Zeros of the Successive Derivatives of Hadamard Gap Series in the Unit Disk

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1. Introduction

Pólya [8] defined the *final set* of an analytic function to be the set of all z in the complex plane such that every neighborhood of z contains zeros of infinitely many derivatives of f. He and others [1, 2, 3, 5, 6, 7, 8] have determined the final sets of various entire and meromorphic functions. Here we examine the final sets of Hadamard gap series with radius of convergence one (and hence with natural boundary $\{|z|=1\}$ [10, p. 223]).

We consider power series of the form

$$f(z)=\sum_{k=1}^{\infty} c_k z^{n_k}, \tag{1.1}$$

satisfying $\lambda>1$, where

$$\lambda=\liminf_{k\to\infty} n_{k+1}/n_k. \tag{1.2}$$

We define

$$H(\lambda)=\begin{cases}(\lambda-1)\lambda^{\lambda/(1-\lambda)} & \text{when } 1<\lambda<\infty,\\ 1 & \text{when } \lambda=\infty.\end{cases} \tag{1.3}$$

THEOREM 1. *Let f, λ, and $H(\lambda)$ be defined as above, and suppose that*

$$|c_k|^{1/n_k}\to 1 \quad \textit{as } k\to\infty. \tag{1.4}$$

(a) *If $1<\lambda<\infty$, then the final set of f is*

$$\{0\}\cup\{H(\lambda)\le|z|\le 1\}. \tag{1.5}$$

(b) *If $\lambda=\infty$, then the final set is contained in* (1.5). *If $\lambda=\infty$, and if* $\limsup n_k^B|c_k|>0$ *for some $B\ge 0$, then the final set is* (1.5).

In Section 4 some functions are constructed which satisfy (1.4), and for which $\lambda=\infty$, but for which the final set is $\{0\}$.

For encouragement, suggestions, and valuable conversations concerning this work I am grateful to W. H. J. Fuchs, W. K. Hayman, L. R. Sons, and W. Bergweiler. I also thank the referee and the editor, whose careful reading and many suggestions helped me to simplify the presentation.

Received March 28, 1988. Revision received April 12, 1989.
Michigan Math. J. 36 (1989).

2. Proof of Theorem 1

Our main lemma (Lemma 1 below) gives information about the location of zeros of $f^{(j)}$ for large j. Its proof, which will be given in Section 3, was suggested by a paper of Fuchs [4] which employed "the idea, due to Hardy and Littlewood, of accentuating the dominance of the largest term [of the series (1.1)] by successive differentiations" [4, p. 167]. It will be shown in the proof of Lemma 1 that the first term of $f^{(j)}$ dominates the sum in a punctured disk centered at the origin, so that $f^{(j)}$ has no zeros there. The outer boundary of the disk is close to the circle where the first and second terms are equal in modulus. On the other hand, the sums of certain pairs of successive terms of $f^{(j)}$ are dominant in certain annuli; Rouché's theorem implies that $f^{(j)}$ has zeros in these annuli.

Choose a function f of the form (1.1), where $\{c_k\}$ satisfies (1.4) and the λ given by (1.2) is in $(1, \infty]$. Pick L in $(1, \lambda)$ such that, for all positive integers s and t satisfying $s \geq t$,

$$n_s/n_t \geq L^{s-t}. \tag{2.1}$$

All statements in this section and the next refer to the f, λ, and L just chosen.

Also define

$$I(j) = \min\{p : n_p \geq j\}. \tag{2.2}$$

Finally, for all positive integers p and j such that $n_p \geq j$, let

$$\gamma(j,p) = \left(\frac{n_p(n_p-1)\cdots(n_p-j+1)}{n_{p+1}(n_{p+1}-1)\cdots(n_{p+1}-j+1)} \right)^{1/(n_{p+1}-n_p)}. \tag{2.3}$$

Then $\gamma(j, p)$ is in $(0, 1)$ and is the value of $|z|$ for which $|(d^j/dz^j)z^{n_p}| = |(d^j/dz^j)z^{n_{p+1}}|$.

LEMMA 1. (a) *For each* $\epsilon > 0$ *there exists* $N > 0$ *such that, if* $j > N$ *and* $0 < |z| < \gamma(j, I(j))e^{-\epsilon}$, *then* $f^{(j)}(z) \neq 0$.

(b) *If* $\lambda = \infty$ *and* $\limsup |c_k| > 1$, *then there exist infinitely many* m *such that, for each real* α, *the set*

$$\{re^{i\theta} : \tfrac{1}{2} < r < 1 \text{ and } \alpha \leq \theta \leq \alpha + 2\pi/(n_{m+1} - n_m)\}$$

contains at least one zero of $f^{(n_m)}$.

