Commutativity Theorems for Banach Algebras

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1. Introduction

A number of theorems in ring theory, mostly due to Herstein, are devoted to showing that certain rings must be commutative as a consequence of conditions which are seemingly too weak to imply commutativity. For surveys of work in this area see [7, Chap. 3] and [10, Chap. X]. Our first aim is to show that in the special case of a Banach algebra some of these results may be sharpened.

Consider the following theorem of Herstein [4, p. 411]. A ring R is commutative if (a) there are no nonzero nil ideals and (b) for each x and y in R there is a positive integer $n(x,y)$ such that $x^{n(x,y)}$ permutes with y.

Let A be a Banach algebra which satisfies the following weakening of (b). Suppose (c) there exists a nonvoid open subset G of A, where for each x and y in G there are positive integers $m=m(x,y)$ and $n=n(x,y)$ such that $x^m y^n = y^n x^m$. If A has a two-sided approximate identity then A is commutative. In general, A need not be commutative but there must exist a positive integer r such that x^r lies in the center of A for all $x \in A$. If A has no nonzero nilpotent ideals, then A is commutative.

Consider also the theorem of Herstein [4, p. 412] which states that a ring R is commutative if for each x and y in R there is a positive integer $n(x,y)>1$ such that $x^{n(x,y)}-x$ permutes with y. For a Banach algebra A we show that $a \in A$ lies in the center if there is a nonvoid open set G where, for each $x \in G$, we have a positive integer $n(x)>1$ so that $x^{n(x)}-x$ permutes with a.

In Section 3 we present theorems in this spirit for Banach *-algebras. Let A be a Banach *-algebra with continuous involution and no nonzero nilpotent ideals (as when A is semi-simple). It is shown that either A is commutative or the set of $x \in A$, where x^n is normal for *no* positive integer n, is dense in A. If A is unital then the requirement on nilpotent ideals can be dropped. Other related results are obtained.

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2. Commuting Properties of Banach Algebras

For notation we let A denote a real or complex Banach algebra, Z the center of A, and E a closed linear subspace of A. As usual $[x,y]=xy-yx$. We shall use several times the following readily established fact. Let $p(t)=\sum_{r=0}^{n} b_r t^r$ be a polynomial in the real variable t with coefficients in A. If $p(t)\in E$ for all t in an infinite subset of the reals, then every b_r lies in E.

LEMMA 2.1. *Let $w\in A$ and let n be a positive integer. If $[w,x^n]\in E$ for all $x\in A$ then $[w^n,x]\in E$ for all $x\in A$.*

Proof. Take $x\in A$. Let B_r denote the sum of the terms in the expansion of $(w+x)^n$ for which the sum of the exponents of the x^j factors is r. Thus $B_0=w^n$ and

$$B_1=\sum_{k=0}^{n-1} w^k x w^{n-1-k}.$$

Inasmuch as $[w,(w+tx)^n]\in E$ for all real t we see that

$$\sum_{k=0}^{n} [w,B_k]t^k$$

lies in E for each real t. Therefore $[w,B_1]\in E$. However,

$$[w,B_1]=[w^n,x]. \qquad \square$$

LEMMA 2.2. *Suppose that there is a nonvoid open set G in A, where for each x and y in G there are positive integers $m=m(x,y)$ and $n=n(x,y)$ so that $[x^m,y^n]\in E$. Then there is a positive integer r where $[x^r,y]\in E$ for all x,y in A.*

Proof. Fix $w\in G$. For each ordered pair (m,n) of positive integers let

$$Q_{m,n}=\{y\in A: [w^m,y^n]\notin E\}.$$

Each $Q_{m,n}$ is open. Suppose that every $Q_{m,n}$ is dense in A. Then, by the Baire category theorem,

$$V=\bigcap_{m,n} Q_{m,n}$$

is dense in A. But V is the set of $y\in A$ such that $[w^m,y^n]\notin E$ for all m and n. This contradicts the existence of the set G.

Hence there are positive integers p and q so that $Q_{p,q}$ is not dense. Let Γ be a nonvoid open set in the complement of $Q_{p,q}$. Let $z\in\Gamma$ and $x\in A$. Then $z+tx\in\Gamma$ for all real t sufficiently small. For these values of t,

$$w^p(z+tx)^q-(z+tx)^q w^p\in E.$$

Expanding this expression we get a polynomial in t. The coefficient of t^q in this polynomial must be in E or $[w^p,x^q]\in E$ for all $x\in A$. By Lemma 2.1 we see that $[w^{pq},x]\in E$ for all $x\in A$.

