A New Proof that a Mapping Is Regular If and Only If It Is Almost Periodic

SAM W. YOUNG

I. Introduction and Definitions

Throughout this paper, X will denote a compact metric space with metric d. The uniform metric on the space of self mappings on X will be denoted by ρ.

A homeomorphism f of a compact metric space onto itself is *regular* if and only if the family of iterates $\{\dots, f^{-2}, f^{-1}, f^0, f^1, f^2, \dots\}$ is an equicontinuous family. A mapping f of a compact metric space onto itself is *almost periodic* if and only if the following is true: If $\epsilon > 0$, then there exists a positive integer N such that every block of N consecutive positive integers contains an integer n such that $d(x, f^n(x)) < \epsilon$ for all $x \in X$ $(\rho(f^n, \mathrm{id}) < \epsilon)$. In case f is a homeomorphism, then the negative iterates are included as well. See Theorem F below.

Motivated by a desire to understand the mechanics of regular mappings, we give a self-contained proof of the theorem in the title of this paper. For an older proof see [1]. It is hoped that the present-day interest in topological dynamics will be served by a fresh proof of this useful old theorem. A clue to the argument is contained in the appendix to [1, p. 146] and is incorporated here into Lemma 1. We will also make use of the following theorem, one proof of which can be found in [3, Lemma 2.2]. But in the spirit of self-containment, we give an outline of the proof here.

THEOREM F. *If f is a mapping of a compact metric space onto itself whose positive iterates form an equicontinuous family (positively regular) then f is a regular homeomorphism (which will henceforth be referred to simply as a regular mapping).*

Outline of proof. We begin by assuming the following proposition, the proof of which is straightforward:

(∗) If the positive iterates of f form an equicontinuous family then either f is a homeomorphism or there exists $\delta > 0$ such that $\rho(f^i, \mathrm{id}) \geq \delta$ for $i = 1, 2, 3, \dots$.

Received August 27, 1987. Revision received September 1, 1988.
Michigan Math. J. 36 (1989).

Using the metric ρ and the operation of functional composition, the Ascoli theorem gives us that $\Gamma(f)=\text{cl}\{f, f^2, f^3, \ldots\}$ is a compact semigroup. It follows from a theorem of Numakura [2, Lemma 3, p. 102] that some subsequence of $\{f, f^2, f^3, \ldots\}$ converges to an idempotent $e=e^2$. And it follows that $e=\text{id}$, and so by (*) we have that f is a homeomorphism. The iterates of f, positive and negative, will now form an equicontinuous family and so f is a regular mapping.

II. Statement and Proof of the Theorem

Notation. Since only one function is involved here, we will replace the expression $f^n(x)$ by x^n.

THEOREM. *If f is a mapping of the compact metric space X onto itself, then f is regular if and only if f is almost periodic.*

Proof that almost periodic implies regular. We will actually show that if f is almost periodic relative to its positive iterates then f is positively regular, and so (by Theorem F) f is a regular mapping.

Suppose $\epsilon > 0$. There exists a positive integer K such that each block of K integers contains an integer n such that $\rho(f^n, \text{id}) < \epsilon/3$. Since the mappings $f^0, f^1, \ldots, f^{K-1}$ are equiuniformly continuous, there exists $\delta > 0$ such that if $x, y \in X$ and $d(x, y) < \delta$ then $d(x^i, y^i) < \epsilon/3$, $0 \le i \le k-1$.