(c) *Suppose that* $1 < \lambda < \infty$, *and let* S *be a set of integers* m *such that* $n_{m+1}/n_m \to \lambda$ *as* $m \to \infty$ *in* S. *Set* $\tau = 1 - (\log L)/(L-1)$. *Then* $\tau > 0$. *For each* A *in* $(0,1)$ *and each* ϵ *in* $(0, A\tau/(3\lambda))$, *there exists* $N > 0$ *such that, if* $m \in S$, *if* $n_m \geq j \geq [An_m] > N$, *and if* α *is real, then the set*

$$\{re^{i\theta} : \gamma(j,m)e^{-\epsilon} < r < \gamma(j,m)e^{\epsilon} \text{ and } \alpha \leq \theta \leq \alpha + 2\pi/(n_{m+1} - n_m)\}$$

is contained in $\{|z| < 1\}$ *and contains at least one zero of* $f^{(j)}$. (Here $[An_m]$ represents the largest integer no greater than An_m.)

We now deduce Theorem 1 from Lemma 1. First, the gap condition ensures that the origin is in the final set of f. Next, we will show that the assumptions that $|c_k|^{1/n_k} \to 1$ and that $1 < \lambda \le \infty$ imply that the final set is contained in the set (1.5). This will follow from Lemma 1(a) once we derive a suitable lower bound on $\gamma(j, I(j))$. Now for fixed q, $\gamma(h,q)$ decreases as h increases, since, by definition of γ,

$$\frac{\gamma(h-1,q)}{\gamma(h,q)} = \left(\frac{n_{q+1}-h+1}{n_q-h+1}\right)^{1/(n_{q+1}-n_q)} > 1. \tag{2.4}$$

But $n_{I(j)} \ge j$ by definition of $I(j)$, so $\gamma(j, I(j)) \ge \gamma(n_{I(j)}, I(j))$. A lower bound on the right-hand side of this last inequality is provided by the following lemma.

LEMMA 2. *Let $H(\lambda)$ be given by* (1.3). *Then:*

(a)
$$\liminf_{p\to\infty} \gamma(n_p, p) = H(\lambda);$$

(b) *if $p \to \infty$ through a sequence of integers such that $n_{p+1}/n_p \to \lambda$, then*

$$\gamma(n_p, p) \to H(\lambda).$$

Proof. By (2.1), $n_{p+1} - n_p \to \infty$. Thus, by Stirling's formula,

$$\begin{aligned}\gamma(n_p,p)^{n_{p+1}-n_p} &= \frac{n_p!\,(n_{p+1}-n_p)!}{n_{p+1}!}\\ &= \left(\frac{n_p}{n_{p+1}-n_p}\right)^{n_p}\left(\frac{n_{p+1}-n_p}{n_{p+1}}\right)^{n_{p+1}}\sqrt{2\pi s_p}\,(1+o(1))\end{aligned}$$

as $p \to \infty$, where $s_p = n_p(n_{p+1}-n_p)/n_{p+1}$. But $s_p^{1/(n_{p+1}-n_p)} \to 1$ since, for p large,

$$1 < n_p(1-L^{-1}) < n_p(1-n_p/n_{p+1}) = s_p < n_{p+1}-n_p.$$

Hence, by the definition (1.3) of $H(\lambda)$,

$$\begin{aligned}\gamma(n_p,p) &= \left(\frac{1}{n_{p+1}/n_p-1}\right)^{1/(n_{p+1}/n_p-1)}\\ &\quad\times\left(\frac{n_{p+1}/n_p-1}{n_{p+1}/n_p}\right)^{(n_{p+1}/n_p)/(n_{p+1}/n_p-1)}(1+o(1))\\ &= H(n_{p+1}/n_p)(1+o(1)),\end{aligned}$$

and (b) follows. Next, $H(x)$ is increasing for $x > 1$, which gives (a) when $\lambda < \infty$. Finally, $H(x) \to 1 = H(\infty)$ as $x \to \infty$, so that (a) also holds when $\lambda = \infty$. □

Lemma 2(a) and the remarks preceding it give the following.

COROLLARY. *The final set of f is contained in* (1.5).

To complete the proof of Theorem 1(b), it is enough to show that the unit circle is in the final set of f whenever $\lambda=\infty$ and there exists $B\geq 0$ such that $\limsup n_k^B|c_k|>0$. We may assume without loss of generality that B is an integer, and, because f and $f^{(B+1)}$ have the same final set, we may assume that $\limsup|c_k|>1$. Then, since $n_{m+1}-n_m\to\infty$, Lemma 1(b) guarantees that the final set intersects $\{re^{i\theta}\colon \frac{1}{2}\leq r\leq 1\}$ for each real θ. But by the Corollary, this intersection can only be $\{e^{i\theta}\}$. This establishes Theorem 1(b).

