Finite Group Actions on the Moduli Space of Self-Dual Connections, II

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1. Introduction

Let G be a finite group, and let M be a simply connected, closed, smooth 4-dimensional manifold with a positive definite intersection form and a smooth action of G on it. Let $\Pi: E \to M$ be a quaternion line bundle with instanton number one and with a G-action on E through bundle isomorphism such that Π is a G-map. Let $\mathfrak{M}$ be the set of self-dual connections on E modulo the group $\mathcal{G}$ of gauge transformations. If we use a G-invariant metric on M then the moduli space $\mathfrak{M}$ is a G-space, but $\mathfrak{M}$ might not be a manifold because of the nonvanishing second cohomology group of the fundamental elliptic complex or because of reducible self-dual connections.

In [5] Donaldson used a compact perturbation of a Fredholm map to make $\mathfrak{M}$ a manifold. In [7] Freed and Uhlenbeck proved that for generic metric on M the moduli space $\mathfrak{M}$ is a manifold. We cannot use their methods directly to make the G-set $\mathfrak{M}$ into a G-manifold, because the perturbation cannot be made G-invariant and so the method of [7] need not yield a G-invariant metric.

In [4] we defined cohomology classes which are obstructions to perturbing the G-set $\mathfrak{M}$ into a G-manifold. In this paper we shall show that when G is the cyclic group of order 2^n, there are classes of metrics on M for which these obstruction classes vanish.

We will follow the notations in [4]; $\hat{}$ stands for irreducibility. Let $\mathcal{C}$ be the set of all connections on E and let $\mathcal{G}$ be the group of gauge transformations on E. Consider the map $\Phi: \mathcal{C}^{\wedge} \times C^G \to \Omega^2_-(\mathcal{G}_E)$ given by $\Phi(\nabla, \psi) = P_- \psi^{-1*} R^\nabla$, where $C^G = C^k(GL(TM))^G$ is the set of G-equivariant C^k-automorphisms of the tangent bundle of M. Here $P_-: \Omega^2(\mathcal{G}_E) \to \Omega^2_-(\mathcal{G}_E)$ is the projection to the anti-self-dual part (with respect to a fixed G-invariant metric on M) of the 2-forms of M with values in the adjoint bundle associated to E, and R^∇ denotes the curvature of the connection ∇. Our result is that there is an open G-set O of $\mathcal{C}^{\wedge} \times C^G$ such that the restriction map $\Phi: O \to \Omega^2_-(\mathcal{G}_E)$ is smooth and has zero as a regular value. The G-set O contains all $(\nabla, \psi) \in \mathcal{C}^{\wedge} \times C^G$ such that $\Pi(\nabla) \in \mathfrak{M}^G$ with respect to the metric $\psi^*(g)$ on M, where $\Pi: \mathcal{C}^{\wedge} \to \mathcal{B}^{\wedge} = \mathcal{C}^{\wedge}/\mathcal{G}$ is the projection. Furthermore, there is an

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open dense set in C^G such that the moduli space $\mathfrak{M}^{\wedge}$ of irreducible self-dual connections is a manifold in a G-neighborhood of $\mathfrak{M}^{\wedge G}$ for each metric in the dense set of C^G. We also have a similar result for the reducible connections.

Combining these results provides a dense set in C^G of the C^∞, G-invariant metrics on M such that the moduli space $\mathfrak{M}$ is a manifold in a G-neighborhood of the fixed point set $\mathfrak{M}^G$ for each metric in the dense set. Moreover, for these metrics the Petrie obstruction classes vanish. This result is true for the cyclic group G of order 2^n.

By perturbing the free part in $\mathfrak{M}$ we obtain a smooth G-manifold $\mathfrak{M}$ of dimension 5 with a finite number λ of singular points, each of which has a cone neighborhood of $\mathbf{C}P^2$, where $\lambda = \operatorname{rank} H^2(M; Z)$.

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2. Generic Metrics on M for the Irreducible Self-Dual Connections

As in [4], let $C = C^k(GL(TM))$ be the set of C^k-automorphisms of the tangent bundle. Throughout this section we fix G to be the group $\mathbf{Z}/2 = \{1, h\}$. Let C^G be the G-fixed point set of C. For a large fixed k, we define a map $\Phi: \mathcal{C}^{\wedge}_{l-1} \times C^G \to \Omega^2_-(\mathcal{G}_E)_{l-2}$ by

$$\Phi(\nabla, \phi) = P_-(\phi^{-1*} R^\nabla).$$

Here the $(l-i)$ means (as usual) the Sobolev norm, and Ω^2_- is the anti-self-dual part with respect to a fixed G-invariant metric g on M. Hereafter we will omit the Sobolev norm notations.