For each positive integer n, let R_n be the complement of the set of $x \in A$ where $[x^n, y] \in E$ for all $y \in A$. We show first the conclusion holds for the integer n if some R_n is not dense. For then there is a nonvoid open set Γ in A such that $[x^n, y] \in E$ for all $x \in \Gamma$ and $y \in A$. Select $z \in \Gamma$ and $w \in A$. For all real t sufficiently small we have $[(z+tw)^n, y] \in E$ for all $y \in A$. We write

$$(z+tw)^n = \sum_{k=0}^{n} a_k t^k,$$

where each $a_k \in A$ and $a_n = w^n$. Then $\sum_{k=0}^{n} [a_k, y] t^k \in E$ for all t sufficiently small, so that $[w^n, y] \in E$ for all $y \in A$ and $w \in A$.

Suppose that every R_n is dense. Clearly R_n is open. By the Baire category theorem, $\bigcap R_n$, the complement of the set $x \in A$ for which there is a positive integer n where $[x^n, y] \in E$ for all $y \in A$, is dense. This is impossible by the existence of the set G. Therefore some R_n is not dense. $\square$

THEOREM 2.3. *Suppose that there is a nonvoid open set G in A, where for each x, y in G we have positive integers $m = m(x,y)$ and $n = n(x,y)$ so that $[x^m, y^n] = 0$. Then there is a positive integer r so that $x^r \in Z$ for all $x \in A$. If A has no nonzero nilpotent ideals it is sufficient to have $[x^m, y^n] \in Z$ and $x, y \in G$, with m and n as above. Then A is commutative.*

Proof. The first conclusion follows from Lemma 2.2 with $E = (0)$. Suppose $E = Z$. By Lemma 2.2 there is a positive integer r so that $[x^r, y] \in Z$ for all x, y in A. Therefore x^r permutes with $[x^r, y]$ for all $y \in A$. If A has no nonzero nilpotent ideals then a "sublemma" of Herstein [8, p. 5] tells us that $x^r \in Z$ for all $x \in A$.

Under these conditions A is commutative if A has no nonzero nilpotent ideals. This follows from the theorem in [11]. Without such an algebraic assumption, A is commutative if A has a two-sided approximate identity, as the following lemma shows. (For the notion of a two-sided approximate identity, see [2].)

LEMMA 2.4. *Suppose that A has a two-sided approximate identity $\{e_\lambda\}$. If there is a positive integer r such that $x^r \in E$ for all $x \in A$, then $E = A$.*

Proof. Let $x \in A$ and let t be real. We may suppose that $r > 1$. Consider, for a fixed index λ, the polynomial $(x + te_\lambda)^r$. The coefficient of t for this polynomial lies in E, or $\sum_{j=0}^{r-1} x^j e_\lambda x^{r-1-j}$ is in E. Taking $\lim_\lambda$ we see that $x^{r-1} \in E$. Continuing in this way we find that $x \in E$. $\square$

THEOREM 2.5. *Let $a \in A$. Suppose that there is a nonempty open set G such that, for each $x \in G$, we have a positive integer $n(x) > 1$ such that*

$$[x^{n(x)} - x, a] \in E.$$

For $E = (0)$ we have $a \in Z$. If A has no nonzero nilpotent ideals and $E = Z$, then we have $a \in Z$.

Proof. For each $n=2,3,\ldots$ let

$$W_n=\{x\in A\colon [x^n-x,a]\notin E\}.$$

From the Baire category theorem and the existence of G we see that some W_m $(m>1)$ is not dense. Let Γ be a nonvoid open set in the complement of W_m. Take $z\in\Gamma$ and $w\in A$. For each t real, consider

$$[(z+tw)^m-(z+tw),a]=\sum_{r=0}^{m} b_k t^k.$$

This polynomial lies in E for all t sufficiently small. Thus $b_m\in E$ or $[w^m,a]\in E$ for all $w\in A$. Inasmuch as $[z^m-z,a]\in E$, we have $[z,a]\in E$ for all $z\in\Gamma$. As Γ is open we see that $[y,a]\in E$ for all $y\in A$. For the case $E=Z$ we again use Herstein's result [8, p. 5]. □

Herstein [5; 6] has shown that a ring R is commutative if it has no nonzero nil ideals and there is a fixed integer $n>1$ such that $(xy)^n=x^ny^n$ for all x,y in R (see also [1]). In the case of a Banach algebra A we can say more.