Finally, we complete the proof of Theorem 1(a). Choose A in $(0,1)$. It follows from Lemma 1(c) that $\{|z|=R\}$ is contained in the final set whenever R is a limit point of numbers $\gamma(j,m)$, where m is in S and j is in $([An_m], n_m)$.

To determine where such $\gamma(j,m)$ accumulate, we first note that, by (2.4) and (2.1),

$$1<\gamma(h-1,m)/\gamma(h,m)<n_{m+1}^{1/(n_{m+1}-n_m)}<n_m^{1/[(L-1)n_m]}\to 1$$

as $m\to\infty$. Hence, for m sufficiently large,

$$\gamma(n_m,m)<\gamma(n_m-1,m)<\cdots<\gamma([An_m],m),$$

and the difference between consecutive numbers $\gamma(j,m)$ in these inequalities becomes uniformly arbitrarily small as $m\to\infty$. Furthermore, by Lemma 2(b), $\gamma(n_m,m)\to H(\lambda)$ as $m\to\infty$ in S. Also, as we will see in a moment,

$$\log\gamma([An_m],m)\geq-(A^{-1}-1+o(1))^{-1}. \tag{2.5}$$

Therefore the final set contains $\{H(\lambda)\leq|z|\leq\exp[(1-A^{-1})^{-1}]\}$, and Theorem 1(a) follows when we allow A to approach 0. □

We will derive (2.5) from the following lemma.

LEMMA 3. *When $n_p\geq j$,*

$$-\sum_{t=0}^{j-1}(n_{p+1}-t)^{-1}\geq\log\gamma(j,p)\geq-\sum_{t=0}^{j-1}(n_p-t)^{-1}.$$

Proof. By the definition (2.3) of γ and the mean value theorem,

$$\log\gamma(j,p)=-\sum_{t=0}^{j-1}\frac{\log(n_{p+1}-t)-\log(n_p-t)}{(n_{p+1}-t)-(n_p-t)}=-\sum_{t=0}^{j-1}\xi_t^{-1},$$

where $n_p-t<\xi_t<n_{p+1}-t$. The lemma follows. □

From Lemma 3 we have

$$\log\gamma([An_m],m)\geq[An_m]\frac{-1}{n_m-[An_m]}=-(A^{-1}-1+o(1))^{-1},$$

which proves (2.5). This completes the derivation of Theorem 1 from Lemma 1. □

3. Proof of Lemma 1

In this section and the next we will write

$$f^{(j)}(z)=\sum_{n_k\geq j}\phi_{jk}(z)=\sum_{k=I(j)}^{\infty}\phi_{jk}(z), \tag{3.1}$$

where

$$\phi_{jk}(z)=c_k n_k(n_k-1)\cdots(n_k-j+1)z^{n_k-j}. \tag{3.2}$$

The first step in the proof of Lemma 1 is to show that each term $\phi_{jq}(z)$ is, for $|z|$ suitably restricted, significantly larger than the sum of the succeeding terms of $f^{(j)}$. This step is taken in the following lemma.

LEMMA 4. (a) *For each* $\epsilon>0$ *there is an* $N>0$ *such that, if* $n_q\geq j>N$ *and* $|z|\leq\gamma(j,q)e^{-\epsilon}$, *then*

$$\left|\sum_{k=q+1}^{\infty}\phi_{jk}(z)\right|\leq\frac{|\phi_{jq}(z)|}{3}.$$

(b) *Suppose that* $\lambda=\infty$. *Then there exists* $N>0$ *such that, if* $n_{q-1}>N$, *if*

$$|z|\leq|c_q|^{-1/n_q}\gamma(n_{q-1},q)e^{-n_{q-1}/(2n_q)},$$

and if $|c_q|^{1/n_q}\geq|c_k|^{1/n_k}$ *whenever* $k>q$, *then*

$$\left|\sum_{k=q+1}^{\infty}\phi_{n_{q-1},k}(z)\right|\leq\frac{|\phi_{n_{q-1},q}(z)|}{3}.$$

Proof. Fix z in $\mathbf{C}$ and j and q such that $n_q\geq j$. Then, for $k>q$,

$$\left|\frac{\phi_{jk}(z)}{\phi_{jq}(z)}\right|=\left|\frac{\phi_{jk}(z)}{\phi_{j,k-1}(z)}\right|\left|\frac{\phi_{j,k-1}(z)}{\phi_{j,k-2}(z)}\right|\cdots\left|\frac{\phi_{j,q+1}(z)}{\phi_{jq}(z)}\right|.$$

