On the Structure of Contraction Operators, III

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1. Introduction

This paper is a continuation of the sequence [18], [13], and is perhaps best described as a sequel to [18] in which we look more deeply into the constructions of that paper. By so doing, we obtain (Theorem 6.2) some surprising and unexpected characterizations of the class $\mathbf{A}_{1,\aleph_0}$ (to be defined below), and this leads to several new sufficient conditions for the reflexivity of a contraction operator (Theorem 7.2). As a corollary, we obtain the following improvement of the main result of [13].

COROLLARY. *Every contraction operator acting on a separable, complex Hilbert space whose spectrum contains the unit circle is either reflexive or has a nontrivial hyperinvariant subspace.*

We shall therefore assume that the reader is familiar with [18], and especially the notation and terminology therefrom, which we continue to use below without extensive review. For the reader's convenience, however, we recall that $\mathcal{H}$ is a separable, infinite-dimensional, complex Hilbert space, and $\mathcal{L}(\mathcal{H})$ is the algebra of all bounded linear operators on $\mathcal{H}$. Moreover, $\mathcal{C}_1(\mathcal{H}) \subset \mathcal{L}(\mathcal{H})$ is the Banach space (and ideal) of trace-class operators under the trace norm, $\mathbf{D}$ is the open unit disc in $\mathbf{C}$, $\mathbf{T} = \partial\mathbf{D}$, and $\mathbf{N}$ is the set of positive integers. The spaces $H^p(\mathbf{T})$ and $L^p(\mathbf{T})$ are the usual Hardy and Lebesgue spaces with respect to normalized Lebesgue measure m on $\mathbf{T}$. Furthermore, $H_0^1(\mathbf{T})$ denotes the subspace of $H^1(\mathbf{T})$ consisting of those functions f whose analytic extension $\hat{f}$ to $\mathbf{D}$ satisfies $\hat{f}(0) = 0$. If Σ is an arbitrary Borel subset of $\mathbf{T}$, we will have occasion to use the (closed) subspace $L^p(\Sigma)$ of $L^p(\mathbf{T})$, $1 \le p \le \infty$, defined as the set of all (classes of) functions f in $L^p(\mathbf{T})$ such that $f = 0$ almost everywhere on $\mathbf{T} \setminus \Sigma$. The space $H^2(\Sigma)$ is the closure in $L^2(\Sigma)$ of the linear manifold consisting of those functions that agree with some polynomial on Σ, and if $m(\mathbf{T}\setminus\Sigma) \neq 0$ then $H^2(\Sigma) = L^2(\Sigma)$.

If $T \in \mathcal{L}(\mathcal{H})$ we write $\mathcal{A}_T$ for the dual algebra generated by T and Q_T for its predual $\mathcal{C}_1(\mathcal{H})/{}^{\perp}\mathcal{A}_T$, so $\mathcal{A}_T = Q_T^*$ under the pairing

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$$\langle A, [L]\rangle = \operatorname{tr}(AL), \quad A \in \mathcal{A}_T, \; L \in \mathcal{C}_1(\mathcal{H}),$$

where $[L]$ (or $[L]_T$) denotes the element of the quotient space Q_T containing the trace-class operator L. Recall that if T is an absolutely continuous contraction then the Sz.-Nagy–Foiaş functional calculus $\Phi_T: H^\infty(\mathbf{T}) \to \mathcal{A}_T$ is a weak* continuous algebra homomorphism with range weak* dense in $\mathcal{A}_T$, and thus is the adjoint of a one-to-one linear contractive map $\varphi_T: Q_T \to L^1(\mathbf{T})/H_0^1(\mathbf{T})$. The class $\mathbf{A} = \mathbf{A}(\mathcal{H})$ is defined to be the set of all absolutely continuous contractions T in $\mathcal{L}(\mathcal{H})$ for which Φ_T is an isometry; it follows easily that in this case Φ_T is a weak* homeomorphism of $H^\infty(\mathbf{T})$ onto $\mathcal{A}_T$ and φ_T is a surjective isometry.

We introduce now, for any cardinal numbers m and n, where $1 \le m, n \le \aleph_0$, some properties $(\mathbf{A}_{m,n})$ and $(\mathbf{A}_{m,n}(r))$. A weak*-closed subspace $\mathcal{A}$ of $\mathcal{L}(\mathcal{H})$ has property $(\mathbf{A}_{m,n})$ if, for every doubly indexed family $\{[L_{ij}]\}_{0\le i<m,\, 0\le j<n}$ of elements of $Q_{\mathcal{A}} = \mathcal{C}_1(\mathcal{H})/{}^\perp\mathcal{A}$, there exist sequences $\{x_i\}_{0\le i<m}$ and $\{y_j\}_{0\le j<n}$ of vectors from $\mathcal{H}$ such that

$$(1) \qquad [L_{ij}] = [x_i \otimes y_j], \quad 0 \le i < m, \; 0 \le j < n.$$

A weak*-closed subspace $\mathcal{A}$ of $\mathcal{L}(\mathcal{H})$ has property $\mathbf{A}_{m,n}(r)$ for some $r \ge 1$ if, for every doubly indexed family $\{[L_{ij}]\}_{0\le i<m,\, 0\le j<n}$ of elements of $Q_{\mathcal{A}}$ such that the rows and columns of the matrix $(\|[L_{ij}]\|)$ are summable, and for every $s > r$, there exist sequences $\{x_i\}_{0\le i<m}$ and $\{y_j\}_{0\le j<n}$ from $\mathcal{H}$ satisfying (1) and also the inequalities

$$(2) \qquad \begin{aligned} \|x_i\| &\le \Big(s \sum_{0\le j<n} \|[L_{ij}]\|\Big)^{1/2}, \quad 0 \le i < m, \\ \|y_j\| &\le \Big(s \sum_{0\le i<m} \|[L_{ij}]\|\Big)^{1/2}, \quad 0 \le j < n. \end{aligned}$$

It is clear that if m and n are finite cardinals and $\mathcal{A}$ has property $(\mathbf{A}_{m,n}(r))$ for some r, then $\mathcal{A}$ also has property $(\mathbf{A}_{m,n})$. Furthermore, it follows from an easy scaling argument that if $\mathcal{A}$ has property $(\mathbf{A}_{1,\aleph_0}(r))$ for some r, then $\mathcal{A}$ also has property $(\mathbf{A}_{1,\aleph_0})$. We define the class $\mathbf{A}_{m,n}$ $(= \mathbf{A}_{m,n}(\mathcal{H}))$ to be the set of those T in $\mathbf{A}(\mathcal{H})$ such that the dual algebra $\mathcal{A}_T$ has property $(\mathbf{A}_{m,n})$, and the class $\mathbf{A}_{m,n}(r)$ similarly. The classes $\mathbf{A}_{n,n}$, $1 \le n \le \aleph_0$, were introduced in [4] and have played a major role in the theory of dual algebras (see, e.g., [5]).

2. Some Equation-Solving Tools Revisited

In this section we take another look at the main construction of Section 3 of [18] and show that, with more careful consideration, Theorem 3.11 of [18] can be improved considerably. Moreover, this improvement will be quite useful in obtaining our new reflexivity results in Sections 6 and 7.

The setting of Section 3 of [18] is that we have under consideration an absolutely continuous contraction T in $\mathcal{L}(\mathcal{H})$ whose minimal co-isometric extension B in $\mathcal{L}(\mathcal{K})$ (with $\mathcal{K}\supset\mathcal{H}$) has a Wold decomposition $B=S^*\oplus R$ corresponding to a decomposition of $\mathcal{K}$ as $\mathcal{S}\oplus\mathcal{R}$, where S is a unilateral shift of some multiplicity in $\mathcal{L}(\mathcal{S})$ if $\mathcal{S}\neq(0)$, S is the zero operator if $\mathcal{S}=(0)$, R is an absolutely continuous unitary operator in $\mathcal{L}(\mathcal{R})$ if $\mathcal{R}\neq 0$, and R is the zero operator if $\mathcal{R}=(0)$. The projection of $\mathcal{K}$ onto $\mathcal{S}$ is denoted by Q, the projection of $\mathcal{K}$ onto $\mathcal{R}$ by A, and the projection of $\mathcal{K}$ onto $\mathcal{H}$ by P. Thus every vector x in $\mathcal{K}$ has a unique decomposition

$$x=Qx+Ax=Qx\oplus Ax,$$

and if $x\in\mathcal{H}$ and $h\in H^\infty$, one has

$$\begin{aligned}h(T)x=h(B)x&=h(S^*)(Qx)\oplus h(R)(Ax)\\&=Q(h(T)x)\oplus A(h(T)x),\end{aligned}$$

so

$$h(S^*)(Qx)=Q(h(T)x),\qquad h(R)(Ax)=A(h(T)x). \tag{3}$$

The following proposition summarizes some facts that we shall need, relating Q_T and Q_B from Lemmas 3.5, 3.6, and 3.7 of [18].

PROPOSITION 2.1. *Suppose $T\in\mathbf{A}(\mathcal{H})$ and has minimal co-isometric extension $B=S^*\oplus R$ in $\mathcal{L}(\mathcal{K})$. Then $B\in\mathbf{A}(\mathcal{K})$, $\Phi_T\circ\Phi_B^{-1}$ is an isometric algebra isomorphism and a weak* homeomorphism from $\mathcal{A}_B$ onto $\mathcal{A}_T$, and $J=\varphi_B^{-1}\circ\varphi_T$ is a linear isometry of Q_T onto Q_B satisfying*

$$J([x\otimes y]_T)=[x\otimes y]_B,\quad x,y\in\mathcal{H}. \tag{4}$$

Moreover, for all $x,y\in\mathcal{H}$ and $w,z\in\mathcal{K}$,

$$\|[x\otimes y]_T\|=\|[x\otimes y]_B\|, \tag{5}$$

$$[x\otimes z]_B=[x\otimes Pz]_B, \tag{6}$$

$$[w\otimes z]_B=[Qw\otimes Qz]_B+[Aw\otimes Az]_B. \tag{7}$$

Furthermore, if $\{x_n\}_{n=1}^\infty$ is a sequence from $\mathcal{H}$ such that

$$\|[x_n\otimes y]_T\|\to 0\quad\forall y\in\mathcal{H},$$

then

$$\|[x_n\otimes z]_B\|\to 0\quad\forall z\in\mathcal{K}, \tag{8}$$

$$\|[Qx_n\otimes z]_B\|\to 0\quad\forall z\in\mathcal{K}, \tag{9}$$

and

$$\|[Ax_n\otimes z]_B\|\to 0\quad\forall z\in\mathcal{K}. \tag{10}$$

Finally, if $\{z_n\}_{n=1}^\infty$ is any sequence in $\mathcal{K}$ converging weakly to zero, then

$$\|[w\otimes z_n]_B\|\to 0\quad\forall w\in\mathcal{S}. \tag{11}$$

Now let us recall some additional notation and terminology from [18]. Suppose that T and $B=S^*\oplus R$ are as above, with $\mathcal{R}\neq(0)$ (and T absolutely continuous). Because R is an absolutely continuous unitary operator, there exists a Borel subset Σ of $\mathbf{T}$ such that the measure $m|\Sigma$, defined on Borel subsets $\mathcal{B}$ of $\mathbf{T}$ by $(m|\Sigma)(\mathcal{B})=m(\Sigma\cap\mathcal{B})$, is a scalar spectral measure for R. For any vectors x and y in $\mathcal{R}$, the complex Borel measure $\mu_{x,y}$ on $\mathbf{T}$, defined by $\mu_{x,y}(\mathcal{B})=(E(\mathcal{B})x,y)$, where E is the spectral measure of R, is clearly absolutely continuous with respect to $m|\Sigma$, and thus we denote by $x\cdot y$ the function in $L^1(\Sigma)$ that is the Radon-Nikodym derivative of $\mu_{x,y}$ with respect to $m|\Sigma$. Clearly,

$$(l(R)x,y)=\int_{\mathbf{T}} l\,d\mu_{x,y}=\int_{\Sigma} l\{x\cdot y\}\,dm \quad \forall l\in L^\infty(\mathbf{T}). \tag{12}$$

Since $L^1(\Sigma)$ is a subspace of $L^1(\mathbf{T})$, we may write $[x\cdot y]$ for the equivalence class of $x\cdot y$ in the quotient space $L^1(\mathbf{T})/H_0^1(\mathbf{T})$. The following proposition summarizes Lemma 3.9 and Proposition 3.10 of [18].

PROPOSITION 2.2. *Suppose T is an absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$, and $B=S^*\oplus R$ is its minimal co-isometric extension in $\mathcal{L}(\mathcal{S}\oplus\mathcal{R})$ with $\mathcal{R}\neq(0)$. Then there exists a Borel set $\Sigma\subset\mathbf{T}$ such that $m|\Sigma$ is a scalar spectral measure for R, and for every pair of vectors $w,z\in\mathcal{R}$ we have*

$$[w\cdot z]=\varphi_B([w\otimes z]_B). \tag{13}$$

Moreover, $\mathcal{R}$ contains a reducing subspace $\mathcal{R}_0$ for R such that:

(a) *$R_0=R|\mathcal{R}_0$ is unitarily equivalent to multiplication by the position function on $L^2(\Sigma)$; and*

(b) *if we denote by $\mathcal{R}_0^+$ the subspace of $\mathcal{R}_0$ corresponding to $H^2(\Sigma)$ under the unitary equivalence in* (a), *then $\mathcal{R}_0^+\subset(A\mathcal{H})^-$.*

Our first new result is a modest generalization of [18, Thm. 3.11] that permits us to approximately solve a "row" of simultaneous equations.

PROPOSITION 2.3. *Suppose T is an absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$ and has minimal co-isometric extension $B=S^*\oplus R$ in $\mathcal{L}(\mathcal{S}\oplus\mathcal{R})$ with $\mathcal{R}\neq(0)$. Let $\Sigma\subset\mathbf{T}$ and $\mathcal{R}_0\subset\mathcal{R}$ be as in Proposition 2.2, and let A_0 denote the projection of $\mathcal{K}=\mathcal{S}\oplus\mathcal{R}$ onto $\mathcal{R}_0$. Let $\epsilon>0$, $0<\rho<1$, and $N\in\mathbf{N}$ be given, along with $a\in\mathcal{K}$, $\{b_j\}_{j=1}^N\subset\mathcal{R}$, and $\{h_j\}_{j=1}^N\subset L^1(\Sigma)$. Then there exist $u\in\mathcal{K}$ and $\{c_j\}_{j=1}^N\subset\mathcal{R}$ such that*

$$\|(Aa\cdot b_j)+h_j-A(a+u)\cdot c_j\|<\epsilon,\quad 1\le j\le N,$$

$$\|Qu\|<\epsilon,$$

$$\|(A-A_0)u\|<\epsilon,$$

$$\|u\|\le 2\Big(\sum_{1\le j\le N}\|h_j\|\Big)^{1/2},$$

$$\|c_j\|\le\frac{1}{\rho}\{\|b_j\|+\|h_j\|^{1/2}\},\quad 1\le j\le N,$$

and

$$c_j - b_j \in \mathcal{R}_0, \quad 1 \le j \le N.$$

Moreover, if $a = 0$ then the vectors u and c_j may be chosen to satisfy

$$\|u\| \le (1+\epsilon)\Big(\sum_{1 \le j \le N} \|h_j\|\Big)^{1/2}$$

and

$$\|c_j\| \le \|h_j\|^{1/2}, \quad 1 \le j \le N.$$

Proof. The Hilbert space isomorphism from $\mathcal{R}_0$ onto $L^2(\Sigma)$ given by (a) of Proposition 2.2 will be indicated by $z \to \{z\}$. Thus objects of the form $\{z\}$ are (classes of) square integrable functions on $\mathbf{T}$ supported on Σ, and $\{z\}(e^{it})$ has the obvious meaning. This being said, the proofs of all but the last assertion are almost word-for-word the same as that of [18, Thm. 3.11], so we content ourselves with making a few comments. As before, the non-trivial case occurs when $\Sigma = \mathbf{T}$. To get started, one sets $g = \sum_{1 \le j \le N} |h_j|$ and then, as in the aforementioned proof, obtains $\{y_1\}$ in $H^2(\mathbf{T})$ and $\{z_1\}$ in $L^2(\mathbf{T})$ such that

$$\{y_1\}\overline{\{z_1\}} = g,$$

$$|\{y_1\}| \ge |g|^{1/2} \text{ a.e. on } \mathbf{T},$$

and

$$\|y_1\| \le \|g\|_1^{1/2}(1+\delta).$$

One continues as in the earlier proof with $\overline{\{z_1^j\}} = h_j/\{y_1\}$, $z_2^j = R^{n_1} z_1^j$, $1 \le j \le N$, and so on; we omit further details except to note that the earlier argument can be simplified considerably when $a = 0$, and this leads easily to the improved inequalities given in the last statement of the proposition. □

We next improve Proposition 2.3 to an "exact" form.