THEOREM 2.6. *Suppose that there are two nonvoid open subsets G_1 and G_2 of A such that for each $w\in G_1$ and $x\in G_2$ we have a positive integer $n=n(w,x)>1$, where $(wx)^n=w^nx^n$. Then there is a fixed integer $r>1$ such that $(xy)^r=x^ry^r$ for all x,y in A.*

Proof. This can be shown by the arguments used above. We omit the details. □

3. On Banach *-Algebras

Henceforth A will be a Banach *-algebra over the complexes with a continuous involution $x\to x^*$. We retain our earlier notation, where E is a closed linear subspace of A and Z is the center of A.

Considerable attention has been paid to commutativity theorems for rings with an involution. See [9, Chap. 3], where further references can also be found. Our results on commutativity for A seem to be rather different.

LEMMA 3.1. *Let n be a positive integer. Suppose that $h^n\in E$ for all self-adjoint elements h. Then $x^n\in E$ for all x in A.*

Proof. Let h and k be self-adjoint in A. We let B_r denote the sum of the terms in the expansion of $(h+k)^n$ for which the sum of the exponents of the k^j factors is r (see the proof of Lemma 2.1). For any real number t,

$$(h+tk)^n=\sum_{r=0}^{n} B_r t^r$$

lies in E. Therefore each B_r is in E. Now consider $x=h+ik$. We have

$$x^n=\sum_{r=0}^{n} i^r B_r$$

in E. □

DEFINITION. We say that $y \in A$ is *normal modulo E* if $[y, y^*] \in E$.

LEMMA 3.2. *Suppose that the set of $x \in A$ for which there is a positive integer $n(x)$ so that $x^{n(x)}$ is normal modulo E has nonvoid interior. Then there is a positive integer n such that x^n is normal modulo E for all $x \in A$.*

Proof. For each positive integer k, let

$$W_k = \{x \in A : x^k \text{ is not normal modulo } E\}.$$

As the involution is continuous, W_k is open. We rule out the possibility that every W_k is dense. For suppose every W_k is dense. By the Baire category theorem, the intersection W of all the W_k is dense. But this is contrary to our hypothesis on normality. Hence there is a positive integer n and a nonvoid open subset Γ of A so that x^n is normal modulo E for all x in Γ.

Let $z \in \Gamma$ and let y be an arbitrary element in A. For all real t sufficiently small,

$$(z+ty)^n(z^*+ty^*)^n-(z^*+ty^*)^n(z+ty)^n \in E.$$

The coefficient of t^{2n} in this polynomial lies in E, or y^n is normal modulo E for all y in A. □

We recall the notation of Lemma 3.1. Let h and k be self-adjoint. Then $B_0 = h^n$ and

$$B_1 = \sum_{j=0}^{n-1} h^j k h^{n-1-j}.$$

LEMMA 3.3. *Suppose that x^n is normal modulo E for all $x \in A$. Let h, k be self-adjoint. Then, in the notation of Lemma 3.1,*

$$[B_0, B_1] \in E.$$

Proof. Let $t \neq 0$ be real and consider

$$(h+itk)^n = \sum_{r=0}^{n} i^r B_r t^r.$$

Let $\alpha(t)$ be the sum of the terms of this expansion for r even and $\beta(t)$ be the sum for r odd. Then

$$(h+itk)^n = \alpha(t)+\beta(t);$$
$$(h-itk)^n = \alpha(t)-\beta(t).$$

As $(h+itk)^n$ is normal modulo E, we see that

$$\alpha(t)\beta(t)-\beta(t)\alpha(t) \in E$$

for all real t. The expression here is a polynomial in t with coefficients in A. Therefore the coefficient of t must lie in E, or

$$B_0B_1 - B_1B_0 \in E. \qquad □$$

We are now ready for the following dichotomy.

THEOREM 3.4. *Let A be unital. Then either $[x,y]\in E$ for all x,y in A or the set S of $x\in A$, for which x^r is normal modulo E for no positive integer r, is dense in A.*

Proof. Let e be the identity of A. Suppose that the set S is not dense. Then, by Lemma 3.2, there is a positive integer n such that x^n is normal modulo E for all $x\in A$. Let h and k be self-adjoint in A. For $t\neq 0$, t real, set

$$u=t^{-1}[(e+th)^n-e];$$

$$v=\sum_{j=0}^{n-1}(e+th)^j k(e+th)^{n-1-j}.$$

By Lemma 3.3 we see that $[u,v]\in E$. Now let $t\to 0$ to see that $[h,k]\in E$. Since h and k are arbitrary self-adjoint elements in A, we have that every $[x,y]\in E$. □