The definitions (3.2) and (2.3) of ϕ and γ give

$$\begin{aligned}\left|\frac{\phi_{j\mu}(z)}{\phi_{j,\mu-1}(z)}\right|&=\left|\frac{c_\mu}{c_{\mu-1}}\right|\left(\frac{n_\mu(n_\mu-1)\cdots(n_\mu-j+1)}{n_{\mu-1}(n_{\mu-1}-1)\cdots(n_{\mu-1}-j+1)}\right)|z|^{n_\mu-n_{\mu-1}}\\&=\left|\frac{c_\mu}{c_{\mu-1}}\right|\left(\frac{1}{\gamma(j,\mu-1)}\right)^{n_\mu-n_{\mu-1}}|z|^{n_\mu-n_{\mu-1}}.\end{aligned}$$

Now by Lemma 3, $\gamma(j,p)$ is, for fixed j, an increasing function of p. Therefore, for $\mu\geq q+1$,

$$\left|\frac{\phi_{j\mu}(z)}{\phi_{j,\mu-1}(z)}\right|\leq\left|\frac{c_\mu}{c_{\mu-1}}\right|\left(\frac{|z|}{\gamma(j,q)}\right)^{n_\mu-n_{\mu-1}}.$$

It follows that, whenever $k>q$,

$$\log\left|\frac{\phi_{jk}(z)}{\phi_{jq}(z)}\right|\leq\log\left|\frac{c_k}{c_q}\right|+(n_k-n_q)\log\left(\frac{|z|}{\gamma(j,q)}\right). \tag{3.3}$$

Suppose now, to begin the proof of part (a), that $|z| \le \gamma(j,q)e^{-\epsilon}$ for some $\epsilon > 0$. Then by (3.3) and (2.1),

$$\log\left|\frac{\phi_{jk}(z)}{\phi_{jq}(z)}\right| \le \log\left|\frac{c_k}{c_q}\right| - n_k\left(1-\frac{n_q}{n_k}\right)\epsilon \le \log\left|\frac{c_k}{c_q}\right| - n_k(1-L^{-1})\epsilon.$$

Let $\delta = (1-L^{-1})\epsilon/4$. Then (1.4) implies that $\log|c_k/c_q| \le \delta n_k + \delta n_q \le 2\delta n_k$ whenever q is sufficiently large and $k > q$. Thus, when j (and hence q) is sufficiently large and $k > q$,

$$\log\left|\frac{\phi_{jk}(z)}{\phi_{jq}(z)}\right| \le 2\delta n_k - 4\delta n_k \le -2\delta k.$$

So whenever j is large enough and $n_q \ge j$,

$$\sum_{k=q+1}^{\infty} |\phi_{jk}(z)| \le |\phi_{jq}(z)| \sum_{k=q+1}^{\infty} e^{-2\delta k} = \frac{e^{-2\delta(q+1)}}{1-e^{-2\delta}} < \frac{1}{3}.$$

This completes the proof of Lemma 4(a). □

For the proof of Lemma 4(b), suppose that

$$|z| \le |c_q|^{-1/n_q}\gamma(n_{q-1}, q)e^{-n_{q-1}/(2n_q)},$$

and that $|c_k|^{1/n_k} \le |c_q|^{1/n_q}$ whenever $k > q$. Then (3.3) gives, for $k > q$,

$$\log\left|\frac{\phi_{n_{q-1},k}(z)}{\phi_{n_{q-1},q}(z)}\right| \le \log\left|\frac{c_k}{c_q}\right| + (n_k - n_q)\left(-\frac{\log|c_q|}{n_q} - \frac{n_{q-1}}{2n_q}\right).$$

Again for $k > q$,

$$\log\left|\frac{c_k}{c_q}\right| = \frac{\log|c_k|}{n_k}n_k - \frac{\log|c_q|}{n_q}n_q \le \frac{\log|c_q|}{n_q}(n_k - n_q),$$

so that, for q large,

$$\log\left|\frac{\phi_{n_{q-1},k}(z)}{\phi_{n_{q-1},q}(z)}\right| \le (n_k - n_q)\left(-\frac{n_{q-1}}{2n_q}\right) = -\frac{n_{q-1}}{2}\left(\frac{n_k}{n_q}-1\right) < -\frac{q}{3}\left(\frac{n_k}{n_q}-1\right).$$

But by (2.1), $n_k/n_q - 1 > L^{k-q} - 1 = e^{(k-q)\log L} - 1 > (k-q)\log L$. Therefore, for q sufficiently large,

$$\begin{aligned}\sum_{k=q+1}^{\infty} |\phi_{n_{q-1},k}(z)| &\le |\phi_{n_{q-1},q}(z)| \sum_{k=q+1}^{\infty} \exp\{-(q/3)(\log L)(k-q)\} \\ &= \frac{|\phi_{n_{q-1},q}(z)|e^{-(q/3)\log L}}{1-e^{-(q/3)\log L}} < \frac{|\phi_{n_{q-1},q}(z)|}{3}.\end{aligned}$$