COROLLARY 2.4. *Suppose T is an absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$, and T has minimal co-isometric extension $B = S^* \oplus R$ in $\mathcal{L}(\mathcal{S} \oplus \mathcal{R})$ with $\mathcal{R} \ne (0)$ and $\Sigma \subset \mathbf{T}$ as in Proposition 2.2. Suppose that $N \in \mathbf{N}$, $a \in \mathcal{H}$, $\{b_j\}_{j=1}^N \subset \mathcal{R}$, and $\{h_j\}_{j=1}^N \subset L^1(\Sigma)$ are given, as well as $\epsilon > 0$. Then there exist $u \in \mathcal{H}$ and $\{c_j\}_{j=1}^N \subset \mathcal{R}$ such that*

$$(Aa \cdot b_j) + h_j = A(a+u) \cdot c_j, \quad 1 \le j \le N, \tag{14}$$

$$\|Qu\| < \epsilon, \tag{15}$$

$$\|(A - A_0)u\| < \epsilon, \tag{16}$$

$$\|u\| \le (2+\epsilon)\Big(\sum_{1 \le j \le N} \|h_j\|\Big)^{1/2}, \tag{17}$$

$$\|c_j\| \le (1+\epsilon)(\|b_j\| + \|h_j\|^{1/2}), \quad 1 \le j \le N, \tag{18}$$

and

$$c_j - b_j \in \mathcal{R}_0, \quad 1 \le j \le N. \tag{19}$$

Moreover, if $a=0$ then the vectors u and c_j may be chosen to satisfy

$$\|u\| \leq (1+\epsilon)\Big(\sum_{1\leq j\leq N} \|h_j\|\Big)^{1/2}$$

and

$$\|c_j\| \leq (1+\epsilon)\|h_j\|^{1/2}, \quad 1\leq j\leq N.$$

Proof. We sketch the proof when $a\neq 0$ and $N=1$; the extension to the case $N>1$ proceeds along obvious lines. If $h=0$ in $L^1(\Sigma)$, then setting $u=0$ and $c=b$ completes the proof. Thus we may suppose that $h\neq 0$. Let $\{s_n\}$ be a strictly decreasing sequence such that $s_0=1$ and $\lim s_n = s > 1/(1+\epsilon)$, and set $\rho_n = s_n/s_{n-1}$, $n\in\mathbf{N}$. Let $\{\epsilon_n\}$ be a decreasing sequence of positive numbers less than one such that

$$\frac{1}{s}\Big(\|b\|+\|h\|^{1/2}+\sum_{n=1}^{\infty}\epsilon_n^{1/2}\Big) < (1+\epsilon)(\|b\|+\|h\|^{1/2})$$

and

$$\sum_{n=1}^{\infty}\epsilon_n^{1/2} < \frac{\max\{\epsilon\|h\|^{1/2},\epsilon\}}{3}.$$

By Proposition 2.3 there exist a_1 in $\mathcal{K}$ and c_1 in $\mathcal{R}$ such that

$$\|Aa\cdot b+h-Aa_1\cdot c_1\| < \epsilon_1, \tag{20}$$

$$\|Q(a_1-a)\| < \epsilon_1, \tag{21}$$

$$\|(A-A_0)(a_1-a)\| < \epsilon_1, \tag{22}$$

$$\|a_1-a\| \leq 2\|h\|^{1/2}, \tag{23}$$

$$\|c_1\| \leq \frac{1}{\rho_1}\{\|b\|+\|h\|^{1/2}\}, \tag{24}$$

and

$$c_1-b\in\mathcal{R}_0. \tag{25}$$

By iterative application of Proposition 2.3 (with h_1 of that proposition set equal to $Aa\cdot b+h-Aa_{n-1}\cdot c_{n-1}$), we obtain sequences $\{a_n\}\subset\mathcal{K}$ and $\{c_n\}\subset\mathcal{R}$ such that for every $n\geq 2$,

$$\|Aa\cdot b+h-Aa_n\cdot c_n\| < \epsilon_n, \tag{26}$$

$$\|Q(a_n-a_{n-1})\| < \epsilon_n, \tag{27}$$

$$\|(A-A_0)(a_n-a_{n-1})\| < \epsilon_n, \tag{28}$$

$$\|a_n-a_{n-1}\| < 2\epsilon_{n-1}^{1/2}, \tag{29}$$

$$\|c_n\| < \frac{1}{\rho_n}\{\|c_{n-1}\|+\epsilon_{n-1}^{1/2}\}, \tag{30}$$

and

$$c_n-c_{n-1}\in\mathcal{R}_0. \tag{31}$$

From (29) one sees that $\{a_n\}$ is a Cauchy sequence, and hence converges to an element a' of $\mathcal{H}$. Furthermore, from (24) and (30) one easily obtains that for $n \geq 2$,

$$\|c_n\| < \frac{1}{s_n}\left\{\|b\| + \|h\|^{1/2} + \sum_{k=1}^{n-1} s_k \epsilon_k^{1/2}\right\}$$
$$< \frac{1}{s}\left\{\|b\| + \|h\|^{1/2} + \sum_{n=1}^{\infty} \epsilon_n^{1/2}\right\} < (1+\epsilon)\{\|b\| + \|h\|^{1/2}\}.$$

Therefore we may extract a subsequence $\{c_{n_k}\}$ of the sequence $\{c_n\}$ that converges weakly in $\mathcal{R}$—say, to c. Thus c satisfies (18), and since subspaces are weakly closed, (19) is also satisfied. If we set $u = a' - a$ then $\{a_n\}$ converges in norm to $a + u$, and it follows easily that $Aa_{n_k} \cdot c_{n_k}$ converges weak* in $L^1(\Sigma)$ to $A(a+u)\cdot c$. Hence we have from (26) that $Aa \cdot b + h = A(a+u)\cdot c$. Moreover, using (21) and (27), we get

$$\|Qu\| = \|Q(a'-a)\| = \lim_n \|Q(a_n - a)\|$$
$$\leq \lim_n \sum_{k=1}^{n} \|Q(a_k - a_{k-1})\| \leq \lim_n \sum_{k=1}^{n} \epsilon_k \leq \frac{\epsilon}{2}$$

(where we have set $a_0 = a$), so (15) is satisfied; a similar argument using (22) and (28) shows that (16) is satisfied. As for (17), we have from (23) and (29) that

$$\|u\| = \lim_n \|a' - a\| \leq \sum_{k=1}^{\infty} \|a_k - a_{k-1}\| \leq 2\|h\|^{1/2} + \sum_{k=1}^{\infty} \epsilon_k^{1/2} < (2+\epsilon)\|h\|,$$

so (17) is satisfied and the proof is complete in the case $a \neq 0$.

When $a = 0$ the proof goes the same way except that at the first step of the induction process we apply the last statement of Proposition 2.3, so the vectors a_1, c_1 satisfy

$$\|a_1\| \leq (1+\epsilon/3)\|h\|^{1/2}, \qquad \|c_1\| \leq \|h\|^{1/2},$$

and one sees easily that the desired inequalities are satisfied. □

When Corollary 2.4 is translated into the language of preduals, the following theorem results.

THEOREM 2.5. *Suppose T is an absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$ and T has minimal co-isometric extension $B = S^* \oplus R$ in $\mathcal{L}(\mathcal{S} \oplus \mathcal{R})$, with $\mathcal{R} \neq (0)$ and $\Sigma \subset \mathbf{T}$ as in Proposition 2.2. If $N \in \mathbf{N}$, $\epsilon > 0$, and elements $\{h_1, \ldots, h_N\}$ in $L^1(\Sigma) \subset L^1(\mathbf{T})$ are given, then there exist vectors x and $\{y_1, \ldots, y_N\}$ in $\mathcal{H}$ such that*

(32) $$\varphi_T([x \otimes y_j]) = [h_j], \quad 1 \leq j \leq N,$$

(33) $$\|x\| \leq (2+\epsilon)\left\{\sum_{j=1}^{N} \|h_j\|\right\}^{1/2},$$

and

$$\|y_j\| \le (1+\epsilon)\|h_j\|^{1/2}, \quad 1 \le j \le N. \tag{34}$$

Moreover, if $R \in \mathbf{A}(\mathcal{R})$ *(or, equivalently, if* $m(\mathbf{T}\backslash\Sigma)=0$*), then* $T \in \mathbf{A}_{1,N}(1)$.

Proof. Taking $a=b_1=\cdots=b_N=0$ in Corollary 2.4, we obtain u in $\mathcal{K}$ and $\{c_1,\dots,c_N\}$ in $\mathcal{R}$ such that

$$h_j = Au\cdot c_j, \quad 1\le j\le N, \tag{35}$$

$$\|u\| \le (2+\epsilon)\left\{\sum_{j=1}^{N}\|h_j\|\right\}^{1/2}, \tag{36}$$

and

$$\|c_j\| \le (1+\epsilon)\|h_j\|^{1/2}, \quad 1\le j\le N. \tag{37}$$

According to Proposition 2.2,

$$[h_j]=[Au\cdot c_j]=\varphi_B([Au\otimes c_j]_B), \quad 1\le j\le N, \tag{38}$$

and since $AB=BA$ and $u\in\mathcal{K}$,

$$[Au\otimes c_j]_B=[u\otimes Ac_j]_B=[u\otimes c_j]_B=[u\otimes Pc_j]_B, \quad 1\le j\le N. \tag{39}$$

Thus, upon combining (38) and (39), we obtain

$$\varphi_B([u\otimes Pc_j]_B)=[h_j], \quad 1\le j\le N. \tag{40}$$

Since u and the Pc_j belong to $\mathcal{K}$, we know that

$$(T^k u, Pc_j)=(B^k u, Pc_j), \quad k\in\mathbf{N},\ 1\le j\le N,$$

and this implies, via [4, Lemma 4.9], that

$$\varphi_B([u\otimes Pc_j]_B)=\varphi_T([u\otimes Pc_j]_T), \quad 1\le j\le N. \tag{41}$$

Upon setting $x=u$ and $y_j=Pc_j, 1\le j\le N$, we obtain (32), (33), and (34) from (40), (41), (36), and (37).

Now suppose $R\in\mathbf{A}(\mathcal{R})$, which is equivalent to saying that $m(\mathbf{T}\backslash\Sigma)=0$ by a theorem of Sullivan [23]. Let $[L_1],\dots,[L_N]$ be arbitrary elements of Q_T, and let $\delta_j>\|[L_j]\|$, $1\le j\le N$. We may choose elements $h_1,\dots,h_N$ of $L^1(\Sigma)=L^1(\mathbf{T})$ and $\eta>0$ such that

$$[h_j]=\varphi_T([L_j]), \quad 1\le j\le N, \tag{42}$$

and

$$(1+\eta)^2\|h_j\|<\delta_j, \quad 1\le j\le N. \tag{43}$$

Applying the last statement of Corollary 2.4 with $\eta=\epsilon$, and repeating the argument above leading to (40) and (41), we obtain vectors $x=u$ and $\{y_j=Pc_j\}_{j=1}^{N}$ in $\mathcal{K}$ such that

$$[h_j]=\varphi_T([x\otimes y_j]_T), \quad 1\le j\le N, \tag{44}$$

$$\|x\|\le(1+\eta)\left(\sum_{j=1}^{N}\|h_j\|\right)^{1/2}\le\left(\sum_{j=1}^{N}\delta_j\right)^{1/2}, \tag{45}$$

and

(46) $$\|y_j\| \le (1+\eta)\|h_j\|^{1/2} \le (\delta_j)^{1/2}, \quad 1 \le j \le N.$$

Since φ_T is always one-to-one, we obtain from (42) and (44) that $[L_j] = [x \otimes y_j]$, $1 \le j \le N$, and it follows from (45) and (46) that $\mathcal{Q}_T$ has property $(\mathbf{A}_{1,N}(1))$. It remains only to prove that $T \in \mathbf{A}$. But from (32) one knows that the range of φ_T is all of $(L^1/H_0^1)(\mathbf{T})$, which implies that $\Phi_T = \varphi_T^*$ is one-to-one and has closed range. By the open mapping theorem, Φ_T is bounded below, and by applying Φ_T to functions of the form f^m and taking mth roots, one sees easily that the lower bound of Φ_T must be 1 (cf. [5, p. 87]). Thus $T \in \mathbf{A}$ and the proof is complete. □

3. A Criterion for Membership in $\mathbf{A}_{1,\aleph_0}$

In [18] it was shown (Theorem 4.7) that if $T \in \mathbf{A}$ and $\mathcal{Q}_T$ has property $E^r_{\theta,\gamma}$ (definition reviewed below) for some $0 \le \theta < \gamma \le 1$ then $T \in \mathbf{A}_1(r(\theta,\gamma))$, where

(47) $$r(\theta,\gamma) = (6/\gamma)(1/\{1-(\theta/\gamma)^{1/2}\})^2.$$

In this section we show that under (a priori) weaker hypotheses on T, one can draw the stronger conclusion that $T \in \mathbf{A}_{1,\aleph_0}(r(\theta,\gamma))$, where $r(\theta,\gamma)$ is as in (47).

We recall from [18] that if $\mathfrak{M}$ is a weak*-closed subspace of $\mathcal{L}(\mathcal{H})$ and $0 \le \theta < 1$, then $\mathcal{E}^r_\theta(\mathfrak{M})$ denotes the set of all $[L]$ in $Q_{\mathfrak{M}}$ for which there exist sequences $\{x_n\}$ and $\{y_n\}$ in the closed unit ball of $\mathcal{H}$ satisfying

(a) $\overline{\lim}\,\|[L]-[x_n \otimes y_n]\| \le \theta$

and

(b^r) $\|[x_n \otimes z]\| \to 0 \;\; \forall z \in \mathcal{H}$,
(c^r) $\{y_n\}$ converges weakly to zero.

The corresponding subset $\mathcal{E}^l_\theta(\mathfrak{M})$ of $Q_{\mathfrak{M}}$ is obtained by replacing conditions (b^r) and (c^r) by

(b^l) $\|[z \otimes y_n]\| \to 0 \;\; \forall z \in \mathcal{H}$,
(c^l) $\{x_n\}$ converges weakly to zero.

REMARK 3.1. The conditions (c^r) and (c^l) in the above definitions are actually superfluous. To see this in the case of (c^r), suppose $[L] \in Q_{\mathfrak{M}}$ and $\{x_n\}$ and $\{y_n\}$ are sequences from the closed unit ball of $\mathcal{H}$ satisfying conditions (a) and (b^r) of the above definition. Let $\{y_{n_k}\}$ be a subsequence of $\{y_n\}$ that is weakly convergent—say, to $y' \ne 0$. Then the sequence $\{y_{n_k} - y'\}$ converges weakly to zero and satisfies

$$\overline{\lim_k}\,\|[L]-[x_{n_k} \otimes (y_{n_k}-y')]\| \le \theta,$$

since $\|[x_{n_k} \otimes y']\| \to 0$ by (b^r). Thus it suffices to show that some tail of the sequence $\{y_{n_k} - y'\}$ belongs to the closed unit ball. But

$$\begin{aligned}\|y_{n_k}-y'\|^2 &= \|y_{n_k}\|^2-2\operatorname{Re}(y_{n_k},y')+\|y'\|^2\\ &\le 1-2\operatorname{Re}(y_{n_k},y')+\|y'\|^2,\end{aligned}$$

and as k becomes large, the right-hand side of this last inequality tends to $1-\|y'\|^2<1$, which establishes the remark.

We next recall from [18] that a weak*-closed subspace $\mathfrak{M}$ of $\mathcal{L}(\mathcal{H})$ is said to have property $E^r_{\theta,\gamma}$ (for some $0\le\theta<\gamma\le 1$) if the closed absolutely convex hull of the set $\mathcal{E}^r_\theta(\mathfrak{M})$ (notation: $\overline{\text{aco}}\{\mathcal{E}^r_\theta(\mathfrak{M})\}$) contains the closed ball in $Q_{\mathfrak{M}}$ centered at 0 with radius γ; property $E^l_{\theta,\gamma}$ is defined similarly. In the case where $\mathfrak{M}$ is the dual algebra generated by an absolutely continuous contraction, we consider now the weaker properties $F^r_{\theta,\gamma}$ and $F^l_{\theta,\gamma}$.

