A Removable Set for Lipschitz Harmonic Functions

NGUYEN XUAN UY

1. Introduction

Let K be a compact set of d-dimensional space $\mathbf{R}^d$ $(d \geq 2)$. We denote the class of bounded harmonic functions defined on $\mathbf{R}^d \backslash K$ by $\mathcal{H}^\infty(K)$ and denote the class of those functions in $\mathcal{H}^\infty(K)$ that satisfy a Lipschitz condition of order α $(0 < \alpha \leq 1)$ by $\mathcal{H}^\infty_\alpha(K)$. The following result on the removable singularities of bounded harmonic functions is well known (see [1, Chap. VII] and [6, Chap. III]): Every $f \in \mathcal{H}^\infty(K)$ is extendable harmonically across K if and only if K has a zero capacity.

Regarding the class of $\mathcal{H}^\infty_\alpha(K)$ for $0 < \alpha < 1$, Carleson [1] proved that K is removable if and only if $\Lambda_{d-2+\alpha}(K) = 0$, where $\Lambda_{d-2+\alpha}$ denotes the $(d-2+d)$-dimensional Hausdorff measure.

The motivation for this paper arises primarily from [7], where the author studied the removable singularities of Lipschitz analytic functions. Our purpose here is to show that there can be a set K with $\Lambda_{d-1}(K) > 0$ even though K is removable for the class $\mathcal{H}^\infty_1(K)$.

This contrasts with the following analogous result obtained for analytic functions: For all α $(0 < \alpha \leq 1)$, a compact set $K \subseteq \mathbf{C}$ is removable for the class of bounded analytic functions satisfying a Lipschitz condition of order α if and only if $\Lambda_{1+\alpha}(K) = 0$ (see [2], [7]). Arguments used in this paper are largely based on a related paper by Garnett [4] on removable singularities of bounded analytic functions.

2. A Multi-Dimensional Cantor Set

In this section we define a d-dimensional Cantor set K with $\Lambda_{d-1}(K) > 0$. First, we form a linear Cantor set E with ratio $\lambda = 2^{-d/(d-1)}$, using an inductive method as in the construction of the well-known one-third Cantor set. We obtain $E = \bigcap_{n=0}^\infty E_n$, where $E_0 = [0,1]$ and E_n $(n = 0, 1, 2, \ldots)$ contains 2^n disjoint intervals of length equal to $2^{-nd/(d-1)}$. Define

$$K_n = \prod_{i=1}^d E_n, \quad n = 0, 1, 2, \ldots, \quad \text{and} \quad K = \bigcap_{n=0}^\infty K_n.$$

Received November 11, 1988.
Michigan Math. J. 37 (1990).

We now show that $\Lambda_{d-1}(K)>0$. For this purpose, denote 2^{nd} cubes of K_n by $K_{n,j}$ ($j=1,2,\dots,2^{nd}$) and let $\mathcal{C}$ be the class of all diadic cubes in $\mathbf{R}^d$. Define

$$M_{d-1}(K)=\inf\left\{\sum_j(\operatorname{side}(S_j))^{d-1}: K\subset\bigcup_j S_j,\ S_j\in\mathcal{C}\right\}.$$

It is easy to see that $\Lambda_{d-1}(K)>0$ if and only if $M_{d-1}(K)>0$.

Set $2^{-kj}=\operatorname{side}(S_j)$. Since the distance between any two cubes $K_{n,p}$ and $K_{n,q}$ is at least $(1-2^{-d/(d-1)})2^{-(n-1)d/(d-1)}$, it follows that S_j may intersect only one cube $K_{n,j}$ if n is sufficiently small relative to k_j—for example,

$$2^{-kj}\sqrt{d}<(1-2^{-1/(d-1)})2^{-(n-1)d/(d-1)}. \tag{2.1}$$

Assume without loss of generality that k_j is sufficiently large, and let n_j be the largest integer satisfying (2.1); that is, n_j satisfies the condition

$$\begin{aligned}&\left(\frac{d-1}{d}\right)\left[k_j+\log_2\left(\frac{1-2^{-1/(d-1)}}{\sqrt{d}}\right)\right]\\ &\qquad\le n_j\le\left(\frac{d-1}{d}\right)\left[k_j+\log_2\left(\frac{1-2^{-1/(d-1)}}{\sqrt{d}}\right)\right]+1.\end{aligned} \tag{2.2}$$