Now suppose that m is a positive integer and that $x, y \in X$ with $d(x, y) < \delta$. There exist nonnegative integers n and p such that $m=n+p$, $\rho(f^n, \text{id}) < \epsilon/3$, and $0 \le p \le k-1$. We have that

$$\begin{aligned} d(x^m, y^m) &= d((x^p)^n, (y^p)^n) \\ &\le d((x^p)^n, x^p) + d(x^p, y^p) + d((y^p), (y^p)^n) \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon. \end{aligned}$$

It has now been shown that the mappings $\{f, f^2, f^3, \ldots\}$ form an equicontinuous family. □

Proof that regular implies almost periodic. This proof will be broken into four lemmas. But first we need some definitions. Given a mapping f of a space X onto itself and $x \in X$, $O(x)=\{x^0, x^1, x^2, \ldots\}$ denotes the *orbit of* x and $\bar{O}(x)$ the *orbit closure of* x. In case f is a homeomorphism, $O(x)=\{\ldots, x^{-2}, x^{-1}, x^0, x^1, x^2, \ldots\}$. Given a mapping f of a space X onto itself, a set $K \subset X$ is called *minimal* if and only if K is minimal with respect to the properties of being closed and mapped into itself by f. If f is a mapping of a space X onto itself and $x \in X$, then f is *almost periodic at* x if and only if the following is true: If $\epsilon > 0$ then there exists a positive integer N such that every block of N integers contains an integer n with $d(x, x^n) < \epsilon$. □

LEMMA 1. *If f maps the compact metric space X onto itself and $x \in X$, then f is almost periodic at x if and only if $\bar{O}(x)$ is a minimal set.*

Proof of $\Rightarrow$. Suppose that f is almost periodic at x, but that there exists a closed proper subset K of $\bar{O}(x)$ such that $f(K) \subset K$. There is a point x^j which is not in K. Let U be an open set containing x^j such that $\bar{U} \cap K = \emptyset$.

Since f is almost periodic at x, it follows that f is almost periodic at x^j; thus there exists a positive integer N such that every block of N integers contains an integer i such that $x^{j+i} \in U$. Also, there exists a sequence of iterates $x^{j+n_1}, x^{j+n_2}, x^{j+n_3}, \dots$ converging to a point of K. So we then consider the following matrix:

$$\begin{array}{lll} x^{j+n_1}, & x^{j+n_2}, & x^{j+n_3}, \dots \\ x^{j+n_1+1}, & x^{j+n_2+1}, & x^{j+n_3+1}, \dots \\ x^{j+n_1+2}, & x^{j+n_2+2}, & x^{j+n_3+2}, \dots \\ \vdots & \vdots & \vdots \\ x^{j+n_1+N}, & x^{j+n_2+N}, & x^{j+n_3+N}, \dots . \end{array}$$

Since $f(K) \subset K$ and the first row converges to a point of K, each row converges to a point of K. But each column contains a point of U and therefore some row contains infinitely many points of U. A sequence that converges to a point of K cannot contain infinitely many points of U because $\bar{U} \cap K = \emptyset$. We have a contradiction. □

Proof of $\Leftarrow$. Suppose that $\bar{O}(x)$ is minimal, but that f is not almost periodic at x. If f is not almost periodic at x, then there exists an open set U containing x and arbitrarily long blocks of integers i such that $x^i \notin U$. Thus there is a 2-block, a 3-block, a 4-block, ... of integers such that if i an integer in any of these blocks then $x^i \notin U$.

Let G be the collection of all increasing sequences of positive integers $\{n_1, n_2, n_3, \dots\}$ such that for each $i \geq 1$, $\{x^{n_i}, x^{n_i+1}, \dots, x^{n_i+i}\} \cap U = \emptyset$. The collection G is not empty since n_1 could be any integer which is the first integer of a 2-block, n_2 any larger integer which is the first integer of a 3-block, and so on. Note that any infinite subsequence of a member of G is a member of G. And note also that if $\{n_1, n_2, n_3, \dots\}$ is in G, then so is $\{n_2+1, n_3+1, n_4+1, \dots\}$.