DEFINITION 3.2. Let T be an absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$ with minimal co-isometric extension $B=S^*\oplus R$ in $\mathcal{L}(\mathcal{K})$, and let $\Sigma\subset\mathbf{T}$ be as in Proposition 2.2 (if $\mathcal{R}=(0)$ then $\Sigma=\emptyset$). We say that the dual algebra $\mathcal{A}_T$ has property $F^r_{\theta,\gamma}$ (for some $0\le\theta<\gamma\le 1$) if

$$\overline{\text{aco}}\{\mathcal{E}^r_\theta(\mathcal{A}_T)\cup\varphi_T^{-1}\{[f]: f\in L^1(\Sigma),\ \|f\|\le 1\}\} \tag{48}$$

contains the closed ball in Q_T of radius γ centered at the origin. Moreover, we say that $\mathcal{A}_T$ has property $F^l_{\theta,\gamma}$ if $\mathcal{A}_{T^*}$ has property $F^r_{\theta,\gamma}$. Obviously, if $\mathcal{A}_T$ has property $E^r_{\theta,\gamma}$ then it has property $F^r_{\theta,\gamma}$.

In connection with property $F^r_{\theta,\gamma}$ it will be convenient to use the following notation. For an arbitrary absolutely continuous contraction T and an element f in $L^1(\mathbf{T})$ whose coset $[f]$ in L^1/H^1_0 belongs to the range of φ_T, we denote by $[[f]]_T$ the (unique) element of Q_T whose image under φ_T is $[f]$.

The following theorem generalizes [18, Thm. 4.7] and plays an essential role in Section 6.

THEOREM 3.3. *Suppose $T\in\mathbf{A}(\mathcal{H})$ and $\mathcal{A}_T$ has property $F^r_{\theta,\gamma}$ for some $0\le\theta<\gamma\le 1$. Then $T\in\mathbf{A}_{1,\aleph_0}(r(\theta,\gamma))$, where $r(\theta,\gamma)$ is as in (47). If, on the other hand, $T\in\mathbf{A}(\mathcal{H})$ and $\mathcal{A}_T$ has property $F^l_{\theta,\gamma}$ for some $0\le\theta<\gamma\le 1$, then $T\in\mathbf{A}_{\aleph_0,1}(r(\theta,\gamma))$.*

It follows easily (as in the proof of [18, Thm. 4.7]) that $T^*\in\mathbf{A}_{\aleph_0,1}(r(\theta,\gamma))$ if and only if $T\in\mathbf{A}_{1,\aleph_0}(r(\theta,\gamma))$, so it suffices to prove the first statement of the theorem. The proof depends on the following approximation scheme, which generalizes [18, Prop. 4.6].

PROPOSITION 3.4. *Suppose $T\in\mathbf{A}(\mathcal{H})$ with minimal co-isometric extension $B\in\mathcal{L}(\mathcal{S}\oplus\mathcal{R})$, and suppose that $\mathcal{A}_T$ has property $F^r_{\theta,\gamma}$ for some $0<\theta<\gamma\le 1$. Suppose further that we are given $0<\rho<1$, $N\in\mathbf{N}$, $\{[V_j]\}_{j=1}^N\subset Q_B$, $a\in\mathcal{H}$, $\{w_j\}_{j=1}^N\subset\mathcal{S}$, $\{b_j\}_{j=1}^N\subset\mathcal{R}$, and positive scalars $\{\mu_j\}_{j=1}^N$ satisfying*

$$\|[V_j]_B-[a\otimes(w_j+b_j)]_B\|<\mu_j,\quad 1\le j\le N. \tag{49}$$

Then there exist $a'\in\mathcal{H}$, $\{w_j'\}_{j=1}^N\subset\mathcal{S}$, and $\{b_j'\}_{j=1}^N\subset\mathcal{R}$ such that

$$\|[V_j]_B-[a'\otimes(w_j'+b_j')]_B\|<(\theta/\gamma)\mu_j,\quad 1\le j\le N, \tag{50}$$

$$\|a'-a\|<\frac{3}{\gamma^{1/2}}\left(\sum_{j=1}^{N}\mu_j\right)^{1/2}, \tag{51}$$

$$\|w_j'-w_j\|<(\mu_j/\gamma)^{1/2},\quad 1\le j\le N, \tag{52}$$

and

$$\|b_j'\|<\frac{1}{\rho}\left\{\|b_j\|+\left(\frac{\mu_j}{\gamma}\right)^{1/2}\right\},\quad 1\le j\le N. \tag{53}$$

Proof. Define $\{[V_j']_B\}_{j=1}^N\subset Q_B$ by

$$[V_j']_B=[V_j]_B-[a\otimes(w_j+b_j)]_B,\quad 1\le j\le N, \tag{54}$$

and define also $\{d_j\}_{j=1}^N$ by

$$d_j=\max\{\|[V_j']_B\|,\mu_j/2\},\quad 1\le j\le N.$$

Choose $\{\epsilon_j\}_{j=1}^N$ to be positive and to satisfy

$$\left(\frac{\theta}{\gamma}\right)d_j+\epsilon_j<\left(\frac{\theta}{\gamma}\right)\mu_j,\quad 1\le j\le N. \tag{55}$$

With $J=\varphi_B^{-1}\circ\varphi_T$, we observe that

$$\left\|\frac{\gamma}{d_j}J^{-1}([V_j']_B)\right\|\le\gamma,\quad 1\le j\le N.$$

Therefore, by hypothesis, there exist integers $0=k_0<k_1<\cdots<k_N$, elements $\{[K_i]_T\}_{i=1}^{k_N}$ in $\mathcal{E}_\theta^r(\mathfrak{A}_T)$, elements $\{\hat{l}_j\}_{j=1}^N$ in $L^1(\Sigma)$, and scalars $\{\hat{\alpha}_i\}_{i=1}^{k_N}$, for which

$$\left\|\frac{\gamma}{d_j}J^{-1}([V_j']_B)-\left([[\hat{l}_j]]_T+\sum_{k_{j-1}<i\le k_j}\hat{\alpha}_i[K_i]_T\right)\right\|<\frac{\epsilon_j\gamma}{2d_j},\quad 1\le j\le N, \tag{56}$$

and

$$\|\hat{l}_j\|_1+\sum_{k_{j-1}<i\le k_j}|\hat{\alpha}_i|<1,\quad 1\le j\le N.$$

Define

$$\alpha_i=(d_j/\gamma)\hat{\alpha}_i,\quad l_j=(d_j/\gamma)\hat{l}_j,\quad k_{j-1}<i\le k_j,\ 1\le j\le N. \tag{57}$$

Then, multiplying (56) by the appropriate d_j/γ, we obtain

$$\left\|J^{-1}([V_j']_B)-\left([[l_j]]_T+\sum_{k_{j-1}<i\le k_j}\alpha_i[K_i]_T\right)\right\|<\frac{\epsilon_j}{2},\quad 1\le j\le N, \tag{58}$$

and

$$\|l_j\|_1+\sum_{k_{j-1}<i\le k_j}|\alpha_i|<\frac{d_j}{\gamma},\quad 1\le j\le N. \tag{59}$$

Since each $[K_i]_T\in\mathcal{E}_\theta^r(\mathfrak{A}_T)$, $1\le i\le k_N$, there exist sequences $\{x_n^i\}_{n=1}^\infty$ and $\{y_n^i\}_{n=1}^\infty$ from the closed unit ball of $\mathcal{H}$ satisfying (a) and (b^r), (c^r) above. After deleting a finite number of terms in each sequence, we may assume that

$$\|[K_i]_T - [x_n^i \otimes y_n^i]_T\| < \theta + (\epsilon_j \gamma / 2d_j), \quad k_{j-1} < i \le k_j, \ 1 \le j \le N. \tag{60}$$

We note that, even though the $[K_i]_T$ are not necessarily distinct, by passing to subsequences we may assume that the sequences $\{x_n^i\}_{n=1}^\infty$ (resp., $\{y_n^i\}_{n=1}^\infty$), $1 \le i \le k_N$, have no terms in common. From (58), (59), and (60) we get, for each choice of the k_N-tuple $\nu = (n_1, \dots, n_{k_N})$ and for each $1 \le j \le N$,

$$\begin{aligned} &\left\| J^{-1}([V_j']_B) - \left([[l_j]]_T + \sum_{k_{j-1} < i \le k_j} \alpha_i [x_{n_i}^i \otimes y_{n_i}^i]_T \right) \right\| \\ &\qquad < \left(\frac{\epsilon_j}{2} \right) + \left(\frac{d_j}{\gamma} \right) \left(\theta + \left(\frac{\gamma \epsilon_j}{2d_j} \right) \right) = \epsilon_j + d_j \left(\frac{\theta}{\gamma} \right). \end{aligned} \tag{61}$$

Then, passing to the predual Q_B by using J (note that $J([[l_j]]_T) = [[l_j]]_B$) and Proposition 2.1, we obtain

$$\begin{aligned} &\left\| [V_j']_B - \left([[l_j]]_B + \sum_{k_{j-1} < i \le k_j} \alpha_i [x_{n_i}^i \otimes y_{n_i}^i]_B \right) \right\| < \epsilon_j + d_j \left(\frac{\theta}{\gamma} \right), \\ &\qquad 1 \le j \le N. \end{aligned} \tag{62}$$

In view of (55) we may choose, for each $1 \le j \le N$, some $\tau_j > 0$ such that

$$5\tau_j < (\theta/\gamma)\mu_j - (d_j(\theta/\gamma) + \epsilon_j). \tag{63}$$

Using (7) and (54) we may deduce from (62) and (63) that, for any choice of ν,

$$\begin{aligned} &\Bigg\| [V_j]_B - [a \otimes (w_j + b_j)]_B - \left(\sum_{k_{j-1} < i \le k_j} \alpha_i [Qx_{n_i}^i \otimes Qy_{n_i}^i]_B \right) \\ &\quad - \left(\sum_{k_{j-1} < i \le k_j} \alpha_i [Ax_{n_i}^i \otimes Ay_{n_i}^i]_B \right) - [[l_j]]_B \Bigg\| < \left(\frac{\theta}{\gamma} \right) \mu_j - 5\tau_j, \quad 1 \le j \le N. \end{aligned} \tag{64}$$

Thus, for any choice of $\nu = (n_1, \dots, n_{k_N})$, we have

$$\begin{aligned} &\left\| [V_j]_B - [Qa \otimes w_j]_B - \left(\sum_{k_{j-1} < i \le k_j} \alpha_i [Qx_{n_i}^i \otimes Qy_{n_i}^i]_B \right) - [M^j(\nu)]_B \right\| \\ &\qquad < \left(\frac{\theta}{\gamma} \right) \mu_j - 5\tau_j, \quad 1 \le j \le N, \end{aligned} \tag{65}$$

where $[M^j(\nu)]_B$ is defined by

$$\begin{aligned} [M^j(\nu)]_B = [Aa \otimes b_j]_B &+ \left(\sum_{k_{j-1} < i \le k_j} \alpha_i [Ax_{n_i}^i \otimes Ay_{n_i}^i]_B \right) \\ &+ [[l_j]]_B, \quad 1 \le j \le N. \end{aligned} \tag{66}$$

Define, for an arbitrary choice of ν,

$$u_\nu = \sum_{1 \le i \le k_N} \beta_i x_{n_i}^i, \qquad v_{\nu,j} = \sum_{k_{j-1} < i \le k_j} \bar{\beta}_i y_{n_i}^i, \qquad 1 \le j \le N, \tag{67}$$

where

$$(\beta_i)^2 = \alpha_i, \quad k_{j-1} < i \le k_j, \ 1 \le j \le N.$$

Proceeding as in the proof of [18, Prop. 4.6], we may make a particular choice of the k_N-tuple $\nu=(n_1,\ldots,n_{k_N})$ so that, for each $1\le j\le N$,

$$\left\|[Qa\otimes w_j]_B+\left(\sum_{k_{j-1}<i\le k_j}\alpha_i[Qx^i_{n_i}\otimes Qy^i_{n_i}]_B\right)-[Q(a+u_\nu)\otimes(w_j+Qv_{\nu,j})]_B\right\|<\tau_j, \tag{68}$$

$$\|[Au_\nu\otimes b_j]_B\|<\tau_j, \tag{69}$$

$$\|v_{\nu,j}\|^2\approx\sum_{k_{j-1}<i\le k_j}|\alpha_i|<\left(\frac{d_j}{\gamma}\right)-\|l_j\|<\frac{\mu_j}{\gamma}, \tag{70}$$

and

$$\|u_\nu\|^2\approx\sum_{1\le i\le k_N}|\alpha_i|<\left(\frac{1}{\gamma}\right)\sum_{1\le j\le N}\mu_j. \tag{71}$$

We define

$$a_1=a+u_\nu,\qquad w_j'=w_j+Qv_{\nu,j},\qquad 1\le j\le N,$$
$$x^i=x^i_{n_i},\qquad y^i=y^i_{n_i},\qquad 1\le i\le k_N,$$

and

$$h_j=\left(\sum_{k_{j-1}<i\le k_j}\alpha_iAx^i\cdot Ay^i\right)+l_j,\quad 1\le j\le N.$$

Note that

$$\|h_j\|\le\left(\sum_{k_{j-1}<i\le k_j}|\alpha_i|\right)+\|l_j\|<\frac{d_j}{\gamma},\quad 1\le j\le N.$$

Furthermore, choose $\epsilon>0$ so that $\epsilon<\min_{1\le j\le N}\{\tau_j/(1+\|w_j'\|)\}$. Applying Proposition 2.3, we obtain $\tilde{u}\in\mathcal{H}$ and $\{b_j'\}_{j=1}^N\subset\mathcal{R}$ such that

$$\|Aa_1\cdot b_j+h_j-A(a_1+\tilde{u})\cdot b_j'\|<\epsilon,\quad 1\le j\le N, \tag{72}$$

$$\|Q\tilde{u}\|<\epsilon, \tag{73}$$

$$\|\tilde{u}\|\le 2\left(\sum_{1\le j\le N}\|h_j\|\right)^{1/2}\le\left(\frac{2}{\gamma^{1/2}}\right)\left(\sum_{1\le j\le N}\mu_j\right)^{1/2}, \tag{74}$$

and

$$\|b_j'\|\le\frac{1}{\rho}\{\|b_j\|+\|h_j\|^{1/2}\}<\frac{1}{\rho}\left\{\|b_j\|+\left(\frac{\mu_j}{\gamma}\right)^{1/2}\right\},\quad 1\le j\le N. \tag{75}$$

From (66) and (72) we get

$$\|[M^j(\nu)]_B+[Au_\nu\otimes b_j]_B-[A(a_1+\tilde{u})\otimes b_j']_B\|<\epsilon,\quad 1\le j\le N. \tag{76}$$

Combining (65), (66), (68), (69), (72), (76), and the definitions of a_1 and the w_j', we obtain

$$\|[V_j]_B-[(a_1+\tilde{u})\otimes(w_j'+b_j')]_B\|<(\theta/\gamma)\mu_j,\quad 1\le j\le N.$$

Thus, with $a'=a_1+u=a+u_\nu+u$, (50) is satisfied, and the inequalities (51) to (53) follow immediately from (71), (74), (70), and (75). Thus the proof of Proposition 3.4 is complete. □

It is obvious that Proposition 3.4 is self-improving, just as was Proposition 2.3, and we will need its exact version to complete the proof of Theorem 3.3.