Let $\Pi: \mathcal{C}^{\wedge} \to \mathcal{B}^{\wedge} = \mathcal{C}^{\wedge}/\mathcal{G}$ be the projection. If the group G acts nontrivially on M, then the free part $M^0 = M \backslash M^G$ is an open dense subset of M.

LEMMA 2.1. *For each $x \in M^0$ there is an open neighborhood U of x such that for each $\sigma \in T(C)$ there exists a $\tau \in T(C^G)$ such that $\sigma | U = \tau | U$.*

Proof. For each $x \in M^0$ choose a neighborhood U of x such that $h(U) \cap U = \varnothing$. Note that

$$\sigma \in T(C) = C^k(\operatorname{End}(TM)) \quad \text{and} \quad T(C^G) = C^k(\operatorname{End}(TM)^G).$$

Choose a cutoff function $f: M^0 \to [0,1]$ such that $f | U \equiv 1$ and $h(\operatorname{supp} f) \cap (\operatorname{supp} f) = \varnothing$. Let $\bar{\sigma} = f\sigma$; then $h[\operatorname{supp}(\bar{\sigma})] \cap [\operatorname{supp}(\bar{\sigma})] = \varnothing$. Since $f | U \equiv 1$, $\sigma = \bar{\sigma}$ on U. By averaging, $\tau \equiv \sum_{h \in G} h^*(\bar{\sigma})$ and $h(\tau) = \tau$. We have $\tau \in T(C^G)$ such that $\tau = \sigma$ on U, because for any $y \in U$

$$\tau(y) = \sum_{h \in G} h^*(\bar{\sigma})(y) = \sum_{h \in G} h \cdot \bar{\sigma}(h^{-1}(y))$$
$$= \bar{\sigma}(y) = \sigma(y). \qquad \square$$

Note that Lemma 2.1 is true for any finite group G. The following theorem is one of our main theorems in this section. To prove this theorem we will follow [7, Thm. 3.4]. However, in our case zero may not be a regular value of Φ because we replace C by C^G. So we must restrict the domain of Φ to a suitably chosen open subset of $\mathcal{C}^\wedge \times C^G$.

LEMMA 2.2 [7]. *Let V be a 4-dimensional oriented Euclidean vector space and W any vector space. Suppose $R \in \Lambda^2_+ V^* \otimes W$ and $\phi \in \Lambda^2_- V^* \otimes W$ satisfies $(r^*R, \phi) = 0$ for all $r \in \mathrm{End}(V)$. Then in $\Lambda^2_+ V^* \otimes \Lambda^2_- V^*$ the images $\mathrm{Im}(R)$ and $\mathrm{Im}(\phi)$ are orthogonal.*

THEOREM 2.3. *There is an open G-set O of $\mathcal{C}^\wedge \times C^G$ such that:*

(i) *the open set O contains all $(\nabla, \psi) \in \mathcal{C}^\wedge \times C^G$ such that $\Pi(\nabla) \in \mathfrak{M}^G$ with respect to the metric $\psi^*(g)$ on M, where $\Pi: \mathcal{C}^\wedge \to \mathcal{B}^\wedge = \mathcal{C}^\wedge/\mathcal{G}$ is the projection; and*

(ii) *the restriction map $\Phi: O \to \Omega^2_-(\mathcal{G}_E)$ is smooth and has zero as a regular value.*

Proof. It is sufficient to prove that the differential $\delta\Phi_{(\nabla,\psi)}$ is surjective whenever $\Phi(\nabla, \psi) = 0$, and that the gauge equivalence class $[\nabla] \in \mathfrak{M}^{\wedge G}$ with respect to the metric $\psi^*(g)$ on M. The differential

$$\delta\Phi_{(\nabla,\psi)}: \Omega^1(\mathcal{G}_E) \times C^k(\mathrm{End}(TM)^G) \to \Omega^2_-(\mathcal{G}_E)$$

splits into two pieces:

$$\delta\Phi_{(\nabla,\psi)} = \delta_1\Phi_{(\nabla)} + \delta_2\Phi_{(\psi)}: \Omega^1(\mathcal{G}_E) \times C^k(\mathrm{End}(TM)^G) \to \Omega^2_-(\mathcal{G}_E),$$

where

$$(\delta_1\Phi_{(\nabla)})(A) = P_-(\psi^{-1*}\nabla A), \qquad (\delta_2\Phi_\psi)(r) = P_-(\psi^{-1*}(r^*R^\nabla))$$

for $A \in \Omega^1(\mathcal{G}_E)$, and $r \in C^k(\mathrm{End}(TM)^G$. We must show that $\mathrm{Coker}(\delta\Phi) = 0$. If $\phi \in \mathrm{Coker}(\delta\Phi)$ then $\phi \in \mathrm{Coker}(\delta_1\Phi)$, so that

$$0 = \int_M (P_-\psi^{-1*}\nabla A, \phi)_g = \int_M (\nabla A, \psi^*(\phi))_{\psi^*(g)} = \int_M (A, \nabla^*\tilde{\phi})_{\psi^*(g)}$$

for all $A \in \Omega^1(\mathcal{G}_E)$, where $\tilde{\phi} = \psi^*(\phi)$. Since ϕ is continuous we have the pointwise equation $\nabla^*\tilde{\phi} = 0$. Since $\phi \in \mathrm{Coker}(\delta_2\Phi)$,

$$0 = \int_M (P_-\psi^{-1*}(r^*R^\nabla), \phi)_g = \int_M (r^*R^\nabla, \tilde{\phi})_{\psi^*(g)}$$

for all $r \in C^k(\mathrm{End}(TM)^G)$. Now, since $\nabla \in \mathfrak{M}^{\wedge G}$, we have $h(\nabla) = g(\nabla)$ for some gauge transformation g. Consider first the case where $g = \mathrm{id}$. Since $h(\nabla) = \nabla$, $(r^*R^\nabla, \tilde{\phi})_{\psi^*(g)} = (r^*R^\nabla, h\tilde{\phi})_{\psi^*(g)}$. So

$$0 = \int_M (r^*R^\nabla, \tilde{\phi} + h\tilde{\phi})_{\psi^*(g)} \quad \text{for all } r \in C^k(\mathrm{End}(TM)^G).$$

Thus we have $(r^*R^\nabla, \tilde{\phi} + h\tilde{\phi})_{\psi^*(g)} = 0$ at each point of M^0.

We also clearly have $(r^*R^\nabla, \tilde{\phi} - h\tilde{\phi})_{\psi^*(g)} = 0$, and since $2\tilde{\phi} = (\tilde{\phi} + h\tilde{\phi}) + (\tilde{\phi} - h\tilde{\phi})$, we obtain

$$0 = (r^*R^\nabla, 2\tilde{\phi})_{\psi^*(g)}, \quad \text{so} \quad (r^*R^\nabla, \tilde{\phi})_{\psi^*(g)} = 0$$

at each point $x \in M^0$.

Next suppose that $\Phi(\nabla, \phi) = 0$ and $h(\nabla) = g(\nabla)$ for some gauge transformation $g \in \mathcal{G}$. Let $\sigma = hg$; then σ^2 is a gauge transformation and $\sigma(\nabla) = \nabla$. Since ∇ is irreducible this means that $\sigma^2 = \pm 1$. Again, since $\phi \in \operatorname{Coker} \delta_2 \Phi$, $0 = \int_M (r^*R^\nabla, \tilde{\phi})_{\psi^*(g)}$, where $\tilde{\phi} = \psi^*(\phi)$. Now, since $\sigma^2\tilde{\phi} = \tilde{\phi}$ and since $\tilde{\phi} + \sigma\tilde{\phi}$ is σ-invariant and h-invariant in M,

$$(r^*R^\nabla, \tilde{\phi})_{\psi^*(g)} = (\sigma r^*R^\nabla, \sigma\tilde{\phi})_{\psi^*(g)} = (r^*R^\nabla, \sigma\tilde{\phi})_{\psi^*(g)}$$