Let K_{n_j,p_j} be the only cube that intersects S_j. Then $K\subseteq\bigcup_j K_{n_j,p_j}$. It follows from (2.2) that

$$\begin{aligned}\sum_j(\operatorname{side}(S_j))^{d-1}&=\sum_j 2^{-(d-1)kj}\\ &\ge C\sum_j 2^{-n_jd}\ge C,\end{aligned}$$

where the last inequality holds because

$$\sum_j 2^{-n_jd}\ge 1\quad\text{and}\quad C=\left(\frac{1-2^{-1/(d-1)}}{\sqrt{d}}\right)^{d-1}.$$

This proves $\Lambda_{d-1}(K)>0$.

3. Removable Singularities

We will assume $d\ge 3$. For the case $d=2$ we need only change the fundamental solution of Laplace's equation from $1/r^{d-2}$ to $\log(1/r)$; all arguments we used here can still apply to this case. We should point out that this particular case can also follow directly from Garnett's result in [4], since if $f(z)$ is harmonic and satisfies a Lipschitz condition then $\partial f/\partial z$ is a bounded analytic function.

Let $f\in\mathcal{H}_1^\infty(K)$ and assume that $f(\infty)=0$. We derive, by Green's identity [5], the following relation:

$$\begin{aligned}f(x)=C_0^{-1}\sum_{j=1}^{2^{nd}}\Bigg\{&\int_{\partial K_{n,j}}f(t)\frac{\partial}{\partial n}\left(\frac{1}{|x-t|^{d-2}}\right)d\sigma(t)\\ &-\int_{\partial K_{n,j}}\frac{\partial f}{\partial n}(t)\frac{1}{|x-t|^{d-2}}\Bigg\}\,d\sigma(t)\end{aligned} \tag{3.1}$$

for $x \in K_n^c$. Here $C_0 = (d-2)\omega_d$, ω_d is the area of the unit sphere in $\mathbf{R}^d$, σ denotes the area measure on $\partial K_{n,j}$, and

$$\frac{\partial f}{\partial n}(t) \quad \text{and} \quad \frac{\partial}{\partial n}\left(\frac{1}{|x-t|^{d-2}}\right)$$

are the directional derivatives in the direction of the inner unit vector normal to $\partial K_{n,j}$ at t ($(\partial f/\partial n)(t)$ exists a.e. as a limit). We need the following definitions:

$$a_{n,j} = \int_{\partial K_{n,j}} \frac{\partial f}{\partial n}(t)\, d\sigma(t);$$

$$\|f\|_* = \sup_{x \in K^c} |f(x)| + \sup_{\substack{x \neq y \\ x, y \in K^c}} \frac{|f(x)-f(y)|}{|x-y|}.$$

From now on we shall denote by C a certain constant depending only on the dimension d. C may have different values at different appearances. Observe that each term inside the summation of (3.1) defines a harmonic function on $(K \cap K_{n,j})^c$, and denote this function by $f_{n,j}$.

LEMMA 1. *Suppose $f \in \mathcal{H}_1^\infty(K)$ and $f(\infty) = 0$. There exists a constant C such that*

(i) $\|f_{n,j}\|_\infty \leq C\|f\|_* 2^{-nd/(d-1)}$ *and*
(ii) $\|\partial f/\partial x_k\|_\infty \leq C\|f\|_*$

for all n, j and $k = 1, 2, \ldots, d$.