COROLLARY 3.5. *Let $T \in \mathbf{A}(\mathcal{H})$ with minimal co-isometric extension $B \in \mathcal{L}(\mathcal{S} \oplus \mathcal{R})$, and suppose that $\mathcal{Q}_T$ has property $F^r_{\theta,\gamma}$ for some $0 \le \theta < \gamma \le 1$. Suppose further that we are given $\beta > 1$, $N \in \mathbf{N}$, $\{[V_j]_B\}_{j=1}^N \subset Q_B$, $a \in \mathcal{H}$, $\{w_j\}_{j=1}^N \subset \mathcal{S}$, $\{b_j\}_{j=1}^N \subset \mathcal{R}$, and positive scalars $\{\mu_j\}_{j=1}^N$, satisfying*

$$\|[V_j]_B - [a \otimes (w_j + b_j)]_B\| < \mu_j, \quad 1 \le j \le N. \tag{77}$$

Then there exist $\hat{a} \in \mathcal{H}$, $\{\hat{w}_j\}_{j=1}^N \subset \mathcal{S}$, and $\{\hat{b}_j\}_{j=1}^N \subset \mathcal{R}$ such that

$$[V_j]_B = [\hat{a} \otimes (\hat{w}_j + \hat{b}_j)]_B, \quad 1 \le j \le N, \tag{78}$$

$$\|\hat{a} - a\| < 3\alpha\Big(\sum_{1 \le j \le N} \mu_j\Big)^{1/2}, \tag{79}$$

$$\|\hat{w}_j - w_j\| < \alpha \mu_j^{1/2}, \quad 1 \le j \le N, \tag{80}$$

and

$$\|\hat{b}_j\| < \beta(\|b_j\| + \alpha \mu_j^{1/2}), \quad 1 \le j \le N, \tag{81}$$

where $\alpha = 1/(\gamma^{1/2} - \theta^{1/2})$.

Proof. (This proof is similar to that of [18, Thm. 4.7].) We first consider the case $\theta > 0$. Let $\{s_n\}_{n=0}^\infty$ be a strictly decreasing sequence of positive numbers such that $s_0 = 1$ and $\lim s_n = 1/\beta$. For $n \in \mathbf{N}$, set $\rho_n = s_n / s_{n-1}$. Via Proposition 3.4 we obtain vectors $a_1 \in \mathcal{H}$, $\{w_{1,j}\}_{j=1}^N \subset \mathcal{S}$, and $\{b_{1,j}\}_{j=1}^N \subset \mathcal{R}$ such that

$$\|[V_j]_B - [a_1 \otimes (w_{1,j} + b_{1,j})]_B\| < (\theta/\gamma)\mu_j, \quad 1 \le j \le N, \tag{82}$$

$$\|a_1 - a\| < \Big(\frac{3}{\gamma^{1/2}}\Big)\Big(\sum_{j=1}^N \mu_j\Big)^{1/2}, \tag{83}$$

$$\|w_{1,j} - w_j\| < (\mu_j/\gamma)^{1/2}, \quad 1 \le j \le N, \tag{84}$$

and

$$\|b_{1,j}\| < (1/\rho_1)(\|b_j\| + (\mu_j/\gamma)^{1/2}), \quad 1 \le j \le N. \tag{85}$$

Upon iterating the procedure, we obtain sequences $\{a_n\}_{n=1}^\infty \subset \mathcal{H}$, $\{w_{n,j}\}_{n=1}^\infty \subset \mathcal{S}$, and $\{b_{n,j}\}_{n=1}^\infty \subset \mathcal{R}$, $1 \le j \le N$, such that

$$\|[V_j]_B - [a_n \otimes (w_{n,j} + b_{n,j})]_B\| < (\theta/\gamma)^n \mu_j, \quad n \in \mathbf{N},\ 1 \le j \le N, \tag{86}$$

$$\|a_n - a_{n-1}\| < \Big(\frac{3}{\gamma^{1/2}}\Big)\Big(\sum_{j=1}^N \mu_j\Big)^{1/2}\Big(\frac{\theta}{\gamma}\Big)^{(n-1)/2}, \quad n \ge 2, \tag{87}$$

$$\|w_{n,j} - w_{n-1,j}\| < (\mu_j/\gamma)^{1/2}(\theta/\gamma)^{(n-1)/2}, \quad n \ge 2,\ 1 \le j \le N, \tag{88}$$

and

$$\|b_{n,j}\| < (1/\rho_n)\{\|b_{n-1,j}\| + (\mu_j/\gamma)^{1/2}(\theta/\gamma)^{(n-1)/2}\}, \quad n \ge 2,\ 1 \le j \le N. \tag{89}$$

Clearly the sequence $\{a_n\}$ and the sequences $\{w_{n,j}\}_{n=1}^{\infty}$, $1 \le j \le N$, are Cauchy, and it is easy to see from (83), (87), (84), and (88) that their limits $\hat{a}$ and $\hat{w}_j$ satisfy (79) and (80). From (89) and the definition of the ρ_n we obtain

$$s_n \|b_{n,j}\| < s_{n-1}\{\|b_{n-1,j}\| + (\mu_j/\gamma)^{1/2}(\theta/\gamma)^{(n-1)/2}\}, \quad n \in \mathbf{N},\ 1 \le j \le N,$$

which leads to

$$\|b_{n,j}\| < \beta\left\{\|b_j\| + \left(\frac{\mu_j}{\gamma}\right)^{1/2}\left(\sum_{k=0}^{n-1} s_k \left(\frac{\theta}{\gamma}\right)^{k/2}\right)\right\}, \quad n \in \mathbf{N},\ 1 \le j \le N. \tag{90}$$

Thus the sequences $\{b_{n,j}\}_{n=1}^{\infty}$, $1 \le j \le N$, are bounded, and by dropping down to appropriate subsequences we may assume that $\{b_{n,j}\}_{n=1}^{\infty}$ converges weakly to $\hat{b}_j$, $1 \le j \le N$. Since $s_k < 1$ for $k \ge 1$, (81) follows from (90). To prove (78), we first observe that

$$\lim_n \|[\hat{a} \otimes (\hat{w}_j + b_{n,j})]_B - [\hat{a}_n \otimes (w_{n,j} + b_{n,j})]_B\| = 0 \tag{91}$$

because $\|[x \otimes y]\| \le \|x\|\,\|y\|$, $\|\hat{a} - a_n\| \to 0$, and the sequences $\{w_{n,j} + b_{n,j}\}_{n=1}^{\infty}$, $1 \le j \le N$, are bounded (being weakly convergent to $\hat{w}_j + \hat{b}_j$). Thus, using (86), we obtain

$$\lim_n \|[V_j]_B - [\hat{a} \otimes (\hat{w}_j + b_{n,j})]_B\| = 0, \quad 1 \le j \le N,$$

and hence, for any $X \in \mathfrak{A}_B$, we have

$$\begin{aligned}\langle X, [\hat{a} \otimes (\hat{w}_j + \hat{b}_j)]_B \rangle &= (X\hat{a}, \hat{w}_j + \hat{b}_j) = \lim_n (X\hat{a}, \hat{w}_j + b_{n,j}) \\ &= \lim_n \langle X, [\hat{a} \otimes (\hat{w}_j + b_{n,j})]_B \rangle = \langle X, [V_j]_B \rangle, \quad 1 \le j \le N.\end{aligned}$$

This completes the proof in the case $\theta > 0$.

In the case $\theta = 0$ we choose $\mu_j' < \mu_j$ such that

$$\|[V_j]_B - [a \otimes (w_j + b_j)]_B\| < \mu_j', \quad 1 \le j \le N,$$

and, by an elementary continuity argument, we choose $0 < \theta' < \gamma$ such that

$$\mu_j'^{1/2} \frac{1}{\gamma^{1/2} - (\theta')^{1/2}} \le \mu_j^{1/2}\left(\frac{1}{\gamma^{1/2}}\right), \quad 1 \le j \le N.$$

Since $\mathfrak{A}_T$ has property $F_{\theta',\gamma}^r$, we may apply what has just been proved with the estimates μ_j', and we do get the result corresponding to $\theta = 0$ and the estimates μ_j. Thus the proof is complete. □

We next observe that what remains to be proved of Theorem 3.3 is a consequence of the following stronger result.

THEOREM 3.6. *Let $T \in \mathbf{A}(\mathcal{H})$, with minimal co-isometric extension $B \in \mathcal{L}(\mathcal{S} \oplus \mathcal{R})$, and suppose that $\mathfrak{A}_T$ has property $F_{\theta,\gamma}^r$ for some $0 \le \theta < \gamma \le 1$. Suppose also we are given vectors $a \in \mathcal{H}$, $\{[L_j]_T\}_{j=1}^{\infty} \subset Q_T$, and positive scalars $\tau > 1$ and $\{\delta_j\}_{j=1}^{\infty}$ such that*

$$\|[L_j]_T\| < \delta_j, \quad j \in \mathbf{N},$$

and

$$\delta = \sum_{j=1}^{\infty} \delta_j < \infty.$$

Then there exist $a' \in \mathcal{H}$ *and* $\{w_j \oplus b_j\}_{j=1}^{\infty} \subset \mathcal{S} \oplus \mathcal{R}$ *satisfying*

$$[L_j]_T = [a' \otimes P(w_j + b_j)]_T, \quad j \in \mathbf{N}, \tag{92}$$

$$\|a' - a\| < 3\alpha\delta^{1/2}, \tag{93}$$

$$\|w_j\| < \alpha\delta_j^{1/2}, \quad j \in \mathbf{N}, \tag{94}$$

$$\|b_j\| < \tau\alpha\delta_j^{1/2}, \quad j \in \mathbf{N}, \tag{95}$$

where $\alpha = 1/(\gamma^{1/2} - \theta^{1/2})$. *In particular, if* $T \in \mathbf{A}$ *and* $\mathcal{Q}_T$ *has property* $F_{\theta,\gamma}^r$, *then* $T \in \mathbf{A}_{1,\aleph_0}(3\sqrt{2}\alpha^2)$. *Furthermore, given* $\{[L_j]_T\}_{j=1}^{\infty} \subset Q_T$, *the set of vectors* $\tilde{a}$ *in* $\mathcal{H}$ *for which there exists a sequence* $\{y_j\}_{j=1}^{\infty} \subset \mathcal{H}$ *satisfying*

$$[L_j]_T = [\tilde{a} \otimes y_j]_T, \quad j \in \mathbf{N},$$

is dense in $\mathcal{H}$.

The first part of Theorem 3.6 will be an immediate consequence of the upcoming Theorem 3.7, which includes some additional technicalities needed in the sequel. Before stating that stronger result, however, we show how to deduce the membership of T in $\mathbf{A}_{1,\aleph_0}(3\sqrt{2}\alpha^2)$ from this first part. To that end let $\{[L_j]_T\}_{j=1}^{\infty}$ be a sequence from Q_T such that $d = \sum_{j=1}^{\infty} d_j < \infty$, where $d_j = \|[L_j]_T\|$. Choose $s > 3\sqrt{2}\alpha^2$ and $\tau > 1$ such that $3(1+\tau^2)^{1/2}\alpha^2 < s$, and also choose, for each integer j, $\delta_j > d_j$ such that $3(1+\tau_j^2)^{1/2}\alpha^2\delta_j < sd_j$. It follows easily that $3(1+\tau^2)^{1/2}\alpha^2\delta < sd$.

By the first part of Theorem 3.6, there exist a vector a' and a sequence $\{y_j' = P(w_j + b_j)\}_{j=1}^{\infty}$ from $\mathcal{H}$ satisfying

$$[L_j]_T = [a' \otimes y_j']_T, \quad j \in \mathbf{N},$$

$$\|a'\| < 3\alpha\delta^{1/2},$$

$$\|y_j'\| \le (\|w_j\|^2 + \|b_j\|^2)^{1/2} < \alpha(1+\tau^2)^{1/2}\delta_j^{1/2}, \quad j \in \mathbf{N}.$$

Setting

$$x = \frac{1}{\sqrt{3}}(1+\tau^2)^{1/4}a', \qquad y_j = \frac{\sqrt{3}}{(1+\tau^2)^{1/4}}y_j', \quad j \in \mathbf{N},$$

we have

$$[L_j]_T = [x \otimes y_j]_T, \quad j \in \mathbf{N},$$

$$\|x\| < \sqrt{3}(1+\tau^2)^{1/4}\alpha\delta^{1/2} < (sd)^{1/2},$$

and

$$\|y_j\| < \sqrt{3}(1+\tau^2)^{1/4}\alpha\delta_j^{1/2} < (sd_j)^{1/2}, \quad j \in \mathbf{N}.$$

Therefore $T \in \mathbf{A}_{1,\aleph_0}(3\sqrt{2}\alpha^2)$, as was to be shown. Moreover, the last statement of Theorem 3.6 (concerning density) follows from the first by a standard argument involving scaling of the $\{[L_j]\}_{j=1}^{\infty}$; we omit the details, as the technique is by now fairly standard and is used, in particular, in the proof of Theorem 5.4.

THEOREM 3.7. *Let $T \in \mathbf{A}(\mathcal{H})$, with minimal co-isometric extension $B \in \mathcal{L}(\mathcal{S} \oplus \mathcal{R})$, and suppose $\mathcal{Q}_T$ has property $F^r_{\theta,\gamma}$ for some $0 \le \theta < \gamma \le 1$. Suppose we are given $a \in \mathcal{H}$, $\{y_j\}_{j=1}^\infty \subset \mathcal{S}$, $\{[L_j]_T\}_{j=1}^\infty \subset Q_T$, scalars $\tau > 1$, $\eta_j > 0$, $\delta_j > \|[L_j]_T\|$, $j \in \mathbf{N}$, with $\delta = \sum_{j=1}^\infty \delta_j < \infty$, and a dense subset $\mathfrak{N}$ of Q_B. Then there exist $\hat{a} \in \mathcal{H}$ and sequences $\{w_j \oplus b_j\}_{j=1}^\infty$, $\{\hat{y}_j \oplus z_j\}_{j=1}^\infty \subset \mathcal{S} \oplus \mathcal{R}$ such that*

(96) $$[L_j]_T = [\hat{a} \otimes P(w_j + b_j)]_T, \quad j \in \mathbf{N},$$

(97) $$[\hat{a} \otimes (\hat{y}_j + z_j)]_B \in \mathfrak{N}, \quad j \in \mathbf{N},$$

(98) $$\|\hat{a} - a\| < 3\alpha\delta^{1/2},$$

(99) $$\|w_j\| < \alpha\delta_j^{1/2}, \quad j \in \mathbf{N},$$

(100) $$\|b_j\| < \tau\alpha\delta_j^{1/2}, \quad j \in \mathbf{N},$$

(101) $$\|\hat{y}_j + z_j - y_j\| < \eta_j, \quad j \in \mathbf{N},$$

where $\alpha = 1/(\gamma^{1/2} - \theta^{1/2})$.

Proof. Let $[V_j]_B = \varphi_B^{-1} \circ \varphi_T([L_j]_T)$ and let $d_j = \|[V_j]_B\|$ for each positive integer j. The proof will require an iteration based on Corollary 3.5 and, to that end, we introduce some auxiliary numerical sequences. By hypothesis there exists a sequence $\{\mu_j\}$ such that $d_j < \mu_j < \delta_j$, $j \in \mathbf{N}$, and we set $\mu = \sum_{j=1}^\infty \mu_j$. We also select a strictly decreasing sequence $\{s_n\}_{n=0}^\infty$ of positive numbers such that $s_0 = 1$ and $\lim_{n\to\infty} s_n = 1/\tau$, and we set $\beta_n = s_{n-1}/s_n$ for each positive integer n. Next we choose a strictly increasing sequence $\{N_n\}_{n=1}^\infty$ of positive integers such that $N_1 = 1$ and

$$\sum_{j=N_n}^\infty \mu_j < \left(\frac{\delta^{1/2} - \mu^{1/2}}{2^n}\right)^2, \quad n \ge 2.$$

Upon setting $\lambda_n = (\sum_{N_n \le j < N_{n+1}} \mu_j)^{1/2}$, we have

$$\sum_{n=1}^\infty \lambda_n < \lambda_1 + (\delta^{1/2} - \mu^{1/2}) \sum_{n=2}^\infty 2^{-n} < \delta^{1/2} - \frac{\delta^{1/2} - \mu^{1/2}}{2}.$$

It follows from this inequality that it is possible to select a sequence $\{\epsilon_n\}_{n=1}^\infty$ of positive numbers such that

(102) $$\sum_{n=1}^\infty (\lambda_n^2 + \epsilon_n)^{1/2} < \delta^{1/2},$$

so suppose this has been done. Next, for each positive integer j, define n_j to be the unique positive integer satisfying

$$N_{n_j} \le j < N_{n_j+1}.$$

For each pair of positive integers (n, j) satisfying $n \ge n_j$ (i.e., $j < N_{n+1}$) we now choose a positive number $\epsilon_{n,j}$ such that

(103) $$\varepsilon_{n,j} \le \inf\{\epsilon_n/N_{n+1}, t_j^2 2^{-2n}\},$$

where

(104) $$t_j = \inf\{(\delta_j^{1/2} - \mu_j^{1/2}), \eta_j/2\tau\alpha\}.$$

We can now sketch the main steps of our iterative procedure.