for all $r \in C^k(\operatorname{End}(TM))^G$. Thus we have $0 = \int_M (r^*R^\nabla, \tilde{\phi} + \sigma\tilde{\phi})_{\psi^*(g)}$, and so $(r^*R^\nabla, \tilde{\phi} + \sigma\tilde{\phi})_{\psi^*(g)} = 0$. As above, $(r^*R^\nabla, \tilde{\phi} - \sigma\tilde{\phi})_{\psi^*(g)} = 0$, and by adding we have $(r^*R^\nabla, \tilde{\phi})_{\psi^*(g)} = 0$ at each point $x \in M^0$. By Lemmas 2.1 and 2.2, the images $\operatorname{Im}(R^\nabla)$ and $\operatorname{Im}(\tilde{\phi})$ are pointwise orthogonal on M^0 and so on M. Thus at each point where R^∇ and $\tilde{\phi}$ are nonzero, one of R^∇ or $\tilde{\phi}$ has rank 1. We shall sketch the rest of the proof, which proceeds exactly as in [7]. Since R^∇ is self-dual and $\tilde{\phi}$ is anti-self-dual with respect to the metric $\psi^*(g)$, we have $\nabla R^\nabla = \nabla^* R^\nabla = \nabla\tilde{\phi} = \nabla^*\tilde{\phi} = 0$.

Suppose $\tilde{\phi} \not\equiv 0$. Suppose $\tilde{\phi} \neq 0$ and R^∇ has rank 2 in some neighborhood of a point. Write $\tilde{\phi} = a \otimes u$, where $a \in \Omega^2_-$ and $u \in \mathcal{G}_E$ with $|u| = 1$.

Then, following the proof in [7, pp. 66–67], we have $\nabla u = 0$. By Lemma 2.2, $(R^\nabla, u) = 0$ and $\nabla^2 u = [R^\nabla, u] = 0$. This cannot happen for nonzero vectors u and R^∇ on R^3. So R^∇ has rank of at most 1. Suppose $R^\nabla = \sigma \otimes u$ with $|u| = 1$ on some open set. Then $\nabla u = 0$, and the complement of $\{R^\nabla = 0\}$ is connected; otherwise, $\nabla^*\nabla + \nabla\nabla^*$ has negative eigenvalues on a domain (see [7]). Then we can extend u on M such that $\nabla u = 0$. Since ∇ is irreducible, this is a contradiction. Also, since $k = 1$, $R^\nabla \not\equiv 0$. Thus we have shown that $\tilde{\phi} \equiv 0$ and so $\phi \equiv 0$. If $\Phi(\nabla, \psi) = 0$, then $\Phi(g(\nabla), \psi) = 0$ and $\Phi(h(\nabla), \psi) = 0$ for all $g \in \mathcal{G}$ and for all $h \in G$, since the metric ψ is G-invariant and the self-duality equation is independent of gauge transformations. Thus there exists an open G-set O in $\mathcal{C}^\wedge \times C^G$ such that (a) the open set O contains all $(\nabla, \psi) \in \mathcal{C}^\wedge \times C^G$ such that $\Pi(\nabla) \in \mathfrak{M}^G$ with respect to the metric $\psi^*(g)$ on M, where

$$\Pi \colon \mathcal{C}^\wedge \to \mathcal{B}^\wedge = \mathcal{C}^\wedge/\mathcal{G}$$

is the projection; and (b) the restriction map $\Phi_1 \colon O \to \mathcal{C}^\wedge \times C^G \to \Omega^2_-(\mathcal{G}_E)$ of $\Phi \colon \mathcal{C}^\wedge \times C^G \to \Omega^2_-(\mathcal{G}_E)$ has zero as a regular value.

We consider the following diagram:

$$\begin{array}{ccccccc}
 & & \Phi_1^{-1}(0) & \hookrightarrow & O & \hookrightarrow & \mathcal{C}^\wedge \times C^G \xrightarrow{\Phi} \Omega^2_-(\mathcal{G}_E) \\
 & & \downarrow \pi & & \downarrow \pi & & \downarrow \pi X \mathrm{id} \\
(\Phi_1^{-1}(0)/\mathcal{G})^G & \to & \Phi_1^{-1}(0)/\mathcal{G} & \to & O/\mathcal{G} & \to & \mathcal{B}^\wedge \times C^G. \\
\downarrow \bar{\bar{\pi}} & & \downarrow \bar{\pi} & & \downarrow & & \downarrow \\
C^G & = & C^G & = & C^G & = & C^G
\end{array}$$