Proof. Let $\tilde{K}_{n,j}$ be a cube having the same center $c_{n,j}$ as $K_{n,j}$, with side $(K_{n,j}) = C\,\text{side}(K_{n,j})$ for some $C > 1$. C is chosen so that $K_{n,j'} \cap \tilde{K}_{n,j} = \emptyset$ for all $j' \neq j$. For $x \in \text{int}(\tilde{K}) \backslash K \cap K_{n,j}$ we obtain, via Green's identity,

$$\begin{aligned} f_{n,j}(x) &= C_0 f(x) \\ &\quad + \int_{\partial \tilde{K}_{n,j}} f(t) \frac{\partial}{\partial n}\left(\frac{1}{|x-t|^{d-2}}\right) d\sigma(t) \int_{\partial \tilde{K}_{n,j}} \frac{\partial f}{\partial n}(t)\left(\frac{1}{|x-t|^{d-2}}\right) d\sigma(t) \\ &= \int_{\partial \tilde{K}_{n,j}} [f(t)-f(x)] \frac{\partial}{\partial n}\left(\frac{1}{|x-t|^{d-2}}\right) d\sigma(t) \\ &\quad - \int_{\partial \tilde{K}_{n,j}} \frac{\partial f}{\partial n}\left(\frac{1}{|x-t|^{d-2}}\right) d\sigma(t), \end{aligned}$$

because

$$\int_{\partial K_{n,j}} \frac{\partial}{\partial n}\left(\frac{1}{|x-t|^{d-2}}\right) d\sigma(t) = -C_0.$$

Therefore, if x is near $K \cap K_{n,j}$ then

$$\begin{aligned} |f_{n,j}(x)| \leq &\int_{\partial \tilde{K}_{n,j}} |f(t)-f(x)|\left(\frac{1}{|x-t|^{d-1}}\right) d\sigma(t) \\ &+ \int_{\partial \tilde{K}_{n,j}} \left|\frac{\partial f}{\partial n}(t)\right|\left(\frac{1}{|x-t|^{d-2}}\right) d\sigma(t) \leq \end{aligned}$$

$$\leq 2\|f\|_* \int_{\partial\tilde{K}_{n,j}} \left(\frac{1}{|x-t|^{d-2}}\right) d\sigma(t)$$
$$\leq C\|f\|_* 2^{-nd/(d-1)}.$$

This proves (i). To prove (ii), use the property

$$\int_{\partial\tilde{K}_{n,j}} \frac{\partial}{\partial n}\left(\frac{x_k-t_k}{|x-t|^d}\right) d\sigma(t) = 0$$

and write

$$\begin{aligned}\frac{\partial f_{n,j}}{\partial x_k}(x) &= C_0\frac{\partial f}{\partial x_k}(x) + C\int_{\partial\tilde{K}_{n,j}} f(t)\frac{\partial}{\partial n}\left(\frac{x_k-t_k}{|x-t|^d}\right) d\sigma(t)\\ &\quad - C\int_{\partial\tilde{K}_{n,j}} \frac{\partial f}{\partial n}(t)\left(\frac{x_k-t_k}{|x-t|^d}\right) d\sigma(t)\\ &= C_0\frac{\partial f}{\partial x_k}(x) + C\int_{\partial\tilde{K}_{n,j}} [f(t)-f(x)]\frac{\partial}{\partial n}\left(\frac{x_k-t_k}{|x-t|^d}\right) d\sigma(t)\\ &\quad - C\int_{\partial\tilde{K}_{n,j}} \frac{\partial f}{\partial n}(t)\left(\frac{x_k-t_k}{|x-t|^d}\right) d\sigma(t).\end{aligned}$$

Hence

$$\begin{aligned}\left|\frac{\partial f_{n,j}}{\partial x_k}(x)\right| &\leq C\|f\|_* + C\|f\|_*\int_{\partial\tilde{K}_{n,j}}\left(\frac{1}{|x-t|^{d-1}}\right) d\sigma(t)\\ &\leq C\|f\|_*.\end{aligned}$$

