Entire Timelike Minimal Surfaces in $E^{3,1}$

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1. Introduction

Calabi was the first to show that an entire *spacelike* minimal surface in Minkowski 3-space $E^{3,1}$ must be a plane (see [1]). However, if w is the timelike coordinate in u, v, w-space, the example

$$v = w \tanh u$$

shows that an entire *timelike* minimal surface in $E^{3,1}$ need not even be flat. The best one can say in this direction is that the surface must be conformally equivalent to the Minkowski 2-plane $E^{2,1}$ (see [6]).

In this paper we generate examples that display considerable variety in the shapes of entire timelike minimal surfaces in $E^{3,1}$. This is done, in part, by describing an analog for the classical construction of associate minimal surfaces in Euclidean 3-space $E^{3,0}$.

Associate minimal surfaces in $E^{3,0}$ are paired in an amusing manner. At corresponding points, they share the same induced metric, Gauss curvature, zero mean curvature, and unit normals. Still, associate minimal surfaces in $E^{3,0}$ can have markedly different shapes, as the helicoid and the catenoid amply illustrate. (For pictures, see [2] or [8].)

To produce associate families of spacelike minimal surfaces from a given spacelike minimal surface in $E^{3,1}$, the original classical construction suffices. But an entirely different construction must be used to generate associate families of timelike minimal surfaces from a given timelike minimal surface in $E^{3,1}$. In both cases, the associate pairing still preserves the Minkowski induced metric, Gauss curvature, zero mean curvature, and unit normals.

While it is pleasing to have the counterpart for a construction based on complex analytic techniques in a situation governed by the wave equation rather than by Laplace's equation, the construction of associate timelike minimal surfaces in $E^{3,1}$ is further justified by the fact that all surfaces associate to an entire timelike minimal surface in $E^{3,1}$ are entire over the same fixed plane. Thus the construction can be used to produce infinite families of isometric entire, timelike minimal surfaces in $E^{3,1}$ no two of which are congruent.

Received February 8, 1989. Revision received October 6, 1989.
Michigan Math. J. 37 (1990).

In Section 4 we explore a connection between timelike minimal immersions of a surface S in $E^{3,1}$ and immersions of S into $E^{3,0}$ that are harmonic with respect to an indefinite prescribed metric g on S. Given a harmonic immersion $\mathcal{Z}: (S, g) \to E^{3,0}$ with indefinite g which is a local graph over some plane, we assign to $\mathcal{Z}$ over the domain D of g-null coordinates a timelike minimal immersion $\tilde{\mathcal{Z}}: D \to E^{3,1}$ whose induced metric is conformally equivalent to g, and which is a local graph over the spacelike coordinate plane $\mathcal{P}$ in $E^{3,1}$. We prove that if $\mathcal{Z}$ is entire, then $\tilde{\mathcal{Z}}: S \to E^{3,1}$ can be globally defined and will be entire over $\mathcal{P}$. Moreover, the assignment procedure can be reversed, in the following sense.

Let $\tilde{I}$ be the Minkowski induced metric for a timelike minimal immersion $\tilde{\mathcal{Z}}: S \to E^{3,1}$ which is a local graph over $\mathcal{P}$, and let D be the domain of $\tilde{I}$-null coordinates on S. Given any smooth map ν from D to the upper open hemisphere of the Euclidean 2-sphere Σ, we obtain a harmonic immersion $\mathcal{Z}: (D, \tilde{I}) \to E^{3,0}$ with Gauss map $\nu: D \to \Sigma$ whose assigned timelike minimal immersion over D is $\tilde{\mathcal{Z}}$.

The assignment construction can be exploited to give a local Weierstrass representation for harmonic immersions $\mathcal{Z}: (S, g) \to E^{3,0}$ with indefinite g, using the Weierstrass functions $\tilde{A}(x)$ and $\tilde{B}(y)$ for the assigned timelike minimal immersion $\tilde{\mathcal{Z}}$ and the $E^{3,0}$ Gauss map $\nu(x, y)$ for $\mathcal{Z}$. (The local Weierstrass representation for timelike minimal immersions in $E^{3,1}$ is described in §2 below.)

The material in Section 5 provides examples of entire timelike minimal surfaces in $E^{3,1}$ with particular properties. Included there is a method we learned from Calabi for generating entire, doubly periodic, timelike minimal surfaces in $E^{3,1}$. Actually, we adapt Calabi's procedure to produce nonplanar timelike minimal surfaces in $E^{3,1}$ which are entire with respect to all three coordinate planes simultaneously. We also give examples of entire timelike minimal surfaces in $E^{3,1}$ on which Gauss curvature is always positive, or always negative. This is done for surfaces entire over a timelike plane and for surfaces entire over a spacelike plane. Of course, for each example described, one has as well the family of associate surfaces.

2. Background

As in [7], we view $E^{3,j}$ as R^3 with the scalar product

$$\langle V, W \rangle^j = v_1 w_1 + v_2 w_2 + (-1)^j v_3 w_3,$$

where $j = 0$ gives Euclidean 3-space and $j = 1$ gives Minkowsi 3-space. Because this paper deals mainly with surfaces in $E^{3,1}$, we delete the index 1 at most points, writing $\langle V, W \rangle$ for $\langle V, W \rangle^1$. A vector V in $E^{3,1}$ is *spacelike* if $\langle V, V \rangle > 0$, *timelike* if $\langle V, V \rangle < 0$, and *null* if $\langle V, V \rangle = 0$.

The surface S is assumed to be C^∞, oriented and connected. Given any immersion $\mathcal{Z}: S \to R^3$, we also write $\mathcal{Z}: S \to E^{3,0}$ and $\mathcal{Z}: S \to E^{3,1}$ since the same underlying map is involved. To study immersions $\mathcal{Z}: S \to E^{3,j}$ for $j = 0, 1$, we use the fundamental forms

$$I^j = \langle d\mathcal{Z}, d\mathcal{Z}\rangle^j, \qquad II^j = \langle d\mathcal{Z}, d\nu^j\rangle^j$$

where the unit normal ν^j is given in terms of local coordinates x, y on S by

$$\sqrt{|\det I^j|}\,\nu^j = \begin{vmatrix} \vec{i} & \vec{j} & (-1)^j\vec{k} \\ & \mathcal{Z}_x & \\ & \mathcal{Z}_y & \end{vmatrix}.$$

Gauss curvature K^j and mean curvature H^j are given by

$$K^j = \det II^j/\det I^j, \qquad H^j = \mathrm{tr}_{I^j}(II^j).$$

Again, we usually write I, ν, II, K, and H for I^1, ν^1, II^1, K^1, and H^1. However, definition of ν, II, K, and H requires that $\det I \neq 0$. Thus we restrict our attention to immersions $\mathcal{Z}: S \to E^{3,1}$ which are spacelike (meaning that $\det I > 0$) or timelike (meaning that $\det I < 0$).

We call an immersion $\mathcal{Z}: S \to E^{3,j}$ *minimal* if $H^j \equiv 0$. Although a minimal immersion is always extremal for the I^j-area integral, spacelike minimal $\mathcal{Z}: S \to E^{3,1}$ actually maximize I-area whereas timelike minimal $\mathcal{Z}: S \to E^{3,1}$ neither maximize nor minimize I-area, even locally.