The set $\Phi_1^{-1}(0)$ is a manifold in O. For each metric $\Psi \in C^G$ the fixed point set $\mathcal{M}_\Psi^{\wedge G} = \{(\nabla, \Psi) \in \mathcal{C}^\wedge \times \{\Psi\} \mid \Phi(\nabla, \Psi) = 0$ for some $g \in \mathcal{G}$ and $h(\nabla) = g(\nabla)\}/\mathcal{G}$. This completes the proof of Theorem 2.3. □

LEMMA 2.4. *$\Phi_1^{-1}/\mathcal{G} \subset O/\mathcal{G}$ is a manifold.*

Proof. First, $O/\mathcal{G}$ is clearly a manifold by the slice theorem. Let $\Phi_1: O/\mathcal{G} \to \Omega_-^2(\mathcal{G}_E)$ be the induced map. For any $(\nabla, \phi) \in \Phi_1^{-1}(0) \subset O$ we have a neighborhood $U_\nabla \times V_\phi \subset \mathcal{C}^\wedge \times C^G$. Because $T\mathcal{C}^\wedge = \Omega^1(\mathcal{G}_E) = \mathrm{Im}\,\nabla \oplus \mathrm{Ker}\,\nabla^*$ and $\delta_1\Phi_1|_{\mathrm{Im}\,\nabla} = 0$, the differential $\delta\bar{\Phi}_1: T(O/\mathcal{G}) \to \Omega^2(\mathcal{G}_E)$ has the same image as $\delta\Phi_1$ on $O \to \Omega_-^2(\mathcal{G}_E)$. Thus $\bar{\Phi}_1$ has zero as a regular value. This completes the proof. □

LEMMA 2.5. *The projection $\Phi_1^{-1}(0)/\mathcal{G} \xrightarrow{\bar{\pi}} C^G$ is a Fredholm map.*

Proof. We consider the construction of $O \subset \mathcal{C}^\wedge \times C^G$. The differential $\delta\bar{\Pi}$: $T_{(\nabla,\phi)}(\Phi_1^{-1}(0)/\mathcal{G}) = \{(A, r) \in \Omega^1(\mathcal{G}_E) \times TC^G \mid \delta_1\Phi_1(A) + \delta_2\Phi_1(r) = 0$ and $\nabla^* A = 0\} \to C^k(\mathrm{End}(TM))^G$. Since $\mathrm{Ker}\,\delta\bar{\Pi} = \{(A, r): \delta_1\Phi_1(A) = \nabla^* A = r = 0\} = H^1_{(\nabla,\phi)}$ and

$$\mathrm{Im}\,\delta\bar{\Pi} = (\delta_2\Phi_1)^{-1}(\mathrm{Im}\,\delta_1\Phi_1|_{\mathrm{Ker}\,\nabla^*}) = (\delta_2\Phi_1)^{-1}(\mathrm{Im}\,\delta_1\Phi_1),$$

we have $\mathrm{Coker}\,\delta\bar{\Pi} \simeq H^2_{(\nabla,\phi)}$ because $\delta\Phi_{1(\nabla,\phi)}$ is onto. Because ∇ is irreducible self-dual, $\mathrm{Ind}\,\bar{\Pi} =$ (index of the fundamental elliptic complex for ∇) $= H^1_\nabla - H^2_\nabla$ has a numerical index 5.

Now we use the Sard–Smale theorem for the Fredholm map

$$\bar{\Pi}: \Phi_1^{-1}(0)/\mathcal{G} \to C^G,$$

between paracompact Banach manifolds. The set of regular values of $\bar{\Pi}$ is an open dense set in C^G, because $\dim(H^2_\nabla)$ is an upper semi-continuous integer valued function on $\Phi_1^{-1}(0)/\mathcal{G}$. If ϕ is a regular value then $\bar{\Pi}^{-1}(\phi)$ is a manifold with dimension 5, which is a neighborhood of $\mathcal{M}_\phi^{\wedge G}$ in $\mathcal{M}_\phi^\wedge$ with respect to the G-invariant metric $\phi^*(g)$ on M. □

THEOREM 2.6. *There is an open dense set in C^G such that the moduli space $\mathcal{M}^\wedge$ of irreducible connections is a manifold in a G-neighborhood of $\mathcal{M}^{\wedge G}$ for each metric in the dense set.*