Let K be the set of all points P such that for some $\{n_1, n_2, n_3, \dots\}$ in G,

$$x^{n_1}, x^{n_2}, x^{n_3}, \dots \to P.$$

We know that K is not empty, and that if $x^{n_1}, x^{n_2}, x^{n_3}, \dots \to P \in K$ then $x^{n_2+1}, x^{n_3+1}, x^{n_4+1}, \dots \to f(P)$ and so $f(P) \in K$. It follows that $f(\bar{K}) \subset \bar{K}$ and $\bar{K} \neq \bar{O}(x)$ since $U \cap \bar{K} = \emptyset$. This contradicts the assumption that $\bar{O}(x)$ is minimal. □

LEMMA 2. *If f is a regular mapping of the compact metric space X onto itself, then each orbit closure is a minimal set and $\{\bar{O}(x) : x \in X\}$ is a continuous collection.*

Proof. Let μ denote the Hausdorff metric on the closed subsets of X. Suppose $\epsilon > 0$. Then there exists $\delta > 0$ such that if $s, t \in X$ and $d(x, t) < \delta$, then $d(s^n, t^n) < \epsilon$ for $n = 0, \pm 1, \pm 2, \pm 3, \dots$.

Suppose that $x, y \in X$ and are such that some point of $\bar{O}(x)$ is at a distance less than δ from some point of $\bar{O}(y)$. Then there exist integers m and n such that $d(x^n, y^m) < \delta$. We have that, for all integers i, $d(x^{n+i}, y^{m+i}) < \epsilon$. It follows that $\mu(\bar{O}(x), \bar{O}(y)) < \epsilon$. So we have established the following:

(**) If $\epsilon > 0$, then there exists $\delta > 0$ such that if $S(\bar{O}(x), \delta) \cap \bar{O}(y) \neq \emptyset$ then $\mu(\bar{O}(x), \bar{O}(y)) < \epsilon$, where $S(\bar{O}(x), \delta)$ denotes the δ-neighborhood of $\bar{O}(x)$ with respect to the metric d.

For the first part of the lemma, suppose to the contrary that for some x, $\bar{O}(x)$ is not minimal. Then there exists $y \in \bar{O}(x)$ such that $\bar{O}(y) \subsetneq \bar{O}(x)$ and (**) leads to a contradiction. For the second part, (**) characterizes a continuous collection and so $\{\bar{O}(x), x \in X\}$ is a continuous collection of disjoint closed minimal sets. □

We note here that Lemmas 1 and 2 imply that a regular mapping on a compact metric space X is almost periodic at each point of X. The next two lemmas show how the crucial inequality $d(x, x^n) < \epsilon$ spreads like a contagious disease through each orbit closure and then through μ-neighborhoods of the continuous collection of orbit closures.

LEMMA 3 (Vertical contagion). *Suppose that f is a regular mapping of the compact metric space X onto itself, $x \in X$ and $\epsilon > 0$. Then there exists $\delta > 0$ such that if n is an integer with $d(x, x^n) < \delta$ then, for all $y \in \bar{O}(x)$, $d(y, y^n) < \epsilon$.*

Proof. Suppose $x \in X$ and $\epsilon > 0$. There exists $\delta > 0$ such that $\epsilon/3 > \delta > 0$, and if $s, t \in X$ and $d(s, t) < \delta$ then $d(s^i, t^i) < \epsilon/3$ for all integers i. Let n be an integer such that $d(x, x^n) < \delta$. Let $y \in \bar{O}(x)$ and let m be an integer such that $d(y, x^m) < \delta$.

We now have that $d(x^m, x^{m+n}) < \epsilon/3$ and $d(y^n, x^{m+n}) < \epsilon/3$. Thus

$$d(y, y^n) \leq d(y, x^m) + d(x^m, x^{m+n}) + d(x^{m+n}, y^n) < \delta + \epsilon/3 + \epsilon/3 < \epsilon. \quad \square$$

We should observe here that Lemmas 1, 2, and 3 have established that if f is regular then $f \mid \bar{O}(x)$ is almost periodic on $\bar{O}(x)$. It is also important to note that if $t \in \bar{O}(x)$ then $\bar{O}(t) = \bar{O}(x)$.