There are always local coordinates x, y on S for a timelike $\mathcal{Z}: S \to E^{3,1}$ in terms of which $I = 2F\,dx\,dy$ for some function $F \neq 0$. These are called *null coordinates,* since the tangential directions $dx \equiv 0$ and $dy \equiv 0$ are null. When null coordinates x, y are used on S, the Christoffel symbols $\Gamma_{12}^1 \equiv \Gamma_{12}^2 \equiv 0$ for I, while $H \equiv 0$ forces the middle coefficient M of II to vanish as well. The Gauss equation

$$\mathcal{Z}_{xy} = \Gamma_{12}^1 \mathcal{Z}_x + \Gamma_{12}^2 \mathcal{Z}_y + M\nu$$

thus shows that a timelike $\mathcal{Z}: S \to E^{3,1}$ is minimal if and only if $\mathcal{Z}_{xy} \equiv 0$, or (equivalently) if and only if $\mathcal{Z}$ has the local expression

$$\mathcal{Z}(x, y) = \mathcal{X}(x) + \mathcal{Y}(y) \tag{1}$$

for any null coordinates x, y on S.

To *normalize* null coordinates x, y on S for a timelike minimal $\mathcal{Z}: S \to E^{3,1}$, reparametrize the curves $\mathcal{X}(x)$ and $\mathcal{Y}(y)$ if necessary so that x and y measure *Euclidean* arc length along $\mathcal{X}(x)$ and $\mathcal{Y}(y)$ respectively. Then $\mathcal{X}'(x)$ and $\mathcal{Y}'(y)$ are unit vectors in $E^{3,0}$ and x, y are *Tchebychev coordinates* for the metric I^0 induced on S by $\mathcal{Z}: S \to E^{3,0}$. Finally, change x to $-x$, or y to $-y$ as needed, and reverse the roles of x and y if this is required to respect the orientation on S, so that

$$\begin{aligned} \mathcal{X}'(x) &= (a(x), b(x), \sqrt{1 - a^2(x) - b^2(x)}), \\ \mathcal{Y}'(y) &= (\alpha(y), \beta(y), \sqrt{1 - \alpha^2(y) - \beta^2(y)}) \end{aligned} \tag{2}$$

for smooth functions $a(x)$, $b(x)$, $\alpha(y)$, and $\beta(y)$. This determines x, y up to additive constants over their domain on S. Because $\mathcal{X}'(x)$ and $\mathcal{Y}'(y)$ are null in $E^{3,1}$, (2) gives

$$a^2(x) + b^2(x) \equiv \alpha^2(y) + \beta^2(y) \equiv 1/2,$$

so that

$$\mathfrak{X}'(x)=(a(x),b(x),1/2), \qquad \mathfrak{Y}'(y)=(\alpha(y),\beta(y),1/2).$$

Thus the values $(a(x),b(x))$ and $(\alpha(y),\beta(y))$ vary along segments C_x and C_y respectively on a circle C. Note that C_x and C_y must be disjoint, since $\mathfrak{X}'(x)$ and $\mathfrak{Y}'(y)$ always span a plane. Either segment C_x or C_y may reduce to a point, but since neither can be empty, neither can be all of C. It follows that there are smooth functions $A(x)$ and $B(y)$ determined up to integral multiples of 2π so that

$$A(x)\neq B(y)+2\pi j \tag{3}$$

for any integer j, while

$$\begin{aligned}\sqrt{2}\mathfrak{X}'(x)&=(\cos A(x),\sin A(x),1),\\ \sqrt{2}\mathfrak{Y}'(y)&=(\cos B(y),\sin B(y),1).\end{aligned} \tag{4}$$

Straightforward computation then gives

$$\begin{aligned}I&=2\sin^2\left(\frac{A-B}{2}\right)dx\,dy,\\ \sqrt{2}II&=-A'(x)\,dx^2+B'(y)\,dy^2,\\ I^0&=dx^2+2\cos^2\left(\frac{A-B}{2}\right)dx\,dy+dy^2,\\ 2\sqrt{\det I^0}\,\nu^0&=(\sin A-\sin B,\ \cos B-\cos A,\ \sin(B-A)),\\ 2\sqrt{-\det I}\,\nu&=(\sin A-\sin B,\ \cos B-\cos A,\ \sin(A-B)).\end{aligned} \tag{5}$$

Using Cartesian coordinates u, v, w in $E^{3,j}$, the immersion $\mathcal{Z}$ above yields a local graph over the u, v-plane wherever $\sin(A-B)\neq 0$, over the u, w-plane wherever $\cos A\neq\cos B$, and over the v, w-plane wherever $\sin A\neq\sin B$. Thus $\mathcal{Z}$ gives a local graph over the u, v-plane wherever

$$A(x)\neq B(y)+j\pi, \tag{6}$$

over the u, w-plane wherever

$$A(x)\neq -B(y)+2j\pi, \tag{7}$$

and over the v, w-plane wherever

$$A(x)\neq -B(y)+(2j-1)\pi, \tag{8}$$

for any integer j. To deal with one null plane, note that Z gives a local graph over the plane $v=w$ wherever

$$\cos B(1+\sin A)\neq\cos A(1+\sin B). \tag{9}$$

REMARK 1. In the situation above,

$$\begin{aligned}\sqrt{2}\mathfrak{X}(x)&=\left(\int_{x_0}^{x}\cos A(t)\,dt,\ \int_{x_0}^{x}\sin A(t)\,dt,\ x-x_0\right),\\ \sqrt{2}\mathfrak{Y}(y)&=\left(\int_{y_0}^{y}\cos B(t)\,dt,\ \int_{y_0}^{y}\sin B(t)\,dt,\ y-y_0\right)\end{aligned} \tag{10}$$

for fixed values of x_0 and y_0. Over the domain D of the null coordinates x, y it is natural to think of (1) and (10) as a *Weierstrass representation* for $\mathcal{Z}$: $D \to E^{3,1}$ with $A(x)$ and $B(y)$ as *Weierstrass functions*. Conversely, any two C^∞ functions $A: J_A \to R$ and $B: J_B \to R$ with $A(x) \neq B(y) \bmod 2\pi$ determine a timelike minimal immersion $\mathcal{Z}: J_A \times J_B \to E^{3,1}$ given by (1) and (10) for any x_0 in J_A and y_0 in J_B. Moreover, x, y are normalized null coordinates for this $\mathcal{Z}$, so that (4) and (5) are valid with

$$\text{sign}\, K = \text{sign}\, A'(x)B'(y) \tag{11}$$

giving control over the sign of Gauss curvature.

REMARK 2. If $\mathcal{Z}(S)$ is the graph of a smooth function over a whole plane, then an immersion $\mathcal{Z}: S \to E^{3,j}$ for $j = 0, 1$ is called *entire*. As explained in Remark 1 of [7], the proof of the Hilbert-Holmgren theorem in [6] shows that for any entire timelike minimal $\mathcal{Z}: S \to E^{3,1}$ there exist global null coordinates x, y on S defined for all real values which are Tchebychev for I^0. These coordinates are easily normalized to give a *global* Weierstrass representation for $\mathcal{Z}$ in terms of functions $A(x)$ and $B(y)$ defined for all real values of x and y. In [5], Magid shows the existence of such a global representation for timelike minimal immersions which are entire over spacelike or timelike planes, using other methods.