3. Generic Metrics on M for the Reducible Self-Dual Connections

Let ∇ be a reducible self-dual G-invariant connection in $\mathcal{M}$. The isotropy group $\Gamma^\nabla = \{g \in \mathcal{G} \mid g(\nabla) = \nabla\}$ of ∇ in the group $\mathcal{G}$ of gauge transformations is isomorphic to the unitary group $U(1)$. There is a nonzero $\psi \in \Omega^0(\mathcal{G}_E)$ with $\nabla(\psi) = 0$. The map $\psi: E \to E$ has global eigenvalues $it, -it$ for some $t \in R$. Then

$$\rho = \begin{pmatrix} it & 0 \\ 0 & -it \end{pmatrix} \in \Omega^0(\mathcal{G}_E)$$

is covariant constant with corresponding gauge transformation

$$g=\begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}.$$

We then get a splitting of $E=l\oplus\bar{l}$ and of the associated Lie algebra bundle $\mathcal{G}_E=R\oplus l$, where the gauge group acts trivially on $\mathbf{R}$ and with weight 2 on l. Also, we can split the fundamental elliptic complex

$$(*)\quad \begin{aligned} 0\to\Omega^0(\mathcal{G}_E)\xrightarrow{\nabla}\Omega^1(\mathcal{G}_E)\xrightarrow{d^\nabla}\Omega^2_-(\mathcal{G}_E)\to 0 &\equiv (0\to\Omega^0\xrightarrow{d}\Omega^1\xrightarrow{d_-}\Omega^2_-\to 0)\\ &\oplus(0\to\Omega^0(l)\xrightarrow{D}\Omega^1(l)\xrightarrow{D_-}\Omega^2_-(l)\to 0), \end{aligned}$$

where d is the usual exterior derivative and the reducible connection $\nabla=D\oplus D$ on $E=l\oplus\bar{l}$.

As Theorem 2.3 shows, the main differences between [7] and our case is that if $h(\nabla)=g(\nabla)$ then $(*)$ is $\sigma=hg$-invariant, where $\sigma=hg$.

To prove that the map

$$Q\colon[\Omega^1(l)\backslash 0]\times C^G\to\Omega^0(l)\oplus\Omega^2_-(l)$$

given by $(A,\phi)\mapsto(D^*\phi^{-1*}A,\ P_-\phi^{-1*}DA)$ is a submersion throughout $Q^{-1}(0)$, we should use the condition of Theorem 2.3, because we restrict to the G-invariant metrics C^G. Then $Q^{-1}(0)$ is a manifold. The projection $\Pi\colon Q^{-1}(0)\to C^G$ has index 6 by considering the split complex. Again using the Sard–Smale theorem and the upper continuity of dim H^2_D, we have the following.

THEOREM 3.1. *There is an open dense set in C^G such that the second cohomology group $H^2_\nabla=0$ in the fundamental elliptic complex of each G-invariant reducible self-dual connection ∇.*

Proof. See Theorem 2.3 and [7]. □

4. Extensions and Conclusion

If G is $\mathbf{Z}/2$, then there is an open dense set in the G-invariant metrics C^G on M such that the space of G-invariant self-dual connections $\mathfrak{M}^G$ has a G-invariant manifold neighborhood in $\mathfrak{M}$. In [4] we have shown that if we have a G-fixed point set $F=\{P_i\}_{i=1}^{n_1}\cup\{T^{\lambda_i}\}_{i=1}^{n_2}$ on M, where P_i is an isolated fixed point and T^{λ_i} is a surface with genus λ_i, then there are cohomology obstruction classes. By Theorems 2.6 and 3.1, we have the following.

THEOREM 4.1. *There is a dense set in the set C^G of G-invariant metrics on M such that the moduli space $\mathfrak{M}$ is a manifold in a G-neighborhood of the fixed point set $\mathfrak{M}^G$ for each metric in the dense set. Moreover, for these metrics, the Petrie obstruction classes vanish.*

In Theorem 2.3, if we do not restrict the map $\Phi\colon\mathcal{C}^{\wedge}\times C^G\to\Omega^2_-(\mathcal{G}_E)$, then zero may not be a regular value. In Theorem 3.1, if we do not choose a G-

invariant reducible connection, then the map Q may not be a submersion. So we need a G-equivariant compact perturbation at the free part to get a G-manifold $\mathfrak{M}$. More generally, if G is a finite cyclic group of order 2^n, then we can extend Theorem 2.3.