The proof of Lemma 4(b) is now complete. □

Proof of Lemma 1(a). Choose $\epsilon > 0$. Let $I(j)$ be defined by (2.2). If j is sufficiently large and $0 < |z| < \gamma(j, I(j))e^{-\epsilon}$, then (3.1) and Lemma 4(a) (with $q = I(j)$) imply that

$$|f^{(j)}(z)| \geq |\phi_{j,I(j)}(z)| - \left|\sum_{k=I(j)+1}^{\infty} \phi_{jk}(z)\right| \geq |\phi_{j,I(j)}(z)| - \frac{|\phi_{j,I(j)}(z)|}{3} > 0.$$

This completes the proof. □

In the proof of parts (b) and (c) of Lemma 1, we need the following application of Rouché's theorem.

LEMMA 5. *Let r_1 and r_2 be in $(0,1)$ and let $j \leq n_m$. Suppose that*

$$2^{1/(n_{m+1}-n_m)} r_1 < |c_m/c_{m+1}|^{1/(n_{m+1}-n_m)} \gamma(j,m) < 2^{-1/(n_{m+1}-n_m)} r_2 \tag{3.4}$$

and that

$$|f^{(j)}(z) - \phi_{jm}(z) - \phi_{j,m+1}(z)| \leq (|\phi_{jm}(z)| + |\phi_{j,m+1}(z)|)/3 \tag{3.5}$$

when $r_1 \leq |z| \leq r_2$. Then, for each real α, $f^{(j)}$ has at least one zero in the set

$$\{re^{i\theta}: r_1 < r < r_2 \text{ and } \alpha \leq \theta \leq \alpha + 2\pi/(n_{m+1}-n_m)\}.$$

Proof. When $|z| = r_1$, the definitions (3.2) and (2.3) of ϕ and γ, along with (3.4), give

$$\left|\frac{\phi_{j,m+1}(z)}{\phi_{jm}(z)}\right| = \left(\frac{|c_{m+1}/c_m|^{1/(n_{m+1}-n_m)}|z|}{\gamma(j,m)}\right)^{n_{m+1}-n_m} < \frac{1}{2}.$$

Hence, again for $|z| = r_1$,

$$\frac{|\phi_{jm}(z)| + |\phi_{j,m+1}(z)|}{|\phi_{jm}(z) + \phi_{j,m+1}(z)|} = \frac{1 + |\phi_{j,m+1}(z)|/|\phi_{jm}(z)|}{|1 + \phi_{j,m+1}(z)/\phi_{jm}(z)|} < \frac{3/2}{1/2} = 3,$$

and thus

$$|f^{(j)}(z) - \phi_{jm}(z) - \phi_{j,m+1}(z)| < |\phi_{jm}(z) + \phi_{j,m+1}(z)|.$$

Similarly, this last inequality holds when $|z| = r_2$. Finally, the same inequality holds for z on each of the $(n_{m+1} - n_m)$ rays

$$\{z: \arg z = (\arg(c_m/c_{m+1}))/(n_{m+1}-n_m)\}.$$

This is because, for such z, $\phi_{jm}(z)$ and $\phi_{j,m+1}(z)$ have the same argument (mod 2π), so that $|\phi_{jm}(z)| + |\phi_{j,m+1}(z)| = |\phi_{jm}(z) + \phi_{j,m+1}(z)|$. Lemma 5 now follows from Rouché's theorem. □

Proof of Lemma 1(b). By hypothesis, there are infinitely many positive integers k such that $|c_k|^{1/n_k} > 1$. Furthermore, $|c_k|^{1/n_k} \to 1$. Thus, by [9, p. 24, problem 108], there is an infinite set T of positive integers m such that

$$|c_{m+1}|^{1/n_{m+1}} \geq |c_k|^{1/n_k} \quad \text{and} \quad |c_{m+1}| > 1 \tag{3.6}$$

whenever $m \in T$ and $k > m+1$. Set

$$r_1 = \tfrac{1}{2} \quad \text{and} \quad r_2 = |c_{m+1}|^{-1/n_{m+1}} \gamma(n_m, m+1) e^{-n_m/(2n_{m+1})}.$$

We will show that $f^{(n_m)}$, r_1, and r_2 satisfy the hypotheses of Lemma 5 for sufficiently large m in T. This will establish Lemma 1(b).

First, $r_2<1$ when $m\in T$ because $|c_{m+1}|>1$ and $\gamma(n_m, m+1)<1$.

Next, (3.1), (3.6), and Lemma 4(b) (with $q=m+1$) give, for large m in T and $|z|$ in (r_1, r_2),

$$|f^{(n_m)}(z)-\phi_{n_m,m}(z)-\phi_{n_m,m+1}(z)|=\left|\sum_{k=m+2}^{\infty}\phi_{n_m,k}(z)\right|\le\frac{|\phi_{n_m,m+1}(z)|}{3}.$$

This gives (3.5).