3. Associative Minimal Surfaces in $E^{3,1}$

We begin by describing the family of associate immersions for a spacelike minimal $\mathcal{Z}: S \to E^{3,1}$. Local coordinates x, y on S are isothermal for a spacelike $\mathcal{Z}: S \to E^{3,1}$ if and only if

$$I = \lambda(dx^2 + dy^2)$$

for some function $\lambda > 0$. Mean curvature $H \equiv 0$ for a spacelike $\mathcal{Z}: S \to E^{3,1}$ if and only if

$$\mathcal{Z}_{xx} + \mathcal{Z}_{yy} \equiv 0$$

for all isothermal coordinates x, y on S. Thus a spacelike $\mathcal{Z}: S \to E^{3,1}$ is minimal if and only if, in terms of isothermal coordinates x, y and $z = x + iy$,

$$\Phi \stackrel{\text{def}}{=} 2\mathcal{Z}_z = (\varphi_k) = \mathcal{Z}_x - i\mathcal{Z}_y, \quad k = 1, 2, 3$$

is holomorphic, so that $\mathcal{Z}$ has the local expression

$$\mathcal{Z} = \text{Re} \int_{z_0}^{z} \Phi\, dz + C_0$$

for a constant vector C_0, while

$$\langle \mathcal{Z}_x, \mathcal{Z}_x \rangle = \langle \mathcal{Z}_y, \mathcal{Z}_y \rangle = \lambda, \qquad \langle \mathcal{Z}_x, \mathcal{Z}_y \rangle = 0. \tag{12}$$

It is known (see, e.g., [3]) that a spacelike minimal $\mathcal{Z}: S \to E^{3,1}$ is the "twin" of a minimal immersion $\tilde{\mathcal{Z}}: D \to E^{3,0}$ defined by taking $\tilde{\Phi} = (\tilde{\varphi}_k)$ with

$$\varphi_1 = \tilde{\varphi}_1, \quad \varphi_2 = \tilde{\varphi}_2, \quad \varphi_3 = i\tilde{\varphi}_3$$

and setting

$$\tilde{\mathcal{Z}} = \operatorname{Re} \int_{z_0}^{z} \tilde{\Phi}\, dz + C_0$$

over the domain D of the coordinates x, y which are isothermal for both I and $\tilde{I}^0$. Classically (see [4]) the associate minimal immersions $\tilde{\mathcal{Z}}_\theta : D \to E^{3,0}$ are given for each real θ by

$$\tilde{\mathcal{Z}}_\theta = \operatorname{Re} e^{i\theta} \int_{z_0}^{z} \tilde{\Phi}\, dz + C_0.$$

Note that $\mathcal{Z}$ is retrieved over D by using the first two coordinate functions of $\tilde{\mathcal{Z}}$, and the third coordinate function of $\tilde{\mathcal{Z}}_{\pi/2}$.

The identical construction applied to $\mathcal{Z}$ yields for each real θ an associate immersion $\mathcal{Z}_\theta : D \to E^{3,1}$ given by

$$\mathcal{Z}_\theta = \operatorname{Re} e^{i\theta} \int_{z_0}^{z} \Phi\, dz + C_0. \tag{13}$$

Here

$$\begin{aligned} (\mathcal{Z}_\theta)_x &= \cos\theta\, \mathcal{Z}_x + \sin\theta\, \mathcal{Z}_y, \\ (\mathcal{Z}_\theta)_y &= \sin\theta\, \mathcal{Z}_x + \cos\theta\, \mathcal{Z}_y, \end{aligned} \tag{14}$$

so $(\mathcal{Z}_\theta)_x$ and $(\mathcal{Z}_\theta)_y$ span the same oriented plane as $\mathcal{Z}_x$ and $\mathcal{Z}_y$, giving $\nu_\theta = \nu$. By (12) and (14), we have

$$\langle (\mathcal{Z}_\theta)_x, (\mathcal{Z}_\theta)_x \rangle = \langle (\mathcal{Z}_\theta)_y, (\mathcal{Z}_\theta)_y \rangle = \lambda, \qquad \langle (\mathcal{Z}_\theta)_x, (\mathcal{Z}_\theta)_y \rangle = 0,$$

so x, y are isothermal for $I_\theta = I$, $\mathcal{Z}_\theta$ is spacelike, and $K_\theta = K$. Finally, $\mathcal{Z}_\theta$: $D \to E^{3,1}$ is minimal since $\Phi_\theta = e^{i\theta}\Phi$ is holomorphic for each θ, giving $H_\theta \equiv H \equiv 0$.

If $\mathcal{Z} : S \to E^{3,1}$ is an entire spacelike minimal immersion, then $\mathcal{Z}(S)$ must be a plane (see [1]). Thus the associate family $\mathcal{Z}_\theta : S \to E^{3,1}$ can be globally defined, with each $\mathcal{Z}_\theta(S)$ a plane.

REMARK 3. Given a harmonic immersion $\mathcal{Z} : (S, g) \to E^{3,0}$ with definite g, the construction (13) yields a family of associate harmonic immersions $\mathcal{Z}_\theta$: $(D, g) \to E^{3,0}$ over the domain D of g-isothermal coordinates x, y with $z = x + iy$, as described in [7]. The situation specializes to the case of a spacelike minimal $\mathcal{Z} : S \to E^{3,1}$ when g is proportional to the metric I induced on S by $\mathcal{Z}$ from $E^{3,1}$.

Suppose now that $\mathcal{Z} : S \to E^{3,1}$ is a timelike minimal immersion. Given any constant $c > 0$, define the *associate immersion* $\mathcal{Z}_c : D \to E^{3,1}$ by setting

$$\mathcal{Z}_c(x, y) = c\mathcal{X}(x) + \mathcal{Y}(y)/c \tag{15}$$

over the domain D of any null coordinates x, y on S. Since $(\mathcal{Z}_c)_x = c\mathcal{Z}_x$ and $(\mathcal{Z}_c)_y = \mathcal{Z}_y / c$, both $(\mathcal{Z}_c)_x$ and $(\mathcal{Z}_c)_y$ are null vectors, with

$$\langle (\mathcal{Z}_c)_x, (\mathcal{Z}_c)_y \rangle = \langle \mathcal{Z}_x, \mathcal{Z}_y \rangle.$$

Thus $\mathcal{Z}_c: S \to E^{3,1}$ is timelike, with $I_c = I$ and $\nu_c = \nu$, so that by (5),

$$\sqrt{2}II_c = -cA'(x)\,dx^2 + (B'(y)/c)\,dy^2,$$

with $K_c \equiv K$ and $H_c \equiv H \equiv 0$. However, $II_c \equiv II$ if and only if $A'(x) \equiv B'(y) \equiv 0$, so the associate pairing is normally not a congruence.

REMARK 4. The choice of different null coordinates x, y on S will leave the family of immersions $\mathcal{Z}_c$ unchanged, but may reindex the maps, exchanging $\mathcal{Z}_c$ with $\mathcal{Z}_{1/c}$.

REMARK 5. If a timelike minimal $\mathcal{Z}: S \to E^{3,1}$ is entire, use of the global null coordinates x, y on S described in Remark 2 allows global definition of the timelike minimal immersions $\mathcal{Z}_c: S \to E^{3,1}$. Theorem 1 below states that each such $\mathcal{Z}_c$ must also be entire.

REMARK 6. Given a harmonic immersion $\mathcal{Z}: (S, g) \to E^{3,0}$ with indefinite g, the construction (15) yields a family of associate harmonic immersions $\mathcal{Z}_c: (D, g) \to E^{3,0}$ over the domain D of g-null coordinates x, y on S, as described in [7]. The situation specializes to the case of a timelike minimal $\mathcal{Z}$: $S \to E^{3,1}$ when g is proportional to the metric I induced on S by $\mathcal{Z}$ from $E^{3,1}$.

Given an entire timelike minimal $\mathcal{Z}: S \to E^{3,1}$, the following theorem provides an infinite family of entire timelike minimal $\mathcal{Z}_c: S \to E^{3,1}$. Normally, no two of the immersions $\mathcal{Z}_c$ are congruent.