COROLLARY 3.5. *Let A be unital. Either A is commutative or the set of x, where x^r is normal for no positive integer r, is dense. If also A has no nonzero nilpotent ideals then either A is commutative or the set of x, where x^r is normal modulo Z for no positive integer r, is dense.*

Proof. The first conclusion follows from Theorem 3.5 with $E=(0)$. Suppose A has no nonzero nilpotent ideals. We employ Theorem 3.5 with $E=Z$. If every $[x,y]\in Z$ it follows from [8, p. 5] that A is commutative. □

THEOREM 3.6. *Suppose that A has no nonzero nilpotent ideals. Either A is commutative or the set S of x, where x^k is normal for no positive integer k, is dense in A.*

Proof. Suppose S is not dense. Then, by Lemma 3.2, there is a positive integer n so that x^n is normal for all x in A.

Let h and k be self-adjoint. By Lemma 3.3, h^n permutes with B_1 and therefore h^n permutes with hB_1-B_1h. However,

$$hB_1-B_1h=[h^n,k].$$

As k is an arbitrary self-adjoint element of A, we see that h^n permutes with $[h^n,x]$ for all x in A. Herstein's sublemma [8, p. 5] then tells us that $h^n\in Z$. In view of Lemma 3.1, we see that $x^n\in Z$ for all $x\in A$. Then A is commutative by the theorem of [11]. □

LEMMA 3.7. (a) *If $(xx^*)^n=(x^*x)^n$ for all $x\in A$ then $x^{2n}\in Z$ for all $x\in A$.*

(b) *If A has no nonzero nilpotent ideals and $(xx^*)^n$ permutes with $(x^*x)^n$ for all $x\in A$, then A is commutative.*

Proof. Let h and k be self-adjoint and let $t\neq 0$ be real. Form $x=h+itk$. Then

$$xx^*=y+tz \quad\text{and}\quad x^*x=y-tz,$$

where $y=h^2+t^2k^2$ and $z=i[k,h]$. Let W_r be the sum of those terms in the expansion of $(y+z)^n$, where r is the sum of the exponents of the z^j factors. Then

$$(xx^*)^n=\sum_{r=0}^{n} W_r t^r \quad \text{and} \quad (x^*x)^n=\sum_{r=0}^{n}(-1)^r W_r t^r.$$

Let Σ' indicate summation over $r=0,1,\dots,n$ for r odd and Σ'' the summation for r even. It is true that each W_r depends on t, but lim W_r exists as $t\to 0$.

Under the hypothesis of statement (a) we must have

$$t^{-1}\Sigma' W_r t^r=0.$$

If we let $t\to 0$, we see that lim $W_1=0$ or

$$V=\sum_{j=0}^{n-1} h^{2j}[k,h]h^{2(n-1-j)}=0.$$

A tedious but straightforward calculation shows that

$$hV+Vh=[k,h^{2n}]=0.$$

Therefore $h^{2n}\in Z$ for all self-adjoint elements h. By Lemma 3.1 we see that $w^{2n}\in Z$ for all $w\in A$.

Consider next the statement (b). There, $t^{-1}\Sigma' W_r t^r$ must permute with $\Sigma'' W_r t^r$ for each $t\neq 0$. Let $t\to 0$ to see that lim $W_0=h^{2n}$ permutes with V. Therefore h^{2n} permutes with $[h^{2n},w]$ for every $w\in A$. As A has no nonzero nilpotent ideals, $h^{2n}\in Z$ by Herstein's sublemma [8, p. 5]. We use Lemma 3.1 and [11] to complete the proof. □

THEOREM 3.8. *If there exists no positive integer r such that $x^r\in Z$ for all $x\in A$ then the set T of x, where $(xx^*)^n\neq(x^*x)^n$ for all positive integers n, is dense in A.*

Proof. Suppose that T is not dense. By the arguments of Lemma 3.2 there is a positive integer n such that $(xx^*)^n=(x^*x)^n$ for all $x\in A$. Then, by Lemma 3.7, $x^{2n}\in Z$ for all x in A.

In the same way we see that if A has no nonzero nilpotent ideals and if no positive integer r exists where $x^r\in Z$ for all $x\in A$, then the set W of x, where $(xx^*)^n$ does not permute with $(x^*x)^n$ for all $n=1,2,\dots$, is dense. □

In particular, for any semi-simple Banach *-algebra A which is not commutative, the sets S and T of Theorems 3.6 and 3.8 are dense.

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