Step 1. Since $\mathfrak{N}$ is dense in Q_B, there exists, for each $1 \le j < N_2$, $[R_j]_B \in \mathfrak{N}$ such that

$$\|[R_j]_B - [a \otimes y_j]_B\| < \epsilon_{1,j}.$$

On the other hand, we have

$$\|[V_j]_B - [a \otimes 0]_B\| = \|[V_j]_B\| < \mu_j, \quad 1 \le j < N_2.$$

Therefore, by Corollary 3.5, we can find vectors a_1 in $\mathcal{H}$ and $\{w_{1,j} \oplus b_{1,j}\}$, $\{y_{1,j} \oplus z_{1,j}\}$ in $\mathcal{S} \oplus \mathcal{R}$, $1 \le j < N_2$, such that

(105) $$[V_j]_B = [a_1 \otimes (w_{1,j} + b_{1,j})]_B, \quad 1 \le j < N_2,$$

(106) $$[R_j]_B = [a_1 \otimes (y_{1,j} + z_{1,j})]_B, \quad 1 \le j < N_2,$$

(107) $$\|a_1 - a\| < 3\alpha\left(\sum_{j < N_2} (\mu_j + \epsilon_{1,j})\right)^{1/2} \le 3\alpha(\lambda_1^2 + \epsilon_1)^{1/2},$$

(108) $$\|w_{1,j}\| < \alpha\mu_j^{1/2}, \quad 1 \le j < N_2,$$

(109) $$\|b_{1,j}\| < \beta_1\alpha\mu_j^{1/2}, \quad 1 \le j < N_2,$$

(110) $$\|y_{1,j} - y_j\| < \alpha\epsilon_{1,j}^{1/2}, \quad 1 \le j < N_2,$$

and

(111) $$\|z_{1,j}\| < \beta_1\alpha\epsilon_{1,j}^{1/2}, \quad 1 \le j < N_2.$$

Step n $(n \ge 2)$. Suppose we have found vectors a_{n-1} in $\mathcal{H}$ and

$$\{w_{n-1,j} \oplus b_{n-1,j}\}, \{y_{n-1,j} \oplus z_{n-1,j}\} \text{ in } \mathcal{S} \oplus \mathcal{R}, \quad 1 \le j < N_n,$$

as well as elements $[R_j]_B$ in $\mathfrak{N}$ such that

$(112)_{n-1}$ $$\begin{aligned} &[V_j]_B = [a_{n-1} \otimes (w_{n-1,j} + b_{n-1,j})]_B, \quad 1 \le j < N_n, \\ &[R_j]_B = [a_{n-1} \otimes (y_{n-1,j} + z_{n-1,j})]_B, \quad 1 \le j < N_n. \end{aligned}$$

For $N_n \le j < N_{n+1}$ we choose $[R_j]_B$ in $\mathfrak{N}$ such that

$$\|[R_j]_B - [a_{n-1} \otimes y_j]_B\| < \epsilon_{n,j}.$$

These inequalities, together with

$$\|[V_j]_B - [a_{n-1} \otimes 0]_B\| < \mu_j, \quad N_n \le j < N_{n+1},$$

$$\|[V_j]_B - [a_{n-1} \otimes (w_{n-1,j} + b_{n-1,j})]_B\| < \tfrac{1}{2}\epsilon_{n,j}, \quad 1 \le j < N_n,$$

and

$$\|[R_j]_B - [a_{n-1} \otimes (y_{n-1,j} + z_{n-1,j})]_B\| < \tfrac{1}{2}\epsilon_{n,j}, \quad 1 \le j < N_n,$$

provide a setting to which we can apply Corollary 3.5. We obtain, thereby, vectors a_n in $\mathcal{H}$ and $\{w_{n,j}\oplus b_{n,j}\}$, $\{y_{n,j}\oplus z_{n,j}\}$ in $\mathcal{S}\oplus\mathcal{R}$, $1\le j<N_{n+1}$, such that the two families of equalities $(112)_n$ are satisfied, as well as the following inequalities on the norms:

$$(113)_n \qquad \|a_n-a_{n-1}\|<3\alpha\Big(\sum_{N_n\le j<N_{n+1}}\mu_j+\sum_{1\le j<N_{n+1}}\epsilon_{n,j}\Big)^{1/2}\le 3\alpha(\lambda_n^2+\epsilon_n)^{1/2},$$

$$(114)_n \qquad \|w_{n,j}\|<\alpha\mu_j^{1/2},\quad \|b_{n,j}\|<\beta_n\alpha\mu_j^{1/2},\quad N_n\le j<N_{n+1},$$

$$(115)_n \qquad \|w_{n,j}-w_{n-1,j}\|<\alpha\epsilon_{n,j}^{1/2},\quad \|b_{n,j}\|<\beta_n(\alpha\epsilon_{n,j}^{1/2}+\|b_{n-1,j}\|),\quad 1\le j<N_n,$$

$$(116)_n \qquad \|y_{n,j}-y_{n-1,j}\|<\alpha\epsilon_{n,j}^{1/2},\quad \|z_{n,j}\|<\beta_n\alpha\epsilon_{n,j}^{1/2},\quad 1\le j<N_n,\ N_n\le j<N_{n+1},$$

$$(117)_n \qquad \|y_{n,j}-y_{n-1,j}\|<\alpha\epsilon_{n,j}^{1/2},\quad \|z_{n,j}\|<\beta_n(\alpha\epsilon_{n,j}^{1/2}+\|z_{n-1,j}\|),\quad 1\le j<N_n.$$

In this fashion we do obtain Cauchy sequences $\{a_n\}_{n=1}^{\infty}$ and (for each $j\in\mathbf{N}$) $\{w_{n,j}\}_{n=n_j+1}^{\infty}$, $\{y_{n,j}\}_{n=n_j+1}^{\infty}$, whose limits $\hat{a}, w_j, \hat{y}_j$ satisfy

$$(118) \qquad \|\hat{a}-a\|<3\alpha\Big(\sum_{n=1}^{\infty}(\lambda_n^2+\epsilon_n)^{1/2}\Big)<3\alpha\delta^{1/2},$$

$$(119) \qquad \|w_j\|<\alpha\Big(\mu_j^{1/2}+\sum_{n>n_j}\epsilon_{n,j}^{1/2}\Big)<\alpha\delta_j^{1/2},\quad j\in\mathbf{N},$$

$$(120) \qquad \|\hat{y}_j-y_j\|<\alpha\sum_{n=1}^{\infty}\epsilon_{n,j}^{1/2}<\frac{\eta_j}{2},\quad j\in\mathbf{N},$$

(where, for example, (118) follows from (107), $(113)_n$, and (102)). Next (since $\beta_n=s_{n-1}/s_n$ and $s_n\le 1$), we observe from $(115)_n$ that

$$s_n\|b_{n,j}\|<s_{n-1}\|b_{n-1,j}\|+\alpha\epsilon_{n,j}^{1/2},\quad n>n_j.$$

Hence, taking into account that $s_{n_j}\|b_{n_j,j}\|<s_{n_j-1}\alpha\mu_j^{1/2}$, and using (103) and (104), we obtain

$$s_n\|b_{n,j}\|<\alpha\Big(\mu_j^{1/2}+\sum_{n>n_j}\epsilon_{n,j}^{1/2}\Big)<\alpha\delta_j^{1/2}.$$

Since $\lim_{n\to\infty}s_n=1/\tau$ we finally get

$$\overline{\lim_{n\to\infty}}\,\|b_{n,j}\|<\tau\alpha\delta_j^{1/2},\quad j\in\mathbf{N}.$$

Similar computations from $(116)_n$, $(117)_n$, (103), and (104) lead to

$$\overline{\lim_{n\to\infty}}\,\|z_{n,j}\|<\tau\alpha\sum_{n=n_j}^{\infty}\epsilon_{n,j}^{1/2}<\frac{\eta_j}{2},\quad j\in\mathbf{N}.$$

Choose, for each $j\in\mathbf{N}$, a weakly convergent subsequence of $\{b_{n,j}\}_{n=n_j+1}^{\infty}$ (resp., $\{z_{n,j}\}_{n=n_j+1}^{\infty}$) and let b_j (resp., z_j) denote its limit. The vectors $\hat{a}, w_j, b_j, \hat{y}_j, z_j$ then satisfy the inequalities (98)–(101).

As in the proof of Corollary 3.5 we see easily that, for all $j\in\mathbf{N}$, $[V_j]_B=[\hat{a}\otimes(w_j+b_j)]_B$, and hence from (4) and (6) that $[L_j]_T=[\hat{a}\otimes P(w_j+b_j)]_T$. Similarly, for each $j\in\mathbf{N}$, we obtain from $(112)_n$ that

$$[R_j]_B=[a_n\otimes(y_{n,j}+z_{n,j})]_B,\quad n\geq j.$$

Therefore,

$$\begin{aligned}[R_j]_B&=\lim_{n\to\infty}[a_n\otimes(y_{n,j}+z_{n,j})]_B\\&=\lim_{n\to\infty}[\hat{a}\otimes(\hat{y}_j+z_{n,j})]_B=[\hat{a}\otimes(\hat{y}_j+z_j)]_B,\end{aligned}$$

which gives (97) and completes the proofs of Theorems 3.3, 3.6, and 3.7. □

4. Analytic Invariant Subspaces

In this section we establish some connections between analytic invariant subspaces of an operator T and reflexivity. (Recall that an operator T in $\mathcal{L}(\mathcal{H})$ is *reflexive* if Alg Lat$(T)=\mathcal{W}_T$, where $\mathcal{W}_T$ is the closure of $\mathcal{A}_T$ in the weak operator topology, and Alg Lat$(T)=\{S\in\mathcal{L}(\mathcal{H})\colon \text{Lat}(S)\supset\text{Lat}(T)\}$.) We begin by recalling the definition and some properties of analytic invariant subspaces as introduced in [20] and [11].

DEFINITION 4.1. Let T be a contraction in $\mathcal{L}(\mathcal{H})$ and suppose $\mathfrak{M}\in\text{Lat}(T)$, the lattice of invariant subspaces of T. We say that $\mathfrak{M}$ is an *analytic invariant subspace* of T if there exists a nonzero conjugate analytic function $e\colon\lambda\to e_\lambda$ from $\mathbf{D}$ into $\mathfrak{M}$ such that

(121) $$(T|_{\mathfrak{M}}-\lambda)^*e_\lambda=0,\quad \lambda\in\mathbf{D}.$$

If, in addition to (121), the function e satisfies

(122) $$\bigvee_{\lambda\in\mathbf{D}}e_\lambda=\mathfrak{M},$$

then $\mathfrak{M}$ is said to be a *full analytic invariant subspace* for T.

The following proposition, for which no proof need be given, develops some elementary facts associated with analytic invariant subspaces.

PROPOSITION 4.2. *If T is a contraction in $\mathcal{L}(\mathcal{H})$ and $\mathfrak{M}$ is an analytic invariant subspace for T with associated map $e\colon\lambda\to e_\lambda$, then there exist vectors $\{y_n\}_{n=0}^\infty$ in $\mathfrak{M}$ such that e has an absolutely convergent power series expansion*

(123) $$e(\lambda)=e_\lambda=\sum_{n=0}^\infty\bar{\lambda}^n y_n,\quad \lambda\in\mathbf{D}.$$

The coefficients y_n are given by

(124) $$y_n=\frac{1}{n!}\left(\frac{d^n e}{d\bar{\lambda}^n}\bigg|_{\lambda=0}\right),\quad n\in\mathbf{N},$$

and satisfy

(125) $$(T|_{\mathfrak{M}})^* y_0 = 0, \qquad (T|_{\mathfrak{M}})^* y_n = y_{n-1}, \quad n \in \mathbf{N}.$$

Moreover,

(126) $$\overline{\lim} \|y_n\|^{1/n} = 1,$$

(127) $$\bigvee_{\lambda \in \mathbf{D}} e_\lambda = \bigvee_{n \geq 0} y_n,$$

(128) $$\dim \ker(T|_{\mathfrak{M}} - \lambda)^* \geq 1, \quad \lambda \in \mathbf{D},$$

and after suitable normalization one may suppose that $y_n \neq 0$ *for each* $n \geq 0$. *Conversely, if* $\mathfrak{M} \in \mathrm{Lat}(T)$ *and there exists a sequence* $\{y_n\}_{n=1}^{\infty}$ *of vectors in* $\mathfrak{M}$ *satisfying* (125) *and* (126), *then the conjugate analytic map e defined by* (123) *turns* $\mathfrak{M}$ *into an analytic invariant subspace for T.*

Let us denote by $H(\mathbf{D})$ the linear space of all complex-valued functions analytic on $\mathbf{D}$, and by M_λ the linear transformation of multiplication by the independent variable on $H(\mathbf{D})$.

PROPOSITION 4.3. *If T is a contraction in* $\mathcal{L}(\mathcal{H})$ *with analytic invariant subspace* $\mathfrak{M}$ *and associated map* $e: \lambda \to e_\lambda$, *and if the map* $F: \mathfrak{M} \to H(\mathbf{D})$ *is defined by*

(129) $$(F(x))(\lambda) = (x, e_\lambda) = \sum_{n=0}^{\infty} (x, y_n)\lambda^n, \quad x \in \mathfrak{M}, \ \lambda \in \mathbf{D},$$

then F is a linear map from $\mathfrak{M}$ *into* $H(\mathbf{D})$ *which is one-to-one if and only if* $\mathfrak{M}$ *is full analytic for T. Moreover,*

(130) $$F \circ (T|_{\mathfrak{M}}) = M_\lambda \circ F,$$

and if $\{x_k\}_{k=1}^{\infty}$ *is any sequence converging weakly to zero in* $\mathfrak{M}$, *then the sequence* $\{F(x_k)\}$ *converges pointwise to zero on* $\mathbf{D}$. *Finally, if* $\mathfrak{M}$ *is full analytic, then*

(131) $$\ker(T|_{\mathfrak{M}} - \mu) = (0), \quad \mu \in \mathbf{D}.$$

Proof. The only statement which is not completely trivial is the last one, and it is a consequence of the fact that

$$\ker(T|_{\mathfrak{M}} - \mu) \perp \ker(T|_{\mathfrak{M}} - \lambda)^*, \quad \forall \lambda, \mu \in \mathbf{D}, \ \mu \neq \lambda. \qquad \square$$

Full analytic invariant subspaces have proved to be very useful in establishing the reflexivity of certain operators (cf. [11], [20]), and we shall see eventually (Theorem 7.2A) that every contraction with *any* analytic invariant subspace is reflexive. But we must boot-strap our way to this result, and we begin with the following two propositions from [11], whose proofs are included here for completeness and because the proof of Proposition 4.5 is somewhat simpler than the corresponding proof in [11].