THEOREM 1. *If the timelike minimal immersion $\mathcal{Z}: S \to E^{3,1}$ is entire over a plane, then for any constant $c > 0$, the timelike minimal immersion $\mathcal{Z}_c$: $S \to E^{3,1}$ is entire over the same plane.*

Proof. Since a minimal immersion $\mathcal{Z}: S \to E^{3,1}$ yields a harmonic immersion $\mathcal{Z}: (S, I^1) \to E^{3,0}$, the result is an immediate corollary of Remark 5 above and the Theorem in [7]. □

4. Assigned Timelike Minimal Immersions

It is easier to find entire harmonic immersions $\mathcal{Z}: (S, g) \to E^{3,0}$ with indefinite prescribed metric g than it is to find entire timelike minimal immersions $\mathcal{Z}: S \to E^{3,1}$. Put another way, it is easier to find smooth functions $\mathcal{X}: R \to E^{3,0}$ and $\mathcal{Y}: R \to E^{3,0}$ so that $\mathcal{Z}: R \times R \to E^{3,0}$ given by

$$\mathcal{Z}(x, y) = \mathcal{X}(x) + \mathcal{Y}(y)$$

is an entire immersion than it is to accomplish the same task with the additional requirement that $\mathcal{X}'(x)$ and $\mathcal{Y}'(y)$ must be null vectors. (See [7] for a detailed study of harmonic maps $\mathcal{Z}: (S, g) \to E^{3,0}$ with indefinite g.)

Suppose now that the harmonic immersion $\mathcal{Z}: (S, g) \to E^{3,0}$ with indefinite g is a local graph over some plane $\mathcal{Q}$, and let x, y be g-null coordinates over the domain D on S. The construction that follows assigns to $\mathcal{Z}$ a timelike minimal immersion $\tilde{\mathcal{Z}}: D \to E^{3,1}$ which is a local graph over the spacelike coordinate plane $\mathcal{P}$ in $E^{3,1}$. If $\mathcal{Z}$ is entire over $\mathcal{Q}$ then $\tilde{\mathcal{Z}}$ can be globally defined over S, and Theorem 2 states that $\tilde{\mathcal{Z}}: S \to E^{3,1}$ must be entire over $\mathcal{P}$.

As explained below in Remark 7, there is no loss of generality in working with g-null coordinates x, y which are I^0-Tchebychev for $\mathcal{Z}$, so that $\mathcal{Z}(x,y) = \mathcal{X}(x) + \mathcal{Y}(y)$ with x and y Euclidean arc length parameters for $\mathcal{X}(x)$ and $\mathcal{Y}(y)$ respectively. Because $\mathcal{Z}$ is a local graph over $\mathcal{Q}$, the normals $\nu^0(x,y)$ over D are never parallel to $\mathcal{Q}$. Thus we can rotate the u, v, w Cartesian coordinate axes in $E^{3,0}$ so $\mathcal{Q}$ is parallel to the horizontal u, v coordinate plane $\mathcal{P}$, with the normals $\nu^0(x,y)$ over D all pointing upward. Then $\mathcal{Z}_x = \mathcal{X}'(x)$, $\mathcal{Z}_y = \mathcal{Y}'(y)$, and the planes they determine are never vertical.

On the 2-sphere Σ given by $u^2+v^2+w^2=1$, let γ be the circle along which $w = \sqrt{2}/2$. Draw great semicircular arcs σ_x and σ_y joining the poles on Σ through the endpoints of $\mathcal{X}'(x)$ and $\mathcal{Y}'(y)$ respectively. Define $\tilde{\mathcal{X}}'(x)$ and $\tilde{\mathcal{Y}}'(y)$ as the vectors pointing to the intersection of γ with σ_x and σ_y respectively.

The $E^{3,0}$ unit vectors $\tilde{\mathcal{X}}'(x)$ and $\tilde{\mathcal{Y}}'(y)$ are null in $E^{3,1}$. Moreover, $\tilde{\mathcal{X}}'(x)$ is never parallel to $\tilde{\mathcal{Y}}'(y)$. Otherwise the vertical plane containing $\mathcal{X}'(x)$ and $\tilde{\mathcal{X}}'(x)$ would also contain $\tilde{\mathcal{Y}}'(y)$, putting $\mathcal{Y}'(y)$ in the same vertical plane as $\mathcal{X}'(x)$, a contradiction. The timelike minimal immersion $\tilde{\mathcal{Z}}: D \to E^{3,1}$ assigned to $\mathcal{Z}$ is given by

$$\tilde{\mathcal{Z}}(x,y) = \tilde{\mathcal{X}}(x) + \tilde{\mathcal{Y}}(y),$$

(16)

$$\tilde{\mathcal{X}}(x) = \int_{x_0}^{x} \tilde{\mathcal{X}}'(x)\,dx, \qquad \tilde{\mathcal{Y}}(y) = \int_{y_0}^{y} \tilde{\mathcal{Y}}'(y)\,dy$$

for a fixed choice of x_0, y_0 in D.

The plane spanned by $\tilde{\mathcal{X}}'(x)$ and $\tilde{\mathcal{Y}}'(y)$ is never vertical. Otherwise $\tilde{\mathcal{X}}'(x)$ and $\tilde{\mathcal{Y}}'(y)$ would lie on the same great circle through the poles on Σ, putting $\tilde{\mathcal{X}}'(x)$ and $\tilde{\mathcal{Y}}'(y)$ in the same vertical plane, a contradiction. Thus $\tilde{\mathcal{Z}}$ is a local graph over $\mathcal{P}$.

Since the $E^{3,0}$ unit normals for $\mathcal{Z}$ point upward, the $E^{3,0}$ unit normals for $\tilde{\mathcal{Z}}$ also point upward. To see this, note that $\mathcal{X}'(x)$ and $\mathcal{Y}'(y)$ can be moved continuously along σ_x and σ_y to coincide with $\tilde{\mathcal{X}}'(x)$ and $\tilde{\mathcal{Y}}'(y)$ respectively, and in the process, the $E^{3,0}$ vector product $\mathcal{X}'(x) \times \mathcal{Y}'(y)$ always points upward.

REMARK 7. Given g-null coordinates $\hat{x}, \hat{y}$ over a domain D in S, the timelike minimal immersion $\tilde{\mathcal{Z}}: D \to E^{3,1}$ for a harmonic immersion $\mathcal{Z}: (S, g) \to E^{3,0}$ with indefinite g is determined once a point p_0 is fixed in D. Indeed, since $\mathcal{Z}(\hat{x}, \hat{y}) = \tilde{\mathcal{X}}(\hat{x}) + \tilde{\mathcal{Y}}(\hat{y})$ over D, one need only use Euclidean arc length parameters x and y for $\mathcal{X}(\hat{x})$ and $\mathcal{Y}(\hat{y})$ respectively (with $x'(\hat{x}) > 0$, $y'(\hat{y}) > \tilde{0}$, and $x = x_0$ and $y = y_0$ at p) to obtain $\tilde{\mathcal{Z}}: D \to E^{3,1}$ from (16). A different choice of p_0 changes $\tilde{\mathcal{Z}}$ by translation.

REMARK 8. $\tilde{\mathcal{Z}}$ can vary with the choice of g-null coordinates in S. For example, use of $\mathring{x}=y$ and $\mathring{y}=-x$ in place of x and y in (16) produces an assigned immersion $\mathring{\tilde{\mathcal{Z}}}(\mathring{x},\mathring{y})=\mathring{\tilde{\mathcal{X}}}(\mathring{x})+\mathring{\tilde{\mathcal{Y}}}(\mathring{y})$, with $\mathring{\tilde{\mathcal{X}}}(\mathring{x})$ describing $\tilde{\mathcal{Y}}(y)$ and $\mathring{\tilde{\mathcal{Y}}}(\mathring{y})$ describing the reflection of $\tilde{\mathcal{X}}(x)$ in the w-axis. Thus any discussion of $\tilde{\mathcal{Z}}$ presumes some fixed choice of g-null coordinates on S (or the switch to the g-null I^0-Tchebychev coordinates for $\mathcal{Z}$ which they determine).