It remains to prove (3.4). By (1.4), the hypothesis that $\lambda=\infty$, and (1.2),

$$\frac{\log|c_m/c_{m+1}|}{n_{m+1}-n_m}=\frac{\log|c_m|}{n_m}\frac{n_m}{n_{m+1}}\Big/\left(1-\frac{n_m}{n_{m+1}}\right)-\frac{\log|c_{m+1}|}{n_{m+1}}\left(1+\frac{n_m}{n_{m+1}-n_m}\right)$$

$$(3.7)\qquad =o(1)\frac{n_m}{n_{m+1}}-\frac{\log|c_{m+1}|}{n_{m+1}}=o(1)$$

as $m\to\infty$. Therefore, by Lemma 2(b),

$$|c_m/c_{m+1}|^{1/(n_{m+1}-n_m)}\gamma(n_m, m)2^{-1/(n_{m+1}-n_m)}r_1^{-1}=(1+o(1))H(\infty)(1+o(1))2$$
$$=2+o(1).$$

This gives the first half of (3.4).

Finally, (3.7) and the definition of r_2 yield

$$\log\{|c_m/c_{m+1}|^{1/(n_{m+1}-n_m)}\gamma(n_m, m)2^{1/(n_{m+1}-n_m)}r_2^{-1}\}$$

$$(3.8)\qquad =o(1)\frac{n_m}{n_{m+1}}-\frac{\log|c_{m+1}|}{n_{m+1}}+\log\gamma(n_m, m)$$
$$+o(1)\frac{n_m}{n_{m+1}}+\frac{\log|c_{m+1}|}{n_{m+1}}-\log\gamma(n_m, m+1)+\frac{n_m}{2n_{m+1}}.$$

We now need an upper bound on $\log\{\gamma(n_m, m)/\gamma(n_m, m+1)\}$; this bound is given in the following lemma, which we will establish after completing the proof of Lemma 1(b).

LEMMA 6. *If $n_p\ge j$, then*

$$\log\left(\frac{\gamma(j,p)}{\gamma(j,p+1)}\right)\le\left(\frac{\log(n_{p+2}/n_{p+1})}{n_{p+2}/n_{p+1}-1}-1\right)\frac{j}{n_{p+1}}.$$

Lemma 6 and the hypothesis that $\lambda=\infty$ imply that the left-hand side of (3.8) is bounded above by

$$o(1)\frac{n_m}{n_{m+1}}+\frac{n_m}{2n_{m+1}}+\log\left(\frac{\gamma(n_m, m)}{\gamma(n_m, m+1)}\right)$$
$$<o(1)\frac{n_m}{n_{m+1}}+\frac{n_m}{2n_{m+1}}+\left(\frac{1}{4}-1\right)\frac{n_m}{n_{m+1}},$$

which is negative for large m. This gives the second half of (3.4), and thus completes the proof of Lemma 1(b).

Proof of Lemma 6. By the proof of Lemma 3,

$$-\log\gamma(j,p+1)=\sum_{t=0}^{j-1}\frac{\log\{(n_{p+2}-t)/(n_{p+1}-t)\}}{(n_{p+2}-t)/(n_{p+1}-t)-1}\,\frac{1}{n_{p+1}-t}.$$

But $(n_{p+2}-t)/(n_{p+1}-t)$ is increasing for t in $[0,n_p)$, while $(\log x)/(x-1)$ is decreasing for $x>1$, so

$$-\log\gamma(j,p+1)\le\sum_{t=0}^{j-1}\frac{\log(n_{p+2}/n_{p+1})}{n_{p+2}/n_{p+1}-1}\,\frac{1}{n_{p+1}-t}.$$

Now $(\log x)/(x-1)<1$ for $x>1$. From Lemma 3 and this inequality, we get

$$\begin{aligned}\log\gamma(j,p)-\log\gamma(j,p+1)&\le\sum_{t=0}^{j-1}\left\{\frac{\log(n_{p+2}/n_{p+1})}{n_{p+2}/n_{p+1}-1}-1\right\}\frac{1}{n_{p+1}-t}\\&\le j\left\{\frac{\log(n_{p+2}/n_{p+1})}{n_{p+2}/n_{p+1}-1}-1\right\}\frac{1}{n_{p+1}}.\end{aligned}$$

This completes the proof of Lemma 6. □

Proof of Lemma 1(c). Choose A in $(0,1)$ and ϵ in $(0,A\tau/(3\lambda))$, where $\tau=1-(\log L)/(L-1)$. (Note that $\tau>0$ because $(\log x)/(x-1)\uparrow 1$ as $x\to 1^+$.) Let $r_1=\gamma(j,m)e^{-\epsilon}$ and $r_2=\gamma(j,m)e^{\epsilon}$. We will verify the hypotheses of Lemma 5 for all sufficiently large m in S and all j in $([An_m],n_m)$.