□

LEMMA 2. *Define*

$$b_{n,j} = a_{n,j}2^{-nd^2/(d-1)}\int_{K_{n,j}} \frac{t_1}{|t|^d}\, dt$$

and let $b_n = \sum_{j=1}^{2^{nd}} b_{n,j}$. *Then there exists a constant C such that*

$$|b_n| \leq C\|f\|_* \quad \textit{for all } n \geq 0.$$

Proof. Let $K_{n,1}$ denote the cube containing the origin. Since $|b_{n,j}| \leq C\|f\|_*$, it suffices to show that

$$\left|\sum_{j\neq 1} b_{n,j}\right| \leq C\|f\|_*.$$

Define $d\mu_{n,j}(t) = a_{n,j}2^{nd^2/(d-1)}\, dt - (\partial f/\partial n)(t)\, d\sigma(t)$. Then

$$\begin{aligned}b_{n,j} &= \int_{K_{n,j}} \frac{t_1}{|t|^d}\, du_{n,j}(t) + \int_{\partial K_{n,j}} \frac{t_1}{|t|^d}\frac{\partial f}{\partial n}(t)\, d\sigma(t)\\ &= b'_{n,j} + b''_{n,j}.\end{aligned}$$

First, we estimate the summation on $b'_{n,j}$. Using the property

$$\int_{K_{n,j}} d\mu_{n,j} = 0,$$

we can write

$$b'_{n,j} = \int_{K_{n,j}} \left\{ \frac{t_1}{|t|^d} - \frac{c_{n,j}^{(k)}}{|c_{n,j}|^d} \right\} d\mu_{n,j}(t),$$

where $c_{n,j} = (c_{n,j}^{(1)}, c_{n,j}^{(2)}, \ldots, c_{n,j}^{(d)})$. Therefore, as $|t|$ and $|c_{n,j}|$ are comparable, we obtain

$$\begin{aligned} |b'_{n,j}| &\le \int_{K_{n,j}} \left| \frac{t_1}{|t|^d} - \frac{c_{n,j}^{(k)}}{|c_{n,j}|^d} \right| d|\mu_{n,j}|(t) \\ &\le C 2^{-nd/(d-1)} \int_{K_{n,j}} \left(\frac{1}{|t|^d} \right) d|\mu_{n,j}|(t) \\ &\le \frac{C\|f\|_* 2^{-nd^2/(d-1)}}{(\operatorname{dist}(0, K_{n,j}))^d} \end{aligned}$$

and

$$\begin{aligned} \sum_{j \ne 1} |b'_{n,j}| &\le C\|f\|_* \sum_{j \ne 1} \frac{2^{-nd^2/(d-1)}}{(\operatorname{dist}(0, K_{n,j}))^d} \\ &\le C\|f\|_* \sum_{k=1}^{n} \frac{2^{kd} 2^{-nd^2/(d-1)}}{2^{(k-n)d^2/(d-1)}} \\ &\le C\|f\|_*. \end{aligned}$$

To estimate the summation on $b''_{n,j}$, we differentiate (3.1) to derive the following relation:

$$\begin{aligned} \sum_{j \ne 1} \int_{\partial K_{n,j}} \frac{\partial f(t)}{\partial n} \left(\frac{x_1 - t_1}{|x-t|^d} \right) d\sigma(t) &= C \frac{\partial f}{\partial x_1}(x) + C \frac{\partial f_{n,j}}{\partial x_1}(x) \\ &\quad + C \int_{\partial K_{n,j}} f(t) \frac{\partial}{\partial n} \left(\frac{x_1 - t_1}{|x-t|^d} \right) d\sigma(t). \end{aligned}$$