REMARK 9. For fixed g-null x, y on S, $\mathcal{Z}$ and its translations or rotations in $E^{3,0}$ all determine the same $\tilde{\mathcal{Z}}$. But harmonic immersions whose images have vastly different shapes can be assigned to the same $\tilde{\mathcal{Z}}$ as well. To see this, let D be the domain of normalized null coordinates x, y for a timelike minimal immersion $\tilde{\mathcal{Z}}: S\to E^{3,1}$. Let ν be any smooth map from D to the open upper hemisphere of Σ. For each x, y in D, let σ be the great circle on Σ cut out by the plane through $(0,0,0)$ perpendicular to $\nu(x,y)$. Since ν avoids the equator, σ is never vertical. Draw great semicircular arcs σ_x and σ_y joining the poles on Σ through the endpoints of $\tilde{\mathcal{X}}'(x)$ and $\tilde{\mathcal{Y}}'(y)$ respectively. Take $\mathcal{X}'(x)$ and $\mathcal{Y}'(y)$ to be the vectors pointing to the intersections of σ with σ_x and σ_y respectively. Then, for any fixed x_0, y_0 in D, $\mathcal{Z}(x,y)=\mathcal{X}(x)+\mathcal{Y}(y)$ with

$$\mathcal{X}(x)=\int_{x_0}^{x}\mathcal{X}'(x)\,dx, \qquad \mathcal{Y}(y)=\int_{y_0}^{y}\mathcal{Y}'(y)\,dy$$

is a harmonic immersion $\mathcal{Z}: (D,\tilde{I})\to E^{3,0}$, where $\tilde{I}$ is the induced metric for $\tilde{\mathcal{Z}}$. By construction, $\nu=\nu(x,y)$ is the $E^{3,0}$ Gauss map for $\mathcal{Z}$, making $\mathcal{Z}$ a local graph over $\mathcal{P}$ and $\tilde{\mathcal{Z}}$ the timelike minimal immersion assigned to $\mathcal{Z}$ over D. If $A(x)$ and $B(y)$ are the Weierstrass functions for $\tilde{\mathcal{Z}}$ as described in Remark 1, $A(x)$, $B(y)$ and $\nu(x,y)$ can be thought of as the Weierstrass functions for $\mathcal{Z}$.

REMARK 10. It might seem that $\tilde{\mathcal{Z}}$ is the same for all associate immersions $\mathcal{Z}_c$ as it is for $\mathcal{Z}$, since at any point of S one uses the same $E^{3,0}$ unit vectors $\mathcal{X}'(x)$ and $\mathcal{Y}'(y)$ to construct $\tilde{\mathcal{X}}_c'(x)$ and $\tilde{\mathcal{Y}}_c'(y)$ as to construct $\tilde{\mathcal{X}}'(x)$ and $\tilde{\mathcal{Y}}'(y)$. But I^0-Tchebychev g-null coordinates x, y for $\mathcal{Z}$ determine I_c^0-Tchebychev g-null coordinates $\hat{x}=cx$ and $\hat{y}=y/c$ for $\mathcal{Z}_c$ so that one integrates $\tilde{\mathcal{X}}_c'(\hat{x})$ and $\tilde{\mathcal{Y}}_c'(\hat{y})$ with respect to $\hat{x}$ and $\hat{y}$ in (16). Thus $(\tilde{\mathcal{Z}})_c$ is the timelike minimal immersion assigned to $\mathcal{Z}_c$, and assignment commutes with the associate construction.

In case the harmonic immersion $\tilde{\mathcal{Z}}: (S,g)\to E^{3,0}$ with indefinite g is entire over a plane $\mathcal{Q}$, use of global g-null I^0-Tchebychev coordinates provided by the Hilbert –Holmgren theorem (see Remark 1 in [7]) gives global definition of $\tilde{\mathcal{Z}}: S\to E^{3,1}$. The next result states that $\tilde{\mathcal{Z}}$ is entire over $\mathcal{P}$. The proof is a variant of the argument establishing the Theorem in [7] for the case of indefinite g. Since considerable reference is made to that argument in the rest of this paper, we denote it by the symbol $(*)$.

THEOREM 2. *If a harmonic immersion* $\mathcal{Z}\colon (S,g)\to E^{3,0}$ *with indefinite* g *is entire, then any timelike minimal* $\tilde{\mathcal{Z}}\colon S\to E^{3,1}$ *globally assigned to* $\mathcal{Z}$ *is entire over the spacelike coordinate plane* $\mathcal{P}$ *in* $E^{3,1}$.

Proof. We assume that $\mathcal{Z}$ is entire over $\mathcal{P}$ with its Euclidean normals pointing upward, since Euclidean motions of $\mathcal{Z}$ do not change $\tilde{\mathcal{Z}}$. Let x, y be the global g-null I^0-Tchebychev coordinates on S used to define $\tilde{\mathcal{Z}}\colon S\to E^{3,1}$. With no loss of generality, we assume that

$$\mathcal{Z}(0,0)=\mathfrak{X}(0)=\mathfrak{Y}(0)=(0,0,0)$$

and take $x_0=y_0=0$ in (16), so that

$$\tilde{\mathcal{Z}}(0,0)=\tilde{\mathfrak{X}}(0)=\tilde{\mathfrak{Y}}(0)=(0,0,0).$$

Let $T\colon E^{3,j}\to\mathcal{P}$ denote orthogonal projection onto $\mathcal{P}$, so that all claims for Z in (∗) apply here to $Z=T\circ\mathcal{Z}$ which is given by $Z(x,y)=X(x)+Y(y)$, where $X=T\circ\mathfrak{X}$ and $Y=T\circ\mathfrak{Y}$.

To prove that $\tilde{\mathcal{Z}}$ is entire over $\mathcal{P}$, we must show that $\tilde{Z}=T\circ\tilde{\mathcal{Z}}$ is a diffeomorphism onto $\mathcal{P}$. We argue much as in (∗), establishing in appropriate order various of the properties {1} through {8} from (∗), substituting $\tilde{Z}$ for Z_c throughout. Of course, $\tilde{Z}(x,y)=\tilde{X}(x)+\tilde{Y}(y)$, where $\tilde{X}=T\circ\tilde{\mathfrak{X}}$ and $\tilde{Y}=T\circ\tilde{\mathfrak{Y}}$.

By the construction of $\tilde{\mathcal{Z}}\colon S\to E^{3,1}$, we know that

$$X'(x)=\lambda\tilde{X}'(x),\quad 0<\lambda\le\sqrt{2},$$
$$Y'(y)=\mu\tilde{Y}'(y),\quad 0<\mu\le\sqrt{2}. \tag{17}$$

Thus {1} holds, and the argument showing {2} in (∗) applies so long as $\tilde{X}(x)$ and $\tilde{Y}(y)$ are simple curves. But with the u, v axes rotated in $\mathcal{P}$ if necessary, as in (∗), $Y'(y)$ points into the half-plane $v>0$ and $X'(x)$ into the half-plane $u>0$. By (17), $\tilde{X}(x)$ and $\tilde{Y}(y)$ are regularly parametrized simple curves. In fact, since (4) applies to $\tilde{\mathcal{Z}}$, x and y are constant speed parametrizations of $\tilde{X}(x)$ and $\tilde{Y}(y)$ respectively.