We first show that $r_2<1$. Lemma 6 and (2.1) imply that, for large m in S,

$$\begin{aligned}\log\left(\frac{r_2}{\gamma(j,m+1)}\right)&<\frac{-j\tau}{n_{m+1}}+\epsilon<-\frac{[An_m]\tau}{n_{m+1}}+\epsilon\\&\le-\frac{A\tau}{\lambda}(1+o(1))+\frac{A}{3\lambda}<0,\end{aligned}\tag{3.9}$$

whence $r_2<\gamma(j,m+1)<1$.

Next we verify (3.4). By (3.7) and (2.1), $|c_m/c_{m+1}|^{1/(n_{m+1}-n_m)}\to 1$ even when $\lambda<\infty$, so

$$2^{1/(n_{m+1}-n_m)}r_1|c_{m+1}/c_m|^{1/(n_{m+1}-n_m)}/\gamma(j,m)=e^{-\epsilon}(1+o(1)).$$

This gives the first half of (3.4); the second half is derived in a similar way.

Finally, we must estimate $f^{(j)}(z)$ for $r_1\le|z|\le r_2$. The first step is to show that the terms of $f^{(j)}(z)$ preceding $\phi_{jm}(z)$ (if any) are small when $|z|\ge r_1$. To accomplish this, we first write, for k satisfying $I(j)\le k<m$,

$$\left|\frac{\phi_{jk}(z)}{\phi_{jm}(z)}\right|=\left|\frac{\phi_{jk}(z)}{\phi_{j,k+1}(z)}\right|\left|\frac{\phi_{j,k+1}(z)}{\phi_{j,k+2}(z)}\right|\cdots\left|\frac{\phi_{j,m-1}(z)}{\phi_{jm}(z)}\right|.$$

Calculations similar to those that led to (3.3) then show that

$$\left|\frac{\phi_{j,\mu-1}(z)}{\phi_{j\mu}(z)}\right|\le\left|\frac{c_{\mu-1}}{c_\mu}\right|\left(\frac{\gamma(j,m-1)}{|z|}\right)^{n_\mu-n_{\mu-1}}$$

for $\mu\le m$, and that

$$\log\left|\frac{\phi_{jk}(z)}{\phi_{jm}(z)}\right| \le \log\left|\frac{c_k}{c_m}\right| + (n_m - n_k)\log\left(\frac{\gamma(j,m-1)}{|z|}\right).$$

Meanwhile, by Lemma 6 and (2.1),

$$\log\left(\frac{\gamma(j,m-1)}{\gamma(j,m)}\right) \le \left(\frac{\log L}{L-1} - 1\right)\frac{j}{n_m} = -\frac{j\tau}{n_m}.$$

So if $|z| \ge r_1$, then

$$\log\left|\frac{\phi_{jk}(z)}{\phi_{jm}(z)}\right| \le \log\left|\frac{c_k}{c_m}\right| + (n_m - n_k)\left(-\frac{j\tau}{n_m} + \epsilon\right).$$

Therefore, because $j \ge [An_m]$ and $\epsilon < A\tau/(3\lambda) < A\tau/3$,

$$\log\left|\frac{\phi_{jk}(z)}{\phi_{jm}(z)}\right| \le \log\left|\frac{c_k}{c_m}\right| + (n_m - n_k)\{-A\tau(1+o(1)) + A\tau/3\}.$$

Let $\delta = (1 - L^{-1})\tau A/8$. Then (1.4) implies that $\log|c_k/c_m| \le \delta n_k + \delta n_m \le 2\delta n_m$ for m sufficiently large and for $k < m$. Thus, by (2.1),

$$\log\left|\frac{\phi_{jk}(z)}{\phi_{jm}(z)}\right| \le 2\delta n_m - n_m(1 - L^{-1})(A\tau/2) = (2\delta - 4\delta)n_m < -2\delta m.$$

Now for large m there are at most $\{-\log(A/2)\}/(\log L)$ terms of $f^{(j)}$ preceding ϕ_{jm}. For let ϕ_{jk} be such a term. Then $n_m > n_k \ge j \ge [An_m]$, so that

$$\log L^{k-m} > \log(n_k/n_m) \ge \log\{[An_m]/n_m\} > \log(A/2),$$

whence $m + \{\log(A/2)\}/(\log L) < k < m$. This proves the claim.