LEMMA 4 (Horizontal contagion). *Suppose that f is a regular mapping of the compact metric space X onto itself, $x \in X$ and $\epsilon > 0$. Then there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that, if n is an integer with $d(t, t^n) < \delta_1$ for some $t \in \bar{O}(x)$, then $d(y, y^n) < \epsilon$ for all y such that $\mu(\bar{O}(x), \bar{O}(y)) < \delta_2$.*

Proof. We are given $x \in X$ and $\epsilon > 0$. From Lemma 3, there exists $\epsilon/3 > \delta_1 > 0$ such that, if $t \in \bar{O}(x)$ and n is an integer such that $d(t, t^n) < \delta_1$, then $d(s, s^n) < \epsilon/3$ for all $s \in \bar{O}(x)$. Let $\epsilon/3 > \delta_2 > 0$ be such that if $u, v \in X$ and $d(u, v) < \delta_2$ then $d(u^i, v^i) < \epsilon/3$ for all i. Let $y \in X$ be such that

$$\mu(\bar{O}(x), \bar{O}(y)) < \delta_2.$$

There exists $s \in \bar{O}(x)$ such that $d(y, s) < \delta_2$.

If n is an integer such that $d(t, t^n) < \delta_1$ for some $t \in \bar{O}(x)$, then $d(y, y^n) \leq d(y, s) + d(s, s^n) + d(s^n, y^n) < \delta_2 + \epsilon/3 + \epsilon/3 < \epsilon$, and this completes the proof. □

Now we finish the proof that regular implies almost periodic. Suppose that f is a regular mapping of a compact metric space X onto itself, and that $\epsilon > 0$. If $x \in X$ then there exist $\delta_1(x) > 0$ and $\delta_2(x) > 0$, according to Lemma 4. The $\delta_2(x)$-neighborhoods cover the decomposition space and so there exists a finite subcover. Hence there exists a finite set $\{x_1, x_2, \ldots, x_p\} \subset X$ which provides such a subcover. Let $\delta_1' = \text{Min}\{\delta_1(x_1), \delta_1(x_2), \ldots, \delta_1(x_p)\}$.

Let $Z = \bar{O}(x_1) \times \bar{O}(x_2) \times \cdots \times \bar{O}(x_p)$ and $F = f \mid \bar{O}(x_1) \times f \mid \bar{O}(x_2) \times \cdots \times f \mid \bar{O}(x_p)$. Clearly each restriction $f \mid \bar{O}(x_i)$ is regular and the product of regular mappings is regular. Therefore F is a regular mapping on Z and, as was pointed out after the proof of Lemma 2, F is pointwise almost periodic on Z. Let $\hat{x} = (x_1, x_2, \ldots, x_p) \in Z$.

Let $\delta_1'' > 0$ be such that, if $d_z((s_1, s_2, \ldots, s_p), (t_1, t_2, \ldots, t_p)) < \delta_1''$, then $d(s_i, t_i) < \delta_1'$ for $i = 1, 2, \ldots, p$. Thus there exists a positive integer N such that every block of N integers contains an integer n such that $d_z(\hat{x}, F^n(\hat{x})) < \delta_1''$.

Choose any $x \in X$. There exists an integer q such that $\mu(\bar{O}(x_q), \bar{O}(x)) < \delta_2(x_q)$. The inequality $d_z(x, F^n(x)) < \delta_1''$ implies $d(x_q, x_q^n) < \delta_1' \leq \delta_1(x_q)$. Applying Lemma 4 we obtain $d(x, x^n) < \epsilon$. So the desired inequality has spread over the entire space X, and the proof is complete. □

References

1. W. H. Gottschalk and G. A. Hedlund, *Topological dynamics,* Amer. Math. Soc. Colloq. Publ., Vol. 36, Amer. Math. Soc., Providence, R.I., 1956.
2. K. Numakura, *On bicompact semigroups,* Math. J. Okayama Univ. 1 (1952), 99–108.
3. S. W. Young, *Concerning the function equation $f(g) = f$, regular mappings and periodic mappings,* Proc. Amer. Math. Soc. 97 (1986), 367–371.

Division of Mathematics
Auburn University
Auburn, AL 36849