Suppose now that $\tilde{u}(x)<\hat{u}$ on $\tilde{X}(x)=(\tilde{u}(x),\tilde{v}(x))$, so that $\hat{u}>0$ since $\tilde{u}(0)=0$. For $x\le 0$, $u(x)\le 0<\hat{u}$ on $X(x)=(u(x),v(x))$ since $u(0)=0$ and $u'(x)>0$. For $x>0$, (17) gives $0<u'(x)\le\sqrt{2}\tilde{u}'(x)$, so that

$$u(x)=\int_0^x u'(x)\,dx\le\sqrt{2}\int_0^x\tilde{u}'(x)\,dx<\sqrt{2}\hat{u},$$

which contradicts ⟨4⟩ from (∗). Assuming $\tilde{u}(x)>\hat{u}$ yields the same contradiction. Thus $\tilde{X}(x)$ crosses every vertical line in $\mathcal{P}$; similarly, $\tilde{Y}(y)$ crosses every horizontal line in $\mathcal{P}$. Thus [5] and [6] from (∗) hold for $\tilde{X}(x)$ and $\tilde{Y}(y)$, from which {3} and {4} follow.

In case $\tilde{Y}'(y)$ is constant, it is always vertical. Then $\tilde{Y}(y)$ describes the whole v-axis, and $\tilde{Z}(x,y)$ covers all points in $\mathcal{P}$ reached by moving the v-axis parallel to itself, with $\tilde{X}(0)=(0,0)$ going to $\tilde{X}(x)$, for all x. By {5}, $\tilde{Z}$ is onto $\mathcal{P}$. If $\tilde{X}'(x)$ is always horizontal, {6} shows that $\tilde{Z}$ is onto $\mathcal{P}$.

Suppose then that $Y'(y)$ is not constant and that $X'(x)$ is not always horizontal. Let $M>0$ be the constant provided by {5} and {6}. For $i=1,2,3,4$ define R^i, R^i_k, $\mathcal{Q}^i$, and $\mathcal{C}^i(p)$ for $\tilde{X}(x)$ and $\tilde{Y}(y)$ as they were for $(cX)(x)$ and $(Y/c)(y)$ in (*). One easily checks that {8} is valid. To prove that $\tilde{Z}$ is onto $\mathcal{P}$, we do not need the full force of {7}. It is enough to show that $\tilde{X}(x)$ and $\tilde{Y}(y)$ cannot both lie to the same (open) side of a line of slope M or $-M$ in $\mathcal{P}$.

Assume first that $\tilde{X}(x)$ lies to the left of the line $v=-Mu+b$. Then

$$\tilde{v}(x)+M\tilde{u}(x)<b$$

for all x, with $b>0$ since $\tilde{u}(0)=\tilde{v}(0)=0$. For $x\le 0$, $X(x)$ lies to the left of $v=-Mu+b$ since, by ⟨4⟩, the ray of $X(x)$ over $(-\infty,0]$ lies in the closed sector bounded by the rays $v=\pm Mu$ with $u\le 0$. If $x>0$, {5} gives $\tilde{v}'(x)+M\tilde{u}'(x)\ge 0$, so by (17) we have

$$\begin{aligned} v(x)+Mu(x) &= \int_0^x \langle v'(x)+Mu'(x)\rangle\,dx \\ &\le \sqrt{2}\int_0^x \langle \tilde{v}'(x)+M\tilde{u}'(x)\rangle\,dx = \sqrt{2}\langle \tilde{v}(x)+M\tilde{u}(x)\rangle < \sqrt{2}b. \end{aligned}$$

Thus $X(x)$ lies to the left of $v=-Mu+\sqrt{2}b$ for all x. The analogous argument shows that $X(x)$ lies to the left of $v=Mu+\sqrt{2}b$ if $\tilde{\mathfrak{X}}(x)$ lies to the left of $v=Mu+b$. Similarly, {6} shows that $Y(y)$ lies to the left of $v=-Mu+\sqrt{2}b$ (resp. $v=Mu+\sqrt{2}b$) if $\tilde{\mathfrak{Y}}(y)$ lies to the left of $v=-Mu+b$ (resp. $v=Mu+b$). Clearly, then, $\tilde{X}(x)$ and $\tilde{Y}(y)$ cannot both lie to the left of a line of the form $v=\pm Mu+b$ unless $X(x)$ and $Y(y)$ both lie to one side of a line of the form $v=\pm Mu+\sqrt{2}b$, contradicting {7}. One argues in the same way that $\tilde{X}(x)$ and $\tilde{Y}(y)$ cannot both be to the right of a line of the form $v=\pm Mu+b$.

The portion of the proof (*) starting with the paragraph before {8} now applies with $\tilde{X}(x)$ and $\tilde{Y}(y)$ in place of $(cX)(x)$ and $(Y/c)(y)$ to show that $\tilde{Z}$ is onto $\mathcal{P}$. □

REMARK 11. The converse of Theorem 2 fails. To see this, take any timelike minimal $\tilde{\mathfrak{Z}}: S\to E^{3,1}$ which is entire over $\mathcal{P}$. Use global normalized null coordinates x,y for $\tilde{\mathfrak{Z}}$ and assume that $\tilde{\mathfrak{Z}}(0,0)=\tilde{\mathfrak{X}}(0)=\tilde{\mathfrak{Y}}(0)=(0,0,0)$ so that $\tilde{Z}(0,0)=\tilde{X}(0)=\tilde{Y}(0)=(0,0)$ for $\tilde{Z}=T\circ\tilde{\mathfrak{Z}}$. Let $\mathfrak{X}'(x)$ point to the closed upper hemisphere of Σ, with $T\circ\mathfrak{X}'(x)=e^{-x^2}\tilde{X}'(x)$. Let $\mathfrak{Y}'(y)$ point to Σ, with $T\circ\mathfrak{Y}'(y)=e^{-y^2}\tilde{Y}'(y)$ and the $E^{3,0}$ vector product $\mathfrak{X}'(x)\times\mathfrak{Y}'(y)$ pointing upward. If g is the induced metric $\tilde{I}$ for $\tilde{\mathfrak{Z}}: S\to E^{3,1}$, then the immersion $\tilde{\mathfrak{Z}}: (S,g)\to E^{3,0}$ defined by $\mathfrak{Z}(x,y)=\mathfrak{X}(x)+\mathfrak{Y}(y)$, with

$$\mathfrak{X}(x)=\int_0^x \mathfrak{X}'(x)\,dx \quad\text{and}\quad \mathfrak{Y}(y)=\int_0^y \mathfrak{Y}'(y)\,dy,$$

is harmonic. Moreover, $\tilde{\mathfrak{Z}}: S\to E^{3,1}$ is the timelike minimal immersion assigned to $\mathfrak{Z}$ using x,y on S. With x and y constant speed parameters on $\tilde{\mathfrak{X}}(x)$

and $\tilde{\mathcal{Y}}(y)$ respectively, the construction forces $\mathcal{X}(x)$ and $\mathcal{Y}(y)$ to have finite length, so that $\mathcal{Z}$ cannot be entire over $\mathcal{P}$.

THEOREM 3. *If the timelike minimal immersion* $\tilde{\mathcal{Z}}: S \to E^{3,1}$ *assigned to a harmonic immersion* $\mathcal{Z}: (S, g) \to E^{3,0}$ *is entire over* $\mathcal{P}$, *and if the* $E^{3,0}$ *unit normals for* $\mathcal{Z}$ *avoid a neighborhood of the equator on* $\mathcal{P}$, *then* $\mathcal{Z}$ *is also entire over* $\mathcal{P}$.