Dominate the first two terms of the right-hand side by $C\|f\|_*$, using Lemma 1, and use the continuity to obtain either

$$\left| \sum_{j \ne 1} \int_{\partial K_{n,j}} \frac{\partial f}{\partial n}(t) \frac{t_1}{|t|^d} \, d\sigma(t) \right| \le C\|f\|_* + \left| \sum_{j \ne 1} \int_{\partial K_{n,j}} f(t) \frac{\partial}{\partial n} \left(\frac{t_1}{|t|^d} \right) d\sigma(t) \right|$$

or

$$\begin{aligned} \left| \sum_{j} b''_{n,j} \right| &\le C\|f\|_* + \sum_{j \ne 1} \int_{\partial K_{n,j}} |f(t) - f(c_{n,j})| \left(\frac{1}{|t|^d} \right) d\sigma(t) \\ &\le C\|f\|_* + C\|f\|_* \sum_{j \ne 1} \frac{2^{-nd^2/(d-1)}}{(\operatorname{dist}(0, K_{n,j}))^d} \\ &\le C\|f\|_*. \end{aligned}$$

as above. This proves Lemma 2. □

LEMMA 3. *Given $M > 0$ and $\epsilon > 0$, there exists $\delta > 0$ such that the condition*

$$\sup_{n,j} |a_{n,j}| > (1+\delta) 2^{-nd} |a_{0,1}|$$

holds for any $f \in \mathcal{H}_1^\infty(K)$ with $f(\infty) = 0$, $\|f\|_* \le M$, and $|a_{0,1}| > \epsilon$.

Proof. Assume by contradiction that there exist $\delta_k \downarrow 0$ and $f_k \in \mathcal{H}_1^\infty(K)$ satisfying $f_k(\infty)=0$, $\|f_k\|_* \le M$, and $|a_{0,1}^{(k)}| > \epsilon$ such that

$$\sup_{n,j} |a_{n,j}^{(k)}| \le (1+\delta_k)2^{-nd}|a_{0,1}^{(k)}|.$$

Let f be the limit of a subsequence $\{f_{k_j}\}$ of $\{f_k\}$ converging uniformly on compact sets of K^c. For this function we obtain

$$\begin{aligned} |a_{n,j}| &= \lim_{i\to\infty} |a_{n,j}^{(k_j)}| \\ &\le 2^{-nd}|a_{0,1}|. \end{aligned}$$

Since $a_{0,1} = \sum_j a_{n,j}$, it follows that $a_{n,j} = 2^{-nd}a_{0,1}$ for all n, j. Thus

$$b_{n,j} = a_{0,1}2^{nd/(d-1)} \int_{K_{n,j}} \frac{t_1}{|t|^d}\, dt$$

and

$$b_n = a_{0,1}2^{nd/(d-1)} \sum_j \int_{K_{n,j}} \frac{t_1}{|t|^d}\, dt.$$

For $p = 1, 2, \ldots, n$, let K_{p,j_p} be a cube of K_p that intersects the diagonal passing through the origin and that has the distance from the origin comparable with $2^{-pd/(d-1)}$. By taking the summation only on those $K_{n,j}$ that are contained in some K_{p,j_p}, we obtain

$$\begin{aligned} |b_n| &> \epsilon 2^{nd/(d-1)} \sum_{p=1}^{n} \frac{2^{(n-p)d}2^{-nd^2/(d-1)}}{2^{-pd}} \\ &> n\epsilon C \end{aligned}$$

tending to ∞ as $n \to \infty$. This contradicts Lemma 3. □

LEMMA 4. *Suppose $f \in \mathcal{H}_1^\infty(K)$ and $f(\infty) = 0$. Then $a_{n,j} = 0$ for all n, j.*

Proof. Consider

$$g_{n,j}(x) = 2^{nd/(d-1)} f_{n,j}\left(c_{n,j} + \frac{x - c_{0,1}}{2^{nd/(d-1)}}\right).$$