PROPOSITION 4.4. *Suppose T is a contraction in* $\mathcal{L}(\mathcal{H})$ *and* $\mathcal{H}$ *is a full analytic invariant subspace for T. Then* $T \in \mathbf{A}$ *and* $\mathrm{Alg\,Lat}(T) = \mathcal{A}_T = \mathcal{W}_T$, *so T is reflexive.*

Proof. Let $e:\lambda\to e_\lambda$ be a nonzero conjugate analytic map satisfying (121) and (122) with $\mathfrak{M}=\mathcal{H}$. By [17, Prop. 2.8], the sequence $\{T^{*n}\}_{n=1}^{\infty}$ converges to zero in the strong operator topology (notation: $T\in C_{\cdot 0}$), and thus T is absolutely continuous. Since $\sigma(T)\cap\mathbf{D}=\mathbf{D}$ from (121), where $\sigma(T)$ denotes the spectrum of T, we know that $T\in\mathbf{A}$ by [5, Prop. 4.6]. Since always

$$\mathcal{A}_T\subset\mathcal{W}_T\subset\operatorname{Alg\,Lat}(T),$$

it suffices to show that $\operatorname{Alg\,Lat}(T)\subset\mathcal{A}_T$. Denote by Λ the set of all λ in $\mathbf{D}$ for which $e_\lambda=0$, and note that at most Λ is a countable set with no point of accumulation in $\mathbf{D}$ (since e is nonzero and conjugate analytic). If $A\in\operatorname{Alg\,Lat}(T)$ then $A^*\in\operatorname{Alg\,Lat}(T^*)$, and from (121) (with $\mathfrak{M}=\mathcal{H}$) we deduce that

$$A^*e_\lambda=\overline{h(\lambda)}e_\lambda,\quad \lambda\in\mathbf{D}\backslash\Lambda, \tag{132}$$

where h is some function defined pointwise. Clearly $|h(\lambda)|\le\|A\|$ for $\lambda\in\mathbf{D}\backslash\Lambda$, so h is bounded on $\mathbf{D}\backslash\Lambda$. A computation using (129) and (132) shows that

$$(F(Ay))(\lambda)=h(\lambda)(F(y))(\lambda),\quad \lambda\in\mathbf{D}\backslash\Lambda, \tag{133}$$

for every $y\neq 0$ in $\mathcal{H}$, and since $F(y)\neq 0$ for any such y by Proposition 4.3, the zeros of $F(y)$ form a set Λ' with the same properties as Λ. Thus one has from (133) that h is a bounded quotient of analytic functions on $\mathbf{D}\backslash(\Lambda\cup\Lambda')$, and it is a routine task to extend h to a bounded function in $H(\mathbf{D})$. Another computation using (130) then shows that

$$(h(T))^*e_\lambda=\overline{h(\lambda)}e_\lambda=A^*e_\lambda,\quad \lambda\in\mathbf{D}\backslash(\Lambda\cup\Lambda'),$$

and using (122) with $\mathfrak{M}=\mathcal{H}$ and the fact that $\Lambda\cup\Lambda'$ is a countable set with no point of accumulation in $\mathbf{D}$, we conclude that $A=h(T)\in\mathcal{A}_T$, so the proof is complete. □

Of course the above proposition concerns a very special situation, but we continue to chip away at the hypotheses. For T a contraction in $\mathcal{L}(\mathcal{H})$, let us denote by $\mathcal{CF}(T)$ the set of all x in $\mathcal{H}$ such that the cyclic invariant subspace $\bigvee_{n\ge 0}T^n x$ for T generated by x is a full analytic invariant subspace for T.

PROPOSITION 4.5. *If T is an absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$ for which $\mathcal{CF}(T)$ is dense in $\mathcal{H}$, then $\mathcal{A}_T=\mathcal{W}_T$ and T is reflexive.*

Proof. Let $A\in\operatorname{Alg\,Lat}(T)$, and note that it suffices to find an h in $H^\infty(\mathbf{T})$ for which $h(T)x=Ax$ for all x in $\mathcal{CF}(T)$. Given nonzero vectors x_1 and x_2 in $\mathcal{CF}(T)$, let $\mathfrak{M}_i=\bigvee_{n\ge 0}T^n x_i$, $i=1,2$. Using Proposition 4.4 on the operators $T|_{\mathfrak{M}_i}$, $i=1,2$, we produce functions h_1 and h_2 in $H^\infty(\mathbf{T})$ such that

$$h_i(T)|_{\mathfrak{M}_i}=h_i(T|_{\mathfrak{M}_i})=A|_{\mathfrak{M}_i},\quad i=1,2, \tag{134}$$

and it clearly suffices to show that $h_1=h_2$. If $\mathfrak{M}_1\cap\mathfrak{M}_2\neq(0)$, let

$$y\in(\mathfrak{M}_1\cap\mathfrak{M}_2)\backslash(0).$$

Then $h_1(T)y = h_2(T)y$ from (134), and therefore

$$(135)\quad \begin{aligned} h_1(\lambda)(y,e_\lambda) &= (y,\tilde{h}_1(\bar{\lambda})e_\lambda) = (y,\tilde{h}_1(T^*)e_\lambda) \\ &= (h_1(T)y,e_\lambda) = (h_2(T)y,e_\lambda) = h_2(\lambda)(y,e_\lambda), \end{aligned}$$

where $e:\lambda \to e_\lambda$ is the map associated with the full analytic invariant subspace $\mathfrak{M}_1$. Since $\lambda \to (y,e_\lambda)$ is a nonzero analytic function on $\mathbf{D}$ (Prop. 4.3), it follows easily from (135) that $h_1 \equiv h_2$, and the argument is complete in case $\mathfrak{M}_1 \cap \mathfrak{M}_2 \neq (0)$. If $\mathfrak{M}_1 \cap \mathfrak{M}_2 = (0)$, choose a sequence $\{u_n\}_{n=1}^{\infty}$ from $\mathcal{CF}(T)$ such that $\lim_n \|u_n - (x_1+x_2)\| = 0$. Using Proposition 4.4 we deduce that for each n in $\mathbf{N}$ there exists a function f_n in $H^\infty(\mathbf{T})$ such that $f_n(T)$ and A agree on $\bigvee_{j\geq 0} T^j u_n$ and $\|f_n\| \leq \|A\|$. By dropping down to a subsequence, we may suppose that $\{f_n\}_{n=1}^{\infty}$ converges weak* to a function f in $H^\infty(\mathbf{T})$. Thus,

$$(136)\quad \begin{aligned} h_1(T)x_1 + h_2(T)x_2 &= Ax_1 + Ax_2 \\ &= \lim_n Au_n \\ &= \lim_n f_n(T)u_n \\ &= f(T)(x_1+x_2), \end{aligned}$$

because $\{f_n(T)\}$ converges in the weak operator topology to $f(T)$ and $\{u_n\}$ converges strongly to $x_1 + x_2$. Since $\mathfrak{M}_1 \cap \mathfrak{M}_2 = (0)$, we obtain from (136) that $h_i(T)x_i = f(T)x_i$, $i = 1, 2$, and then by a repetition of the argument used above we get $h_1 \equiv h_2$, so the proof is complete. □

In Section 7 we will prove the much better theorem that every contraction with an analytic invariant subspace is reflexive, but first we must connect the topic of analytic invariant subspaces to the material of Sections 2 and 3.

5. Solving Equations in Q_T and Analytic Invariant Subspaces

In this section we develop a connection between the solution of certain $1 \times \aleph_0$ systems of simultaneous equations in a predual Q_T and the existence of analytic invariant subspaces for the operator T. In particular, we show that if $T \in \mathbf{A}(\mathcal{H})$ and $\mathcal{Q}_T$ has some property $F_{\theta,\gamma}^r$ $(\theta < \gamma)$, then either T has an isometric part or a supply of cyclic full analytic invariant subspaces that is sufficiently rich for Proposition 4.5 to apply.

DEFINITION 5.1. If $T \in \mathbf{A}$ and n is a nonnegative integer, we set $[C_0^{(n)}] = \varphi_T^{-1}([e^{-int}])$. It follows easily that $\|[C_0^{(n)}]\| = 1$ and $[C_0^{(n)}]$ is the unique element of Q_T satisfying

$$(137)\quad \langle h(T), [C_0^{(n)}] \rangle = \hat{h}(n), \quad h \in H^\infty,$$

where $\hat{h}(n)$ is the nth Fourier coefficient of h. We denote by $\mathfrak{N}_T$ the linear manifold in Q_T spanned (algebraically) by the set $\{[C_0^{(n)}]: n = 0, 1, \ldots\}$, and note that it follows easily from the definitions and elementary Fourier analysis that $\mathfrak{N}_T$ is norm-dense in Q_T.

The following two propositions are taken from [11], and their proofs are included here for completeness.

PROPOSITION 5.2. *Let $T \in \mathbf{A}(\mathcal{H})$ and suppose that there exist vectors x and $\{t_j\}_{j=0}^{\infty}$ in $\mathcal{H}$ such that*

$$[x \otimes t_j]_T = [C_0^{(j)}]_T, \quad j = 0, 1, \ldots, \tag{138}$$

and such that

$$\overline{\lim_{j\to\infty}} \|t_j\|^{1/j} \le 1. \tag{139}$$

Then the cyclic invariant subspace $\mathfrak{M}$ for T generated by x is an analytic invariant subspace for T. In particular, if $T \in \mathbf{A}_{1,\aleph_0}(r)$ for some $r \ge 1$, then T has a (cyclic) analytic invariant subspace.

Proof. Since $\mathfrak{M} \in \operatorname{Lat}(T)$ and $x \in \mathfrak{M}$, we have

$$[x \otimes \tilde{y}]_T = [x \otimes y]_T, \quad y \in \mathcal{H}, \tag{140}$$

where $\tilde{y}$ is the projection of y onto $\mathfrak{M}$. Moreover, since $\mathfrak{M} = \bigvee_{n\ge 0} T^n x$, the conjugate linear map $y \to [x \otimes y]_T$ from $\mathfrak{M}$ into Q_T is one-to-one. From (138) and (137) we obtain

$$\begin{aligned} &[Tx \otimes t_0]_T = 0, \\ &[Tx \otimes t_j]_T = [C_0^{(j-1)}]_T = [x \otimes t_{j-1}]_T, \quad j \in \mathbf{N}. \end{aligned} \tag{141}$$

Of course, we also have

$$[Tx \otimes t_j]_T = [Tx \otimes \tilde{t}_j]_T = [x \otimes (T|_{\mathfrak{M}})^* \tilde{t}_j]_T, \quad j = 0, 1, \ldots, \tag{142}$$

and it follows easily from (141) and (142) that each $\tilde{t}_j$ is nonzero and that

$$\begin{aligned} &(T|_{\mathfrak{M}})^* \tilde{t}_0 = 0, \\ &(T|_{\mathfrak{M}})^* \tilde{t}_j = \tilde{t}_{j-1}, \quad j \in \mathbf{N}. \end{aligned} \tag{143}$$

Since $\|\tilde{t}\| \le \|t\|$ for all t in $\mathcal{H}$, we obtain from (139) that $\overline{\lim} \|\tilde{t}_j\|^{1/j} \le 1$, and the reverse inequality follows from (143). Thus, by Proposition 4.2, $\mathfrak{M}$ is an analytic invariant subspace for T.

Suppose next that $T \in \mathbf{A}_{1,\aleph_0}(r)$ for some $r \ge 1$. Then, by definition there exist vectors x and $\{z_j\}_{j=0}^{\infty}$ in $\mathcal{H}$ such that

$$[x \otimes z_j]_T = [C_0^{(j)}]_T / (j+1)^2, \quad j = 0, 1, \ldots, \tag{144}$$

and

$$\|z_j\| < 2r^{1/2}/(j+1), \quad j = 0, 1, \ldots. \tag{145}$$

Therefore, upon setting $t_j = (j+1)^2 z_j$, we obtain (138). Moreover, from (145) we deduce that (139) is valid, so by what has already been proved, T has a cyclic analytic invariant subspace. □

PROPOSITION 5.3. *Let $T \in \mathbf{A}(\mathcal{H})$, and suppose that there exist a vector x and sequences $\{t_j\}_{j=0}^{\infty}$ and $\{s_j\}_{j=1}^{\infty}$ in $\mathcal{H}$ such that*

$$[x \otimes t_j]_T = [C_0^{(j)}]_T, \quad j = 0, 1, \ldots, \tag{146}$$

$$\overline{\lim} \|t_j\|^{1/j} \le 1, \tag{147}$$

$$[x \otimes s_j]_T \in \mathfrak{N}_T, \quad j = 1, 2, \ldots, \tag{148}$$

and

$$\bigvee_{j \ge 1} s_j = \mathcal{H}. \tag{149}$$

Then the cyclic invariant subspace $\mathfrak{M}$ for T generated by x is a full analytic invariant subspace for T.

Proof. It follows from Proposition 5.2 that $\mathfrak{M} = \bigvee_{n \ge 1} T^n x$ is an analytic invariant subspace for T. Furthermore, from the definition of $\mathfrak{N}_T$ it follows that for each fixed $j \in \mathbf{N}$, $[x \otimes s_j]_T$ is a linear combination of the $[C_0^{(i)}]_T$—say,

$$[x \otimes s_j]_T = \sum_{i=0}^{N_j} \alpha_i^{(j)} [C_0^{(i)}]_T = \left[x \otimes \sum_{i=0}^{N_j} \bar{\alpha}_i^{(j)} t_i \right]_T$$

Thus, from the one-to-one character of the mapping $\tilde{y} \to [x \otimes \tilde{y}]_T$, where $\tilde{y}$ is (as before) the projection of y onto $\mathfrak{M}$, we see that

$$\tilde{s}_j = \sum_{i=0}^{N_j} \bar{\alpha}_i^{(j)} \tilde{t}_i, \quad j \in \mathbf{N}.$$

Thus $\bigvee_{j \ge 1} \tilde{s}_j \subset \bigvee_{j \ge 0} \tilde{t}_j$, and from (149) we obtain $\bigvee_{j \ge 0} \tilde{t}_j = \mathfrak{M}$. Since the sequence $\{\tilde{t}_j\}$ satisfies (143), just as in the proof of Proposition 5.2, it follows from Proposition 4.2 (cf. (127)) that $\mathfrak{M}$ is a full analytic invariant subspace for T. □

The next step in our boot-strapping process is crucial.

THEOREM 5.4. *Suppose T is a completely nonunitary contraction in $\mathbf{A}(\mathcal{H})$ with minimal co-isometric extension $B \in \mathcal{L}(\mathcal{S} \oplus \mathcal{R})$, $\mathcal{A}_T$ has property $F_{\theta,\gamma}^r$ for some $0 \le \theta < \gamma \le 1$, and there exists no nonzero subspace $\mathfrak{M} \in \mathrm{Lat}(T)$ such that $T | \mathfrak{M}$ is a pure isometry (i.e., a unilateral shift). Then the set $\mathcal{CF}(T)$ of those vectors x in $\mathcal{H}$ each of which generates a full analytic invariant subspace for T is dense in $\mathcal{H}$, and hence T is reflexive.*

Proof. We assert that $P(\mathcal{S} \oplus (0))$ is dense in $\mathcal{H}$. Indeed, a calculation shows that

$$\mathcal{H} \ominus \overline{P(\mathcal{S} \oplus (0))} \subset \mathcal{R} \cap \mathcal{H} \in \mathrm{Lat}(T) \cap \mathrm{Lat}(R).$$

Since $T | (\mathcal{R} \cap \mathcal{H}) = R | (\mathcal{R} \cap \mathcal{H})$ is an isometry, and since T is completely nonunitary and has no pure isometric part, the density of $P(\mathcal{S} \oplus 0)$ in $\mathcal{H}$ is established. Next, let $\{y_j\}_{j=1}^{\infty}$ be a sequence dense in $\mathcal{S} \oplus (0)$, and let $a \in \mathcal{H}$ and $\epsilon > 0$ be given. Fix $\tau > 1$ and choose $\nu > 0$ such that

$$4\nu\left(\sum_{j=1}^{\infty}\frac{1}{j^2}\right)^{1/2}<\epsilon(\gamma^{1/2}-\theta^{1/2}). \tag{150}$$

Then we may apply Theorem 3.7 with $\mathfrak{N}=\varphi_B^{-1}\circ\varphi_T(\mathfrak{N}_T)$,

$$\delta_j=\frac{16\nu^2}{9j^2},\quad \eta_j=\frac{1}{j},\quad [L_j]_T=\frac{\nu^2[C_0^{(j-1)}]_T}{j^2},\quad j\in\mathbf{N}, \tag{151}$$

and we obtain a vector $\hat{a}$ in $\mathcal{H}$ and sequences $\{\hat{y}_j\oplus z_j\}_{j=1}^{\infty}$, $[w_j\oplus b_j\}_{j=1}^{\infty}$ in $\mathcal{S}\oplus\mathcal{R}$ such that (96)–(101) are satisfied. It follows easily that by setting

$$t_{j-1}=\frac{j^2}{\nu^2}P(w_j+b_j),\quad s_j=P(\hat{y}_j+z_j),\quad j\in\mathbf{N}, \tag{152}$$

and using (4) and (6), we obtain

$$[\hat{a}\otimes t_j]_T=[C_0^{(j)}]_T,\quad j=0,1,\dots, \tag{153}$$

and

$$[\hat{a}\otimes s_j]_T\in\mathfrak{N}_T,\quad j\in\mathbf{N}. \tag{154}$$

Moreover, from (99), (100), (151), and (152) we have

$$\|t_j\|\le\frac{(j+1)^2}{\nu^2}(\|w_{j+1}\|+\|b_{j+1}\|)\le\left(\frac{\tau+1}{\gamma^{1/2}-\theta^{1/2}}\right)\left(\frac{4}{3\nu}\right)(j+1),\quad j\in\mathbf{N},$$

so

$$\overline{\lim}\|t_j\|^{1/j}\le 1.$$

On the other hand, from (152) and (101) we know that

$$\|s_j-Py_j\|\le 1/j,\quad j\in\mathbf{N},$$

and since $\{y_j\}$ is dense in $\mathcal{S}\oplus(0)$ and $P(\mathcal{S}\oplus(0))$ is dense in $\mathcal{H}$, the sequences $\{Py_j\}$ and $\{s_j\}$ are dense in $\mathcal{H}$. Thus, by Proposition 5.3, $\mathfrak{M}=\bigvee_{k\ge 0}T^k\hat{a}$ is a full analytic invariant subspace for T. Since from (98) and (151) we know that

$$\|a-\hat{a}\|<\left(\frac{3}{\gamma^{1/2}-\theta^{1/2}}\right)\left(\frac{4\nu}{3}\right)\left(\sum\frac{1}{j^2}\right)^{1/2}<\epsilon,$$

it follows that $\mathcal{CF}(T)$ is dense in $\mathcal{H}$. Therefore, by Proposition 4.5, T is reflexive. □

In Section 7 we will use Theorem 5.4 to obtain our new sufficient conditions for the reflexivity of a contraction operator, but first we must bring together the various concepts treated so far.