Hence, for sufficiently large m in S, for j in $([An_m], n_m)$, and for $|z| \ge r_1$,

$$\left|\sum_{k=I(j)}^{m-1} \phi_{jk}(z)\right| < -\frac{\log(A/2)}{\log L} e^{-2\delta m}|\phi_{jm}(z)| \le \frac{|\phi_{jm}(z)|}{3}. \tag{3.10}$$

Next, by (3.9), $r_2 < \gamma(j, m+1)e^{-A/(2\lambda)}$ for large m in S. Therefore we may apply Lemma 4(a) (with $q = m+1$) to conclude that

$$\left|\sum_{k=m+2}^{\infty} \phi_{jk}(z)\right| \le \frac{|\phi_{j,m+1}(z)|}{3} \tag{3.11}$$

for all large m in S, all j in $([An_m], n_m)$, and all z such that $|z| \le r_2$.

From (3.10), (3.11), and (3.1), we see that (3.5) of Lemma 5 holds when m, j, and $|z|$ are restricted as above. Lemma 1(c) now follows from Lemma 5. □

4. Counterexamples for Large Gaps

The following result shows that, when $\lambda = \infty$, it is possible for the coefficients of f to decrease so rapidly that, for all large j, the first term of $f^{(j)}$ is dominant in the entire unit disk. In this case, the only zeros of $f^{(j)}$ are those at the origin.

THEOREM 2. *Let f be given by* (1.1) *and suppose that* (1.4) *holds. If*

$$\limsup n_{k+1}|c_{k+1}|^{1/n_k} < 1, \tag{4.1}$$

then $n_{k+1}/n_k \to \infty$ and the final set of f is $\{0\}$.

Proof. We first show that $n_{k+1}/n_k \to \infty$. By (1.4),

$$n_{k+1}|c_{k+1}|^{1/n_k} = n_{k+1}(|c_{k+1}|^{1/n_{k+1}})^{n_{k+1}/n_k} = n_{k+1}(1+o(1))^{n_{k+1}/n_k}.$$

By (4.1), the left-hand side is bounded; the right-hand side can only be bounded if $n_{k+1}/n_k \to \infty$.

Next we show that the final set is $\{0\}$. By (4.1) and (1.4), there exist σ in $(0,1)$ and $N>0$ such that $|c_{k+1}| < (\sigma^2/n_{k+1})^{n_k}$ and $|c_k| > \sigma^{n_k}$ when $n_k > N$. Thus, whenever $n_k \geq j > N$ and $|z|$ is in $(0,1)$,

$$\begin{aligned}\left|\frac{\phi_{j,k+1}(z)}{\phi_{jk}(z)}\right| &< \left|\frac{c_{k+1}}{c_k}\right| \frac{n_{k+1}(n_{k+1}-1)\cdots(n_{k+1}-j+1)}{n_k(n_k-1)\cdots(n_k-j+1)} \\ &\leq \left|\frac{c_{k+1}}{c_k}\right| \left(\frac{n_{k+1}-j+1}{n_k-j+1}\right)^j \\ &< (\sigma/n_{k+1})^{n_k}(n_{k+1}-n_k+1)^{n_k} < \sigma^{n_k} \leq \sigma^j.\end{aligned}$$

Therefore, for j sufficiently large and $|z|$ in $(0,1)$,

$$\sum_{p=I(j)+1}^{\infty} \left|\frac{\phi_{jp}(z)}{\phi_{j,I(j)}(z)}\right| < \sum_{p=I(j)+1}^{\infty} (\sigma^j)^{p-I(j)} = \frac{\sigma^j}{1-\sigma^j} < 1,$$

so that

$$|f^{(j)}(z)| \geq |\phi_{j,I(j)}(z)| - \sum_{p=I(j)+1}^{\infty} |\phi_{jp}(z)| > 0.$$

This completes the proof of Theorem 2. □

The following corollary to Theorem 2 shows that the factor n_k^B in Theorem 1(b) cannot be replaced by a larger function of n_k.

COROLLARY. *Suppose that $\psi: \mathbf{Z}^+ \to \mathbf{R}$, that $\psi(n)^{1/n} \to 1$, and that*

$$\psi(n)/n^B \to \infty \quad \textit{for each } B \geq 0.$$

Then there exists a sequence $\{n_k\}$, satisfying $n_{k+1}/n_k \to \infty$, such that the final set of $f(z) = \sum_{k=1}^{\infty} \{\psi(n_k)\}^{-1} z^{n_k}$ is $\{0\}$.

Proof. We define $\{n_k\}$ inductively. The choice of n_1 is arbitrary. Having chosen n_k, we pick n_{k+1} so that $\psi(n_{k+1})/n_{k+1}^{n_k} > 2^{n_k}$. (This is possible because $\psi(n)/n^{n_k} \to \infty$ as $n \to \infty$.) Then for each k, $n_{k+1}\{1/\psi(n_{k+1})\}^{1/n_k} < \frac{1}{2}$, so that (4.1) holds. The corollary now follows from Theorem 2. □

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