Then, by Lemma 1, $g_{n,j} \in \mathcal{H}_1^\infty(K)$ and

$$\|g_{n,j}\|_* \le C\|f\|_*$$

for all n, j. Furthermore, by changing variables, we obtain

$$\begin{aligned} \int_{\partial K_{0,1}} \frac{\partial g_{n,j}}{\partial n}\, d\sigma &= 2^{nd} \int_{\partial K_{n,j}} \frac{\partial f_{n,j}}{\partial n}\, d\sigma \\ &= 2^{nd} \int_{\partial K_{n,j}} \frac{\partial f}{\partial n}\, d\sigma \\ &= 2^{nd} a_{n,j}. \end{aligned}$$

Therefore, to prove this lemma it suffices to show $a_{0,1} = 0$. Suppose $a_{0,1} \ne 0$. Let $M = C\|f\|_*$ and $\epsilon = |a_{0,1}|$. Repeating application of Lemma 3 to $g_{n,j}$ we obtain a subsequence $\{a_{n_k,j_k}\}$ such that

$$|a_{n_k, j_k}| > (1+\delta)^k 2^{-n_k d} |a_{0,1}|$$

for all k. This is a contradiction for $|a_{n,j}| < \|f\|_* 2^{-nd}$. □

We will now show $f(x) \equiv 0$ for any $f \in \mathcal{H}_1^\infty(K)$ with $f(\infty) = 0$. By Lemma 4, we can rewrite (3.1) as

$$f(x) = C_0^{-1} \sum_j \int_{\partial K_{n,j}} (f(t) - f(c_{n,j})) \frac{\partial}{\partial n} \left(\frac{1}{|x-t|^{d-2}} \right) d\sigma(t)$$
$$- C_0^{-1} \sum_j \int_{\partial K_{n,j}} \frac{\partial f}{\partial n}(t) \left\{ \frac{1}{|x-t|^{d-2}} - \frac{1}{|x-c_{n,j}|^{d-2}} \right\} d\sigma(t)$$

for $x \in K_n^c$. When x is fixed and n is sufficiently large, we obtain

$$|f(x)| \le C\|f\|_* 2^{-nd/(d-1)} \sum_j \int_{\partial K_{n,j}} \left(\frac{1}{|x-t|^{d-1}} \right) d\sigma(t)$$
$$+ C\|f\|_* \sum_j \int_{\partial K_{n,j}} \left| \frac{|x-c_{n,j}|^{d-2} - |x-t|^{d-2}}{|x-t|^{d-2}|x-c_{n,j}|^{d-2}} \right| d\sigma(t)$$
$$\le C\|f\|_* 2^{-nd/(d-1)} \sum_j \int_{\partial K_{n,j}} \left(\frac{1}{|x-t|^{d-1}} \right) d\sigma(t)$$
$$+ C\|f\|_* 2^{-nd/(d-1)} \int_{\partial K_{n,j}} \left(\frac{1}{|x-t|^{d-1}} \right) d\sigma(t)$$
$$\le \frac{C\|f\|_* 2^{-nd/(d-1)}}{\eta^{d-1}},$$

where $\eta = \operatorname{dist}(x, K)$. Since the last expression tends to 0 as $n \to \infty$, $f(x) = 0$.

References

1. L. Carleson, *Selected problems on exceptional sets,* Van Nostrand, Princeton, N.J., 1967.
2. E. P. Dolzenko, *On the removal of singularities of analytic functions,* Uspekhi Mat. Nauk 18 (1967), 135–142. English translation: Trans. Amer. Math. Soc. 97 (1970), 33–41.
3. J. Garnett, *Analytic capacity and measures,* Lecture Notes in Math., 297, Springer, New York, 1972.
4. ———, *Positive length but zero analytic capacity,* Proc. Amer. Math. Soc. 24 (1970), 696–699.
5. L. L. Helms, *Introduction to potential theory,* Wiley, New York, 1969.
6. M. Tsuji, *Potential theory in modern function theory,* Maruzen, Tokyo, 1959.
7. N. X. Uy, *Removable sets of analytic functions satisfying a Lipschitz condition,* Ark. Mat. 17 (1969), 19–27.

Department of Mathematics and Computer Science
State University of California, Los Angeles
Los Angeles, CA 90032