6. The Wheel of Equivalences

In this section we give several characterizations of the class $\mathbf{A}_{1,\aleph_0}$ of a geometric and spectral nature. We begin with a well-known lemma which was brought to our attention by Hari Bercovici. Recall that a subspace $\mathcal{J}\subset\mathcal{H}$ is semi-invariant for an operator T in $\mathcal{L}(\mathcal{H})$ if there exist $\mathfrak{M}, \mathfrak{N} \in \text{Lat}(T)$ such that $\mathfrak{M} \supset \mathfrak{N}$ and $\mathcal{J} = \mathfrak{M} \ominus \mathfrak{N}$.

LEMMA 6.1. *Every absolutely continuous contraction T in $\mathcal{L}(\mathcal{H})$ is similar to a contraction $\tilde{T}$ in $\mathcal{L}(\mathcal{H})$ with the property that $\|\tilde{T}x\| < \|x\|$ for every nonzero x in $\mathcal{H}$.*

Proof. It is well known that the minimal unitary dilation of an absolutely continuous contraction is an absolutely continuous unitary operator (cf. [24, Ch. II, Thm. 6.4]), and it follows easily that T has a unitary dilation W, acting on a Hilbert space $\mathcal{W} \supset \mathcal{H}$, which is a bilateral shift of some multiplicity. It is also well known that W is similar to a weighted bilateral shift $\tilde{W}$ all of whose weights are positive numbers less than one. It is obvious that $\tilde{W}$ has the property that $\|\tilde{W}x\| < \|x\|$ for all nonzero x, and this property clearly carries over to the compression of $\tilde{W}$ to any semi-invariant subspace. An elementary calculation shows that T, which is the compression of W to a semi-invariant subspace, is similar to the compression $\tilde{T}$ of $\tilde{W}$ to some semi-invariant subspace, and the result follows. □

We now introduce some notation that will be useful in the remainder of the paper. If T belongs to $\mathcal{L}(\mathcal{H})$, we write $\sigma_r(T)$ for the right spectrum of T, and $\mathfrak{F}'_+(T)$ for the set of all points λ in $\mathbf{C}$ such that $T-\lambda$ is a Fredholm operator with (strictly) positive (Fredholm) index. The compression of T to a semi-invariant subspace $\mathcal{J}$ is denoted by $T_{\mathcal{J}}$, and the set of all semi-invariant subspaces for T is denoted by $\mathcal{S}\mathcal{J}(T)$.

The following result, which establishes an equivalence between various a priori unrelated concepts, is one of our two principal theorems.

THEOREM 6.2. *If T is an absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$ then the following are equivalent:*

(a) $T \in \mathbf{A}$ *and* $\mathcal{Q}_T$ *has property* $E^r_{0,1}$;
(b) $T \in \mathbf{A}$ *and* $\mathcal{Q}_T$ *has property* $E^r_{\theta,\gamma}$ *for some* $0 \le \theta < \gamma \le 1$;
(c) $T \in \mathbf{A}$ *and* $\mathcal{Q}_T$ *has property* $F^r_{\theta,\gamma}$ *for some* $0 \le \theta < \gamma \le 1$;
(d) $T \in \mathbf{A}_{1,\aleph_0}(\rho)$ *for some* $\rho \ge 1$;
(e) $T \in \mathbf{A}_{1,\aleph_0}$;
(f) *the set* $\mathcal{C}\mathfrak{F}(T)$ *of those vectors x which generate a full analytic invariant subspace for T is dense in* $\mathcal{H}$;
(g) *T has an analytic invariant subspace*;
(h) *there exists* $\mathfrak{M} \in \text{Lat}(T)$ *such that* $(T|_{\mathfrak{M}}) \in \mathbf{A}(\mathfrak{M})$ *and* $(\sigma_r(T|_{\mathfrak{M}}) \cap \mathbf{D}) \cup (\mathbf{D} \setminus \mathfrak{F}'_+(T|_{\mathfrak{M}}))$ *is dominating for* $\mathbf{T}$;
(i) *there exists* $\mathcal{J}$ *in* $\mathcal{S}\mathcal{J}(T)$ *such that* $T_{\mathcal{J}} \in \mathbf{A}(\mathcal{J})$ *and* $(\sigma_r(T_{\mathcal{J}}) \cap \mathbf{D}) \cup (\mathbf{D} \setminus \mathfrak{F}'_+(T_{\mathcal{J}}))$ *is dominating for* $\mathbf{T}$;
(j) *the j-fold ampliation* $T^{(j)} \in \mathbf{A}_{1,\aleph_0}$ *for some* $j \in \mathbf{N}$.

Proof. We establish first the wheel of implications

$$\text{(a)} \Rightarrow \text{(b)} \Rightarrow \text{(c)} \Rightarrow \text{(d)} \Rightarrow \text{(e)} \Rightarrow \text{(a)}. \tag{155}$$

Of these, that (a) implies (b), (b) implies (c), and (d) implies (e) are trivial. Moreover, that (c) implies (d) is the content of Theorem 3.3. To show that (e) implies (a) and thus that (155) is valid, suppose now that $T \in \mathbf{A}_{1,\aleph_0}$, and for each λ in $\mathbf{D}$ let $[C_\lambda]_T$ be the unique element of Q_T such that

$$\langle f(T), [C_\lambda]_T \rangle = f(\lambda), \quad f \in H^\infty(\mathbf{T}).$$

We will show that each $[C_\lambda]_T$ belongs to $\mathcal{E}_0^r(\mathfrak{A}_T)$, and since $\overline{\mathrm{aco}}\{[C_\lambda] : \lambda \in \mathbf{D}\}$ is the closed unit ball in Q_T (cf. [5, Prop. 1.21]), this will, indeed, show that $\mathfrak{A}_T$ has property $E_{0,1}^r$. By applying the familiar technique of taking Möbius transforms (cf. [13, proof that Theorem 2.4 implies Theorem 2.3]), we see that it suffices to show that $[C_0]_T \in \mathcal{E}_0^r(\mathfrak{A}_T)$. For this purpose, let x and $\{t_j\}_{j=0}^\infty$ be vectors in $\mathcal{H}$ such that

$$[C_0^{(j)}]_T = [x \otimes t_j], \quad j = 0, 1, \ldots,$$

and set $\mathfrak{M} = \bigvee_{n \geq 0} T^n x$. As in the proof of Proposition 5.2, we may suppose that each $t_j \in \mathfrak{M}$, and from (143) we learn that

$$t_j \in \mathrm{Ker}(T|_{\mathfrak{M}})^{*(j+1)}, \quad (T|_{\mathfrak{M}})^{*j} t_j = t_0 \neq 0, \quad j \in \mathbf{N}.$$

Thus there exists an orthonormal sequence $\{x_n\}_{n=1}^\infty$ such that

$$x_n \in \mathrm{Ker}(T|_{\mathfrak{M}})^{*(n+1)} \ominus \mathrm{Ker}(T|_{\mathfrak{M}})^{*n}, \quad n \in \mathbf{N},$$

and from [17, Lemma 5.2], with T and T^* interchanged, we deduce that $[C_0] \in \mathcal{E}_0^r(\mathfrak{A}_T)$. This establishes the equivalence of (a)–(e), and we next verify that (j) is also equivalent to this group. Obviously (e) implies (j), and if $T^{(j)} \in \mathbf{A}_{1,\aleph_0}$ for some $j \in \mathbf{N}$, then $\mathfrak{A}_{T^{(j)}}$ has property $E_{0,1}^r$ by (155). The technique used in the proof of [5, Thm. 3.8] then shows that $\mathfrak{A}_T$ has property $E_{(j-1)/j,1}^r$, and since $T \in \mathbf{A}$ along with $T^{(j)}$, we have that (j) implies (b).

To complete the proof of the theorem, we will establish the chain of implications

$$\text{(156)} \qquad \text{(e)} \Rightarrow \text{(f)} \Rightarrow \text{(g)} \Rightarrow \text{(h)} \Rightarrow \text{(i)} \Rightarrow \text{(b)}.$$

Of these, only the first and last are nontrivial. To show that (e) implies (f), suppose $T \in \mathbf{A}_{1,\aleph_0}$. By Lemma 6.1 there exist a contraction T_1 and an invertible operator B in $\mathcal{L}(\mathcal{H})$ such that T_1 reduces the norm of every nonzero vector and $T_1 = B^{-1}TB$. Since $f(T) = Bf(T_1)B^{-1}$ for every f in $H^\infty(\mathbf{T})$, we have

$$\text{(157)} \qquad \|f\|_\infty = \|f(T)\| \leq \|B\| \, \|B^{-1}\| \, \|f(T_1)\|,$$

and applying this relation to the functions f^n, $n \in \mathbf{N}$, and taking nth roots, we see that $T_1 \in \mathbf{A}$. That $\mathfrak{A}_{T_1}$ also has property $(\mathbf{A}_{1,\aleph_0})$ is proved as in [5, Remark 2.1], so $T_1 \in \mathbf{A}_{1,\aleph_0}$, and since (155) is valid, $\mathfrak{A}_{T_1}$ has property $E_{0,1}^r$. Since T_1 reduces the norm of every nonzero vector, T_1 has no nontrivial isometric part, and thus by Theorem 5.4, $\mathcal{CF}(T_1)$ is dense in $\mathcal{H}$. If $\mathfrak{M}_1$ is a cyclic full invariant subspace for T_1 generated by x then, since $T_1|_{\mathfrak{M}_1}$ is similar to $T|_{B\mathfrak{M}_1}$, it is clear that $B\mathfrak{M}_1$ is a full analytic invariant subspace for T generated by

Bx, and so (e) implies (f). To show that (i) implies (b), we suppose that $\mathcal{J}$ is a semi-invariant subspace for T such that

(158) $\quad T_{\mathcal{J}} \in \mathbf{A}(\mathcal{J})$ and $(\sigma_r(T_{\mathcal{J}}) \cap \mathbf{D}) \cup (\mathbf{D} \setminus \mathcal{F}'_+(T_{\mathcal{J}}))$ is dominating for $\mathbf{T}$,

and we show that

(159) $\quad \mathcal{Q}_{T_{\mathcal{J}}}$ has property $E^r_{\theta,1}$ for some $0 \le \theta < 1$.

Assume for the moment that this has been done. Then, using (155), we see that $T_{\mathcal{J}} \in \mathbf{A}_{1,\aleph_0}$, and since $\mathcal{J}$ is a semi-invariant subspace for T it follows easily (cf. [5, Prop. 4.11]) that $T \in \mathbf{A}_{1,\aleph_0}$. Using (155) again, we conclude that $\mathcal{Q}_T$ has property $E^r_{\theta,\gamma}$ for some $\theta < \gamma$, so (b) is established.

To show that (158) implies (159) and thus complete the proof of the theorem, we may as well take $\mathcal{J} = \mathcal{H}$ for simplicity of notation, and we separate out what is to be proved as a proposition.

PROPOSITION 6.3. *Suppose that* $T \in \mathbf{A}(\mathcal{H})$ *and* $(\sigma_r(T) \cap \mathbf{D}) \cup (\mathbf{D} \setminus \mathcal{F}'_+(T))$ *is dominating for* $\mathbf{T}$. *Then there exists* $0 \le \theta < 1$ *such that* $\mathcal{Q}_T$ *has property* $E^r_{\theta,1}$.

Proof. By definition, it suffices to show that there exists $0 \le \theta < 1$ such that $\overline{\text{aco}}\{\mathcal{E}^r_\theta(\mathcal{Q}_T)\}$ contains the unit ball in Q_T. For any subset $\Lambda \subset \mathbf{D}$, let $\text{NTL}(\Lambda)$ denote the (Borel) subset of $\mathbf{T}$ consisting of all nontangential limit points of Λ. It follows from the hypothesis that, disregarding a set of measure zero, we have

(160) $$\text{NTL}(\mathbf{D} \setminus \mathcal{F}'_+(T)) \cup \text{NTL}(\sigma_r(T) \cap \mathbf{D}) = \mathbf{T}.$$

Moreover, it follows from the same calculation as that made in the proof of [12, Prop. 2.8] that if Λ_1 is any subset of $\mathbf{D}$, then

(161) $$\overline{\text{aco}}\{[C_\lambda] : \lambda \in \Lambda_1\} \supset \{[[f]] \in Q_T : f \in L^1(\text{NTL}(\Lambda_1)),\ \|f\| \le 1\},$$

where $[[f]]$ is as defined after Definition 3.2; it is elementary that if $\Lambda_2 \subset \mathbf{D}$ then

(162) $$\overline{\text{aco}}\{[[\chi_\Gamma/m(\Gamma)]] : \Gamma \subset \text{NTL}(\Lambda_2)\} \supset \{[[f]] \in Q_T : f \in L^1(\text{NTL}(\Lambda_2)),\ \|f\| \le 1\},$$

where $\Gamma \subset \text{NTL}(\Lambda_2)$ means that Γ runs through the Borel subsets of $\text{NTL}(\Lambda_2)$. Therefore, from (160), (161), and (162), to prove the proposition it suffices to show that

(163) $$[C_\lambda]_T \in \mathcal{E}^r_{1/2}(\mathcal{Q}_T) \quad \forall \lambda \in \mathbf{D} \setminus \mathcal{F}'_+(\mathcal{Q}_T)$$

and

(164) $$\exists 0 \le \theta < 1 : [[\chi_\Gamma/m(\Gamma)]] \in \mathcal{E}^r_\theta(\mathcal{Q}_T) \quad \forall \Gamma \subset \text{NTL}(\sigma_r(T) \cap \mathbf{D}).$$

That (164) is valid is a direct consequence of [14, Ch. IV, Thm. 4], so we concentrate on (163). It follows easily from elementary Fredholm theory that $\mathbf{D} \setminus \mathcal{F}'_+(T)$ can be written as the disjoint union

(165) $$\mathbf{D}\backslash\mathfrak{F}'_+(T)=(\sigma_e(T)\cap\mathbf{D})\cup\mathfrak{F}_-(T)\cup\rho_b(T)\cup\sigma_{if}(T),$$

where $\rho_b(T)$ is the union of the holes in $\sigma(T)$, $\mathfrak{F}_-(T)$ is the union of those holes H in the essential spectrum $\sigma_e(T)$ of T such that $H\subset\sigma(T)$ and the Fredholm index $i(H)$ associated with H is nonpositive, and

$$\sigma_{if}(T)=\{\lambda\in\sigma(T)\backslash(\sigma_e(T)\cup\mathfrak{F}_-(T)): i(T-\lambda)=0\}.$$

One knows that $\sigma_{if}(T)$ consists at most of a countable set of isolated points γ of $\sigma(T)$, each of which has a punctured neighborhood $\mathfrak{N}_\gamma\subset\mathbf{D}\backslash\mathfrak{F}'_+(T)$, and thus

$$\Lambda_3=(\sigma_e(T)\cap\mathbf{D})\cup\mathfrak{F}_-(T)\cup\rho_b(T)$$

satisfies $\mathrm{NTL}(\Lambda_3)=\mathrm{NTL}(\mathbf{D}\backslash\mathfrak{F}'_+(T))$. Hence we may replace the task of establishing (163) by that of establishing