THEOREM 4.2. *If G is a finite cyclic group of order 2^n and M has a G-action, then there is a dense set in the set C^G of G-invariant metrics on M such that the moduli space $\mathfrak{M}$ is a manifold in a G-neighborhood of the fixed point set $\mathfrak{M}^G$ for each metric in the dense set.*

Sketch of Proof. As in Theorem 2.3, we can easily have that

$$(r^*R^\nabla, \tilde{\phi}+h\tilde{\phi}+\cdots+h^{2^n-1}\tilde{\phi})_{\Psi^*(g)}=0$$

and

$$(r^*R^\nabla, \tilde{\phi}-h\tilde{\phi}+h^2\tilde{\phi}-\cdots-h^{2^n-1}\tilde{\phi})_{\Psi^*(g)}=0.$$

Adding and dividing by 2,

$$(r^*R^\nabla, \tilde{\phi}+h^2\tilde{\phi}+\cdots+h^{2^n-2}\tilde{\phi})_{\Psi^*(g)}=0.$$

Continuing this process, we have

$$(r^*R^\nabla, \tilde{\phi}+h^{2^{n-1}}\tilde{\phi})_{\Psi^*(g)}=0;$$
$$(r^*R^\nabla, \tilde{\phi}-h^{2^{n-1}}\tilde{\phi})_{\Psi^*(g)}=0.$$

So we have $(r^*R^\nabla, \tilde{\phi})_{\Psi^*(g)}=0$. If $h(\nabla)=g(\nabla)$, $g\neq\pm1$, and ∇ is irreducible, then $(hg)^{2^n}(\nabla)=\nabla$, $(hg)^{2^n}=hgh^{2^n-1}\cdot h^2gh^{2^n-2}\cdots g=\pm1$. If $h(\nabla)=g(\nabla)$, $g\notin\Gamma_\nabla$, and ∇ is reducible, then $hg=g_1h_1$ for some $g_1\in\Gamma_\nabla$ and for some $h_1\in G$.

There is a G-invariant metric on M such that the moduli space $\mathfrak{M}$ is a manifold in a G-neighborhood of the set $\mathfrak{M}^G$ of the G-invariant self-dual connections. Under the G-invariant metric the cross section $\mathcal{B}=\mathcal{C}/\mathcal{G}\xrightarrow{\Psi}\mathcal{C}\times_{\mathcal{G}}\Omega^2_-(\mathcal{G}_E)$ given by $\Psi(\nabla)=(\nabla, R^\nabla_-)$ is transversal throughout the G-neighborhood; this was proved in [4].

THEOREM 4.3. *There is a compact G-equivariant perturbation $\Psi_1=\Psi+\sigma$ of Ψ such that the perturbed moduli space $\mathfrak{M}_1=\{\nabla\in\mathcal{C}/\mathcal{G}\mid\Psi_1(\nabla)=0\}$ is a smooth 5-dimensional G-manifold with a finite number λ of singularities, each of which has a neighborhood which is diffeomorphic to a cone on $\mathbf{C}P^2$, where $\lambda=\operatorname{rank} H^2(M:\mathbf{Z})$.*

References

1. M. F. Atiyah, N. Hitchin, and I. Singer, *Self duality in four-dimensional Riemannian geometry,* Proc. Roy. Soc. London Ser. A 362 (1978), 425–461.
2. M. F. Atiyah and I. Singer, *The index of elliptic operators III,* Ann. of Math. (2) 87 (1968), 546–604.
3. G. Bredon, *Introduction to compact transformation groups,* Pure Appl. Math., 46, Academic Press, New York, 1972.

4. Y. S. Cho, *Finite group actions on the moduli space of self-dual connections (I)*, Trans. Amer. Math. Soc., to appear.
5. S. Donaldson, *An application of gauge theory to four-manifold theory,* J. Differential Geom. 18 (1983), 279–315.
6. R. Fintushel and R. Stern, *Pseudofree orbifolds,* Ann. of Math. (2) 122 (1985), 335–364.
7. D. Freed and K. Uhlenbeck, *Instantons and four-manifolds,* Math. Sci. Res. Inst. Publ., 1, Springer, New York, 1984.
8. M. Freedman, *The topology of four dimensional manifolds,* J. Differential Geom. 17 (1983), 357–454.
9. S. Smale, *An infinite dimensional version of Sard's theorem,* Amer. J. Math. 87 (1968), 861–866.
10. C. Taubes, *Self-dual connections on non-self-dual 4-manifolds,* J. Differential Geom. 17 (1982), 139–170.

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