Proof. Let x, y be the global g-null I^0-Tchebychev coordinates on S used to construct $\tilde{\mathcal{Z}}$. Assume that $\mathcal{Z}(0,0) = \mathcal{X}(0) = \mathcal{Y}(0) = (0,0,0)$ with $x_0 = y_0 = 0$ in (16), so that $\tilde{\mathcal{Z}}(0,0) = \tilde{\mathcal{X}}(0) = \tilde{\mathcal{Y}}(0) = (0,0,0)$. Then the claims ⟨1⟩ through ⟨5⟩ in (∗) for $\mathcal{Z}$ apply here to $\tilde{Z} = T \circ \tilde{\mathcal{Z}}$, since $\tilde{\mathcal{Z}}$ is entire over $\mathcal{P}$.

To show that $\mathcal{Z}$ is entire over $\mathcal{P}$, adapt the argument in (∗) with Z_c replaced by $Z = T \circ \mathcal{Z}$. The new reasoning needed is the same as that above in the proof of Theorem 2. The one difference is that here,

$$\tilde{X}'(x) = \lambda X'(x), \quad \sqrt{2}/2 \le \lambda \le C,$$
$$\tilde{Y}'(y) = \mu Y'(y), \quad \sqrt{2}/2 \le \mu \le C,$$

where the constant C depends upon the positive infimum of the distance of $\nu^0(x, y)$ from the equator on Σ. Thus C plays the role in this argument that $\sqrt{2}$ played in the proof of Theorem 2.

5. Examples

Calabi noted that suitable periodic Weierstrass functions $A(x)$ and $B(y)$ produce an entire "doubly periodic" timelike minimal immersion $\mathcal{Z}: x, y$-plane $\to E^{3,1}$. To see this, let $A(x)$ and $B(y)$ be smooth functions with periods α and β respectively such that

$$\frac{\pi}{12} < A < \frac{\pi}{6}, \qquad \frac{7\pi}{12} < B < \frac{2\pi}{3}.$$

Since

$$0 < \sin\left(\frac{\pi}{12}\right) < \sin A < \sin\left(\frac{\pi}{6}\right),$$
$$0 < \cos\left(\frac{\pi}{6}\right) < \cos A < \cos\left(\frac{\pi}{12}\right),$$
$$0 < \sin\left(\frac{2\pi}{3}\right) < \sin B < \sin\left(\frac{7\pi}{12}\right),$$
$$0 > \cos\left(\frac{7\pi}{12}\right) > \cos B > \cos\left(\frac{2\pi}{3}\right),$$

we know by (6), (7), (8), and (9) that the $\mathcal{Z}$ defined by $A(x)$ and $B(y)$ in Remark 1 is a local graph over the three coordinate planes and the null plane $v = w$. Suppose we write $Z = T \circ \mathcal{Z}$, where here T represents the Euclidean orthogonal projection of $E^{3,j}$ onto the particular plane under discussion. Since $Z(x, y) = X(x) + Y(y)$, projection to the u, v-plane gives

$$\sqrt{2}X'(x) = (\cos A, \sin A), \qquad \sqrt{2}Y'(y) = (\cos B, \sin B);$$

projection to the u, w-plane gives

$$\sqrt{2}X'(x) = (\cos A, 1), \qquad \sqrt{2}Y'(y) = (\cos B, 1);$$

projection to the v, w-plane gives

$$\sqrt{2}X'(x) = (\sin A, 1), \qquad \sqrt{2}Y'(y) = (\sin B, 1);$$

and projection to the $v = w$ plane with Cartesian coordinates $u, \sqrt{2}v$ gives

$$\sqrt{2}X'(x) = \left(\cos A, \frac{1+\sin A}{\sqrt{2}}\right), \qquad \sqrt{2}Y'(y) = \left(\cos B, \frac{1+\sin B}{\sqrt{2}}\right).$$

In all cases, the coordinate functions for $X(x)$ and $Y(y)$ are strictly monotonic, so $X(x)$ and $Y(y)$ are simple curves. Since $X'(x)$ is never parallel to $Y'(y)$, the argument showing {2} in (*) applies here to Z in place of Z_c to show that Z is one-to-one. To see that Z is onto the plane in question, note that the fundamental forms I and II for $\mathcal{Z}$ depend only on $A(x)$ and $B(y)$ and are thus periodic in x and y. Since the fundamental theorem for timelike surfaces in $E^{3,1}$ is just like the classical version in $E^{3,0}$ (see [8]), it follows that $\mathcal{Z}$ over any period rectangle $[j\alpha, (j+1)\alpha] \times [k\beta, (k+1)\beta]$ for integers j and k is congruent to $\mathcal{Z}$ over $[0, \alpha] \times [0, \beta]$. Thus $\mathcal{Z}$ is entire over any of the planes considered. More generally, $\mathcal{Z}$ is entire over any plane for which $\mathcal{Z}$ is a local graph, so long as $\mathcal{Z}$ is one-to-one over $[0, \alpha] \times [0, \beta]$.

REMARK 12. Suppose $\mathcal{Z}: S \to E^{3,1}$ is a timelike minimal immersion, entire over the u, w-plane. Assume that $\mathcal{Z}(S)$ contains $(0,0,0)$ and take global normalized null coordinates x, y on S with $\mathcal{X}(0) = \mathcal{Y}(0) = (0,0,0)$. Because $\cos A(x) \neq \cos B(y)$, we have $-A(x) \neq B(y) \bmod 2\pi$. Hence the timelike minimal immersion $\hat{\mathcal{Z}}: S \to E^{3,1}$, given by the Weierstrass functions $\hat{A}(x) = -A(x)$ and $\hat{B}(y) = B(y)$ with $x_0 = y_0$, is well defined. The projections of $\mathcal{Z}$ and $\hat{\mathcal{Z}}$ to the u, w-plane are identical, making $\hat{\mathcal{Z}}$ entire over the u, w-plane. But the Gauss curvatures K and $\hat{K}$ for $\mathcal{Z}$ and $\hat{\mathcal{Z}}$ are related by

$$\text{sign}\, \hat{K}(x,y) = -\text{sign}\, K(x,y). \tag{18}$$

The example

$$v = w \tanh u$$

of a timelike minimal surface in $E^{3,1}$ on which $K > 0$ thereby shows the existence of a convex ($\hat{K} < 0$) timelike minimal immersion $\hat{\mathcal{Z}}: S \to E^{3,1}$ that is entire over the u, w-plane. The role of the u, w-plane is not special here to the extent that any timelike plane can be taken to the u, w-plane by a motion of $E^{3,1}$.

The simple "flip trick" in Remark 11 can fail if applied to a timelike minimal immersion $\mathcal{Z}: S \to E^{3,1}$ entire over the u, v-plane. To see why, suppose the u, v-plane is rotated as necessary in (*) for $Z = T \circ \mathcal{Z}$. Using the notation in (*), trouble can arise if the disjoint arcs C_x and C_y share a common endpoint. If C_y contains its left endpoint, and $C_x \cap \bar{C}_y \neq \emptyset$, then $\hat{\mathcal{Z}}: S \to E^{3,1}$

(defined by using $\hat{A}(x) = -A(x)$ and $\hat{B}(y) = B(y)$) is not a local graph over the u, v-plane, because somewhere $\hat{X}'(x) = -\hat{Y}'(y)$. Even when C_x and C_y avoid their common endpoint, $\hat{X}'(x) = -\hat{Y}'(y)$ might lie to one side of a line parallel to the diameter through the left endpoint of $\hat{C}_y = C_y$, so that $\hat{\mathcal{Z}}$ would not be entire over the u, v-plane. We show in Remark 13 that the flip trick works if C_x, C_y, and $-C_y$ have no common endpoint. The role of the u, v-plane is not special here, to the extent that any spacelike plane can be taken to the u, v-plane by a motion of $E^{3,1}$.

