subsegment of L as above and let φ_j be a Möbius transformation of $\mathbf{H}_n$ onto itself such that $\varphi_j(x_j) = e_n$; then $\varphi_j(L_j)$ lies in the closed hyperbolic ball $\bar{D}(e_n, c_0)$. By normality, we may assume that $\{f \circ \varphi_j^{-1}\}$ converges uniformly in $\bar{D}(e_n, 2c_0)$ to a quasimeromorphic mapping f_0. Since the hyperbolic diameter of each $\varphi_j(L_j)$ is 1, the sequence $\{\varphi_j(L_j)\}$ has a nondegenerate limit continuum in $\bar{D}(e_n, c_0)$ on which $f_0 \equiv 0$. The discreteness of f_0 would then imply that $f_0 \equiv 0$ in $\bar{D}(e_n, 2c_0)$, but this is impossible because $|f_0(e_n)| \geq \epsilon > 0$.

Theorem 2.1 is thereby established. □

4. Proof of Theorem 2.3

The following argument (cf. [9]) is based on the fact that the oscillation of f on each individual sphere $\partial B_n(0, R)$ tends to zero as $R \to 0$. Indeed, if $x, y \in \mathbf{B}_n^*$ with $|x| = |y|$, then

$$k_\delta(x, y) \leq \pi |x| \max_{|z| = |x|} \delta(z) = o(1);$$

whence, through the uniform continuity assumption, it is no loss of generality to assume that

$$q(f(x), f(y)) < c < \tfrac{1}{4} \tag{4.1}$$

whenever x and y lie on the same sphere centered at 0.

Let us now make the antithesis that 0 is an essential singularity. The big Picard theorem for quasimeromorphic mappings [13, Thm. 1.2] ensures that f assumes all but at most a finite number of points in each neighborhood of 0. We may therefore assume that f has a zero on a sphere $\partial B_n(0, R)$. Thus, by (4.1),

$$q(f(x), 0) < c \tag{4.2}$$

for all x with $|x| = R$. Let $r < R$ be the largest radius such that

$$q(f(x), 0) \leq 2c$$

for all x in the ring domain $B_n(0, R) \setminus \bar{B}_n(0, r)$ but

$$q(f(z), 0) = 2c \tag{4.3}$$

for some z with $|z| = r$; such an r exists because f is unbounded in each neighborhood of 0.

Denote by S_R and S_r the images of $\partial B_n(0, R)$ and $\partial B_n(0, r)$, respectively. Then S_R and S_r are compact, disjoint subsets of $\mathbf{R}^n$; indeed, if $w \in S_R \cap S_r$, then by (4.2) and (4.3)

$$q(f(z), w) \geq q(f(z), 0) - q(w, 0) > c,$$

contradicting (4.1). Since (4.1) holds for any two points on S_r, a similar argument shows that S_r cannot separate S_R from the point at infinity, and it is easy to see how this leads to a desired contradiction. In fact, because the image of any line segment from $\partial B_n(0,r)$ to $\partial B_n(0,R)$ joins S_r and S_R, there is a point y in $B_n(0,R)\backslash\bar{B}_n(0,r)$ such that $f(y)$ can be joined to ∞ by an arc L in the complement of $S_r \cup S_R$; but such an arc would have a lifting starting from y and meeting either $\partial B_n(0,R)$ or $\partial B_n(0,r)$ (see [12, 3.12]), which is impossible since L meets neither S_R nor S_r.

The proof of Theorem 2.3 is complete. □

5. Remarks

As the proof of Theorem 2.1 reveals, the approach region P can be replaced by various other sets. Suppose, for instance, that P is the union of $(n-1)$-dimensional cubes $\{Q_i\} \subset \mathbf{R}^{n-1} = \partial\mathbf{H}_n$ such that $0 \in \bar{P}$ and that

$$\operatorname{dist}(Q_i, Q_{i+1}) \leq c\ell(Q_i); \tag{5.1}$$

here $\ell(Q)$ designates the side length of Q. Then one can build a sequence of "pyramids" Ω_i based on Q_i such that the height of Ω_i is proportional to $\ell(Q_i)$ and that f is bounded in $\Omega = \bigcup \Omega_i$. Applying a harmonic measure estimate in each individual pyramid, one finds a sequence of line segments $L_i \subset \Omega_i$ such that $L_0 = \bigcup L_i$ lies in a Stolz cone C and that $f(x) \to 0$ as $x \to 0$ along L_0. Because of (5.1), the set L_0 shares all the required properties of the line segment L in the proof of Theorem 2.1, whence $f(x) \to 0$ along any Stolz cone.

As a second example, we let P be the graph of a positive continuous function $u\colon \{x_n = 0, x_1 > 0\} \to \mathbf{R}$, $u(x) \to 0$ as $x_1 \to 0$. Since P blocks off the set $\{x_n = 0, x_1 > 0\}$, the monotonicity of the $\mathfrak{A}$-harmonic measure guarantees that the required estimates are retained, and the theorem remains true in this case as well (cf. [1, pp. 40–41]).

5.1. QUESTION. We wish to pose the question whether the general condition given in [4, Thm. 4.21] is sufficient to imply Lindelöf's theorem for normal quasimeromorphic mappings. More precisely, suppose that $f(x) \to 0$ as $x \to 0$ along a relatively closed subset E of $\bar{\mathbf{H}}_n$, and that

$$\lim_{r\to 0} \liminf_{\substack{x\to 0\\ x\in C}} \omega(\bar{B}_n(0,r)\cap E, \mathbf{H}_n; \mathfrak{A})(x) > 0,$$

where C is a Stolz cone with vertex at 0. Is it then true that $f(x) \to 0$ as $x \to 0$ along C?

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