(166) $$[C_\lambda]_T\in\mathcal{E}^r_{1/2}(\mathfrak{A}_T)\quad\forall\lambda\in(\sigma_e(T)\cap\mathbf{D})\cup\mathfrak{F}_-(T)\cup\rho_b(T).$$

If $\lambda\in(\sigma_e(T)\cap\mathbf{D})\cup\mathfrak{F}_-(T)$, the argument is exactly like that in the proof of [18, Thm. 5.3], so it suffices to do business with those λ in $\rho_b(T)$. In this case, we need the fact (established in [15]) that the two-fold ampliation $T^{(2)}$ of T acting on the Hilbert space $\mathcal{H}^{(2)}$ belongs to $\mathbf{A}_1(r)$ for some $r\geq 1$. Thus, for any such λ, there exist vectors $\tilde{x}$ and $\tilde{y}$ in $\mathcal{H}^{(2)}$ such that $[C_\lambda]_{T^{(2)}}=[\tilde{x}\otimes\tilde{y}]$. It follows that, if we define $\tilde{\mathfrak{M}}=\bigvee_{k=0}^\infty(T^{(2)})^k\tilde{x}$, then $\tilde{\mathfrak{M}}\in\mathrm{Lat}(T^{(2)})$ and $(T^{(2)}-\lambda)|_{\tilde{\mathfrak{M}}}$ is a semi-Fredholm operator with index -1. It is therefore clear that there exists an orthonormal sequence $\{\tilde{z}_n\}_{n=1}^\infty$ in $\tilde{\mathfrak{M}}$ satisfying

(167) $$\tilde{z}_n\in\mathrm{Ker}(T^{(2)}|_{\tilde{\mathfrak{M}}}-\lambda)^{*n+1}\ominus\mathrm{Ker}(T^{(2)}|_{\tilde{\mathfrak{M}}}-\lambda)^{*n},\quad n\in\mathbf{N},$$

and by [18, Lemma 5.2] we have

(168) $$[\tilde{z}_n\otimes\tilde{z}_n]_{T^{(2)}}=[C_\lambda]_{T^{(2)}},\quad n\in\mathbf{N},$$

and

(169) $$\|[\tilde{z}_n\otimes\tilde{w}]_{T^{(2)}}\|\to 0,\quad \tilde{w}\in\mathcal{H}^{(2)}.$$

Let us write $\tilde{z}_n=z_n^{(1)}\oplus z_n^{(2)}$ for each $n\in\mathbf{N}$, where $z_n^{(1)}\in\mathcal{H}\oplus(0)$ and $z_n^{(2)}\in(0)\oplus\mathcal{H}$, and note that since

$$1=\|\tilde{z}_n\|^2=\|z_n^{(1)}\|^2+\|z_n^{(2)}\|^2,\quad n\in\mathbf{N},$$

we may suppose (without loss of generality) that for infinitely many values of n—say, along the subsequence $\{n_k\}_{k=1}^\infty$—we have

(170) $$\|z_{n_k}^{(1)}\|^2\geq 1/2,\quad k\in\mathbf{N}.$$

Equation (168) can be passed to the predual Q_T by [17, Lemma 2.4], and becomes

$$[C_\lambda]_T=[z_n^{(1)}\otimes z_n^{(1)}]_T+[z_n^{(2)}\otimes z_n^{(2)}]_T,\quad n\in\mathbf{N},$$

which, along with (170), yields

(171) $$\|[C_\lambda]_T-[z_{n_k}^{(1)}\otimes z_{n_k}^{(1)}]_T\|\leq 1/2,\quad k\in\mathbf{N}.$$

Note that the number 1/2 in (171) is independent of the λ in $\rho_b(T)$, so the proof can be completed by showing that

(172) $$\|[z_{n_k}^{(1)} \otimes y]_T\| \to 0, \quad y \in \mathcal{H},$$

and this follows immediately from (169) by taking $\tilde{w} = y \oplus 0$. Thus the proofs of Proposition 6.3 and Theorem 6.2 are complete. □

7. The Results on Reflexivity

In this section we use Theorem 5.4 and Theorem 6.2 to establish a long list of sufficient conditions that a contraction operator T in $\mathcal{L}(\mathcal{H})$ be reflexive. Since Theorem 6.2 pertains only to absolutely continuous contractions, one needs to recall that an arbitrary contraction T in $\mathcal{L}(\mathcal{H})$ has a unique decomposition $T = T_a \oplus T_s$, where T_a is an absolutely continuous contraction and T_s is a singular unitary operator. (Of course, either summand may act on the space (0).)

The following lemma is well known. A proof can be found, for example, in [3].

LEMMA 7.1. *If $T = T_a \oplus T_s$ is an arbitrary contraction in $\mathcal{L}(\mathcal{H})$, then T is reflexive if and only if T_a is reflexive.*

We are now prepared to establish several new sufficient conditions for the reflexivity of a contraction—our second principal theorem. Recall that $C_{0\cdot}$ [resp., $C_{1\cdot}$] denotes the set of all contractions T in $\mathcal{L}(\mathcal{H})$ such that the sequence $\{\|T^n x\|\}$ converges to 0 for all x in $\mathcal{H}$ [resp., only for $x = 0$] and $C_{\cdot 0} = (C_{0\cdot})^*$, $C_{\cdot 1} = (C_{1\cdot})^*$.

THEOREM 7.2. *Each of the following is a sufficient condition for an arbitrary contraction T in $\mathcal{L}(\mathcal{H})$ to be reflexive:*

(A) T *(or T^*) satisfies any one of the conditions* (a)–(j) *of Theorem* 6.2;
(B) T_a *(or T_a^*) satisfies any one of the conditions* (a)–(e) *of Theorem* 6.2;
(C) $T_a \in (C_{0\cdot} \cup C_{\cdot 0}) \cap \mathbf{A}$;
(D) $T_a \in (C_{1\cdot} \cup C_{\cdot 1}) \cap \mathbf{A}$;
(E) *T is hyponormal and $T_a \in \mathbf{A}$.*

Proof. To prove that each of the conditions in (A) is sufficient, suppose that T is an arbitrary contraction in $\mathcal{L}(\mathcal{H})$ satisfying one of the conditions (a)–(j) of Theorem 6.2. If the condition is one of (a)–(e), or (j), then $T = T_a$; whereas if the condition is one of (f)–(i), then one verifies easily that T_a satisfies the same condition. Thus we may suppose that T_a satisfies one of the conditions (a)–(j), and therefore *all* of the conditions (a)–(j), by Theorem 6.2. In particular $T_a \in \mathbf{A}_{1,\aleph_0}$, and, just as in the proof that (e) implies (f) of Theorem 6.2, one shows that T_a is similar to a contraction T' in $\mathbf{A}_{1,\aleph_0}$ that reduces the norm of every nonzero vector. Thus $\mathcal{Q}_{T'}$ has property $E_{0,1}^r$ by Theorem 6.2, and T' is reflexive by Theorem 5.4. Hence T_a is reflexive, and that T is reflexive now follows from Lemma 7.1. This proves also that each condition in (B) is sufficient.

To treat (C), suppose that $T_a \in C_{\cdot 0} \cap \mathbf{A}$. It follows from [1, Thm. 4] and [17, Prop. 2.7] that $\mathcal{A}_{T_a}$ has property $E^r_{\theta,1}$ for some $\theta < 1$, and thus by what was proved in (B), T is reflexive. The case in which $T_a \in C_{0\cdot} \cap \mathbf{A}$ is done by taking adjoints.

To treat (D), suppose that $T_a \in C_{1\cdot} \cap \mathbf{A}$. Then $\operatorname{Ker}(T_a - \lambda) = (0)$ for each λ in $\mathbf{D}$, and hence $\mathfrak{F}'_+(T_a) = \emptyset$. Thus $\mathbf{D} \backslash \mathfrak{F}'_+(T_a)$ is dominating for $\mathbf{T}$, so T_a satisfies condition (h) of Theorem 6.2, and therefore all of the conditions of Theorem 6.2. That T is reflexive now follows from (B) above. The case in which $T_a \in C_{\cdot 1} \cap \mathbf{A}$ is dealt with by taking adjoints.

To treat (E), we note that the hyponormality of T implies that of $T_a - \lambda$ for each λ in $\mathbf{D}$, and this in turn shows that $\operatorname{Ker}(T_a - \lambda) \subset \operatorname{Ker}(T_a - \lambda)^*$ for each λ in $\mathbf{D}$. Thus if $T_a - \lambda$ is semi-Fredholm for some such λ, then $i(T_a - \lambda) \leq 0$, so $\mathfrak{F}'_+(T_a) = \emptyset$, and that T is reflexive follows as in (D). Hence the theorem is proved. □

As a first corollary of Theorem 7.2 we have the following improvement of the main theorem of [13] that was noted in Section 1.

COROLLARY 7.3. *If T is a contraction in $\mathcal{L}(\mathcal{H})$ such that $\sigma(T) \supset \mathbf{T}$, then either T is reflexive or T has a nontrivial hyperinvariant subspace.*

Proof. One knows from a theorem of Apostol [0, Thm. 2.2] that if $T \notin \mathbf{A}$ then T has a nontrivial hyperinvariant subspace. Thus we may suppose that $T \in \mathbf{A}$, and if $T \notin C_{0\cdot} \cup C_{1\cdot}$ then $\{x \in \mathcal{H} : \|T^n x\| \to 0\}$ is a nontrivial hyperinvariant subspace for T. Thus we may suppose that $T \in \mathbf{A} \cap (C_{0\cdot} \cup C_{1\cdot})$, and that T is reflexive now follows from (C) and (D) of Theorem 7.2. □

It is worth pointing out that this corollary cannot easily be generalized.

EXAMPLE 7.4. Let W be a singular unitary operator such that $\sigma(W) = \mathbf{T}$, and let T be any nonreflexive and absolutely continuous contraction. Then $\sigma(T \oplus W) \supset \mathbf{T}$, but by Lemma 7.1, $T \oplus W$ fails to be reflexive. Thus it is easy to find a nonreflexive contraction whose spectrum contains the unit circle.

As additional corollaries of Theorem 7.2 one recovers easily many known results on reflexivity from [2], [3], [8], [11], [25], [26], and [27]. In particular, we obtain the following.

COROLLARY 7.5 (Bercovici–Takahashi–Wu). *If T, S, and X are operators in $\mathcal{L}(\mathcal{H})$ such that T is a contraction, S is a unilateral shift, $X \neq 0$, and $XT = SX$, then T is reflexive.*

Proof. Let $\lambda \to e_\lambda$ be a nonzero conjugate analytic function on $\mathbf{D}$ such that $S^* e_\lambda = \bar{\lambda} e_\lambda$ for λ in $\mathbf{D}$ and $\bigvee_{\lambda \in \mathbf{D}} e_\lambda = \mathcal{H}$. Then $T^*(X^* e_\lambda) = \bar{\lambda}(X^* e_\lambda)$ for

λ in $\mathbf{D}$ and the function $\lambda \to X^* e_\lambda$ is clearly a nonzero conjugate analytic function on $\mathbf{D}$. Since the zeros of such a function are isolated, it follows that the point spectrum $\sigma_p(T^*) \cap \mathbf{D}$ is dominating for $\mathbf{T}$, so $T^* \in \mathbf{A}$, and T is reflexive by (A) of Theorem 7.2 (using (h) of Theorem 6.2). □

COROLLARY 7.6 ([3], [2]). *Every operator in the class* $\mathbf{A}_{\aleph_0}$ *is reflexive.*

Proof. The class $\mathbf{A}_{\aleph_0}$ is a subset of $\mathbf{A}_{1,\aleph_0}$. □

We close this paper with two conjectures that result from Theorem 7.2.

CONJECTURE 7.7. Every operator in $\mathbf{A}$ is reflexive.

CONJECTURE 7.8. Every hyponormal operator is reflexive.

The results in this paper were presented at the meeting of the American Mathematical Society in Lincoln, Nebraska in October, 1987, and were announced in [16]. These results were obtained in the spring of 1987 while the second and third authors were visiting the Department of Mathematics of the University of Bordeaux I, and we wish to express our gratitude to this department for its kind hospitality to the authors.

Added in proof. Very recently, the first author and Scott Brown have shown that every contraction operator in $\mathbf{A}$ is reflexive, thus establishing Conjecture 7.7 (cf. "Toute contraction à calcul isométrique est reflexive," C. R. Acad. Sci. Paris 307 (1988), Series I, 185–188).

References

0. C. Apostol, *Ultraweakly closed operator algebras,* J. Operator Theory 2 (1979), 49–61.
1. H. Bercovici, *A contribution to the structure theory of the class* **A**, J. Funct. Anal., to appear.
2. H. Bercovici, B. Chevreau, C. Foiaş, and C. Pearcy, *Dilation theory and systems of simultaneous equations in the predual of an operator algebra, II,* Math. Z. 189 (1984), 97–103.
3. H. Bercovici, C. Foiaş, J. Langsam, and C. Pearcy, (BCP)-*operators are reflexive,* Michigan Math. J. 29 (1982), 371–379.
4. H. Bercovici, C. Foiaş, and C. Pearcy, *Dilation theory and systems of simultaneous equations in the predual of an operator algebra, I,* Michigan Math. J. 30 (1983), 335–354.
5. ———, *Dual algebras with applications to invariant subspaces and dilation theory,* CBMS Regional Conference Series in Math. No. 56, Amer. Math. Soc., Providence, R.I., 1985.
6. ———, *Factoring trace-class operator valued functions with applications to the class* $\mathbf{A}_{\aleph_0}$, J. Operator Theory 14 (1985), 351–389.
7. ———, *On the reflexivity of algebras and linear spaces of operators,* Michigan Math. J. 33 (1986), 119–126.
8. H. Bercovici and K. Takahashi, *On the reflexivity of contractions on Hilbert space,* J. London Math. Soc. (2) 32 (1985), 149–156.

9. S. Brown, *Some invariant subspaces for subnormal operators,* Integral Equations Operator Theory 1 (1978), 310–333.
10. ———, *Contractions with spectral boundary,* Integral Equations Operator Theory 11 (1988), 49–63.
11. ———, *Full analytic subspaces for contractions with rich spectrum,* Pacific J. Math. 132 (1988), 1–10.
12. S. Brown, B. Chevreau, and C. Pearcy, *Contractions with rich spectrum have invariant subspaces,* J. Operator Theory 1 (1979), 123–136.
13. ———, *On the structure of contraction operators, II,* J. Funct. Anal. 76 (1988), 30–55.
14. B. Chevreau, *Algèbres duales et sous-espaces invariants,* Thèse d'Etat, Université de Bordeaux I, 1987.
15. ———, *Sur les contractions à calcul fonctionnel isométrique, I,* C. R. Acad. Sci. Paris Sér. A 305 (1987), 537–540.
16. B. Chevreau, G. Exner, and C. Pearcy, *Structure theory and reflexivity of contraction operators,* Bull. Amer. Math. Soc. 19 (1988), 299–301.
17. B. Chevreau and C. Pearcy, *On the structure of contraction operators with applications to invariant subspaces,* J. Funct. Anal. 67 (1986), 360–379.
18. ———, *On the structure of contraction operators, I,* J. Funct. Anal. 76 (1988), 1–29.
19. D. Hadwin and E. Nordgren, *Subalgebras of reflexive algebras,* J. Operator Theory 7 (1982), 3–23.
20. R. Olin and J. Thompson, *Algebras of subnormal operators,* J. Funct. Anal. 37 (1980), 271–301.
21. H. Radjavi and P. Rosenthal, *On invariant subspaces and reflexive algebras,* Amer. J. Math. 91 (1969), 683–692.
22. G. Robel, *On the structure of* (BCP)*-operators and related algebras, I and II,* J. Operator Theory 12 (1984), 23–45 and 235–245.
23. P. Sullivan, Thesis, Univ. of Michigan, 1986.
24. B. Sz.-Nagy and C. Foiaş, *Harmonic analysis of operators on Hilbert space,* North-Holland, Amsterdam, 1970.
25. K. Takahashi, *On the reflexivity of contractions with isometric parts,* Acta. Sci. Math. (Szeged), to appear.
26. P. Wu, *Contractions with a unilateral shift summand are reflexive,* Integral Equations Operator Theory 7 (1984), 899–904.
27. ———, *On the reflexivity of* $C_{1\cdot}$ *contractions and weak contractions,* J. Operator Theory 8 (1982), 209–217.

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