REMARK 13. Given a timelike minimal $\mathcal{Z}: S \to E^{3,1}$ entire over the u, v-plane $\mathcal{P}$, reorient S if necessary so the Euclidean normals for $\mathcal{Z}$ point upward. Take global normalized null coordinates x, y on S for $\mathcal{Z}$, and assume with no loss of generality that $\mathcal{Z}(0,0) = \mathcal{X}(0) = \mathcal{Y}(0) = (0,0,0)$. Rotate the u, v-axes in $\mathcal{P}$ as specified in $(*)$. Then $Z = T \circ \mathcal{Z}$ given by $Z(x,y) = X(x) + Y(y)$ is a diffeomorphism onto $\mathcal{P}$, with the properties $\langle 1 \rangle$ through $\langle 5 \rangle$ in $(*)$. If $\bar{C}_x \cap \bar{C}_y = \emptyset$ and $\bar{C}_x \cap -\bar{C}_y = \emptyset$, one can take $M = M_1$ in $\langle 4 \rangle$ and $M = M_2$ in $\langle 5 \rangle$ with $0 < M_1 < M_2$, so both $\langle 4 \rangle$ and $\langle 5 \rangle$ hold for any $M = \hat{M}$ with $M_1 < \hat{M} < M_2$. If $\hat{\mathcal{Z}}: x, y\text{-plane} \to E^{3,1}$ is now defined by the flip trick, then $\hat{Z} = T \circ \hat{\mathcal{Z}}$ has the form $\hat{Z}(x,y) = \hat{X}(x) + \hat{Y}(y)$, with $\hat{X}(x)$ the reflection of $X(x)$ in the u-axis and $\hat{Y}(y) = Y(y)$. Thinking of $\hat{Z}$ in place of Z_c in $(*)$, $\{1\}$ is obvious, and $\{2\}$ can be argued as in $(*)$ since $\hat{X}(x)$ and $\hat{Y}(y)$ are simple curves. Properties $\{3\}$ and $\{4\}$ are clear. Moreover, one can take $M = M_1$ in $\{5\}$ and $M = M_2$ in $\{6\}$ with $0 < M_1 < M_2$, so both $\{5\}$ and $\{6\}$ hold for any $M = \hat{M}$ with $M_1 < \hat{M} < M_2$. Form the rays $R^1(x)$, $R^2(y)$, $R^3(x)$, and $R^4(y)$ for $\hat{X}(x)$ and $\hat{Y}(y)$ just as they were for $(cX)(x)$ and $(Y/c)(y)$ in $(*)$. Fix $\hat{M}$ with $M_1 < \hat{M} < M_2$. Any line ℓ of slope $\pm\hat{M}$ which crosses the positive u-axis crosses $R^1(x)$. Otherwise, $R^1(x)$ would lie in the closed triangular region bounded by ℓ and the lines $v = \pm\hat{M}u$, contradicting $\{4\}$. Similarly, any line of slope $\pm M$ that crosses the positive v-axis crosses $R^2(y)$, any line of slope $\pm\hat{M}$ that crosses the negative u-axis crosses $R^3(x)$, and any line of slope $\pm\hat{M}$ that crosses the negative v-axis crosses $R^4(y)$. The final argument in $(*)$ can now be adapted to show that $\hat{Z}$ is onto $\mathcal{P}$. One uses $M = M_1$ to define $\mathcal{C}^1$ and $\mathcal{C}^3$ and $M = M_2$ to define $\mathcal{C}^2$ and $\mathcal{C}^4$. Any line of slope $-\hat{M}$ through $p \in \mathcal{Q}^1$ crosses both the positive u-axis and the positive v-axis, and thus hits both $R^1(x)$ and $R^2(y)$. Analogous remarks apply if p lies in $\mathcal{Q}^2$, $\mathcal{Q}^3$, or $\mathcal{Q}^4$. Of course, (18) holds for $\mathcal{Z}$ and $\hat{\mathcal{Z}}$.

The flip trick can provide an example of a timelike minimal immersion $\hat{\mathcal{Z}}$: $x, y\text{-plane} \to E^{3,1}$ with $\hat{K}(x,y) < 0$ which is entire over the u, v-plane. Since $\hat{\mathcal{Z}}: x, y\text{-plane} \to E^{3,0}$ is complete with Euclidean Gauss curvature $\hat{K}^0(x,y) > 0$, the image of $\hat{\mathcal{Z}}$ lies to one side of its tangent plane at every point. Magid gives an explicit example of such an immersion in [5]. We obtain $\hat{\mathcal{Z}}$ in Remark 14 by giving an example of a timelike minimal $\mathcal{Z}: x, y\text{-plane} \to E^{3,1}$ with $K(x,y) > 0$ which is entire over the u, v-plane, and which has the properties shown in Remark 13 to produce a $\hat{\mathcal{Z}}$ of the sort just described.

REMARK 14. Use the Weierstrass functions $A(x)$ and $B(y)$ given by

$$4A(x) = \arc\tan x, \qquad 4B(y) = 2\pi + \arc\tan y,$$

so that

(19) $$\frac{-\pi}{8} < A(x) < \frac{\pi}{8}, \qquad \frac{3\pi}{8} < B(y) < \frac{5\pi}{8}.$$

Define the timelike minimal immersion $\mathcal{Z}: x, y\text{-plane} \to E^{3,1}$ with $x_0 = y_0 = 0$ in (10) so that $Z = T \circ \mathcal{Z}$ is given by $Z(x, y) = X(x) + Y(y)$ with

(20) $$\sqrt{2}X'(x) = (\cos A, \sin A), \qquad \sqrt{2}Y'(y) = (\cos B, \sin B).$$

Since $\bar{C}_x \cap \bar{C}_y = \emptyset$ and $\bar{C}_x \cap -\bar{C}_y = \emptyset$, Remark 13 will apply if we show that Z is a diffeomorphism onto $\mathcal{P}$. We think of Z in place of Z_c in (*). By (19) and (20), $X'(x)$ is never parallel to $Y'(y)$, so that Z is a local diffeomorphism giving the relevant fact in {1}. Moreover, $\cos A > 0$ and $\sin B > 0$ force $X(x)$ and $Y(y)$ to be simple curves, so the argument in (*) gives {2}. Indeed, (19) and (20) show that there are constants M_1 and M_2 with $0 < M_1 < 1 < M_2$, so that $X(x)$ is the graph of a function $v = F(u)$ with $|F'(u)| < M_1$ and $Y(y)$ is the graph of a function $u = G(v)$ with $|G'(v)| < 1/M_2$. Since x and y are constant speed parameters for $X(x)$ and $Y(y)$ (respectively) defined for all real values, $F(u)$ and $G(v)$ are defined for all real values, giving {5} and {6} from which {3} and {4} easily follow. We can now adapt (*) as we did in Remark 13, using $M = 1$ in place of $\hat{M}$. Hence $\mathcal{Z}$ is entire, and since

$$\operatorname{sign} K(x, y) = \operatorname{sign} A'(x)B'(y) > 0,$$

the flip trick provides a timelike minimal $\hat{\mathcal{Z}}: x, y\text{-plane} \to E^{3,1}$ entire over the u, v-plane which is convex.

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