PROPER HOLOMORPHIC MAPS BETWEEN BALLS OF DIFFERENT DIMENSIONS

John P. D'Angelo

Introduction. The proper holomorphic mappings from the unit ball in **C** to itself are exactly the finite Blaschke products. The higher-dimensional result seems quite different. Alexander [1] proved that a proper holomorphic map of the unit ball B_n in $\mathbf{C}^n$ to itself is actually biholomorphic. This result is actually a special case of a general result on proper maps of smoothly bounded strongly pseudoconvex domains, or even certain weakly pseudoconvex domains of finite type. It is not hard to see that there can be no proper maps from B_n to B_m for m less than n, so it remains to classify them for m larger than n. The boundary behavior of such maps appears to get worse as the codimension $(m-n)$ increases; see Low [10]. Thus it is reasonable to ask to classify only those proper maps that are sufficiently regular at the boundary (the unit sphere). Webster [14] has proved that a proper holomorphic map from B_n to B_{n+1} that is three times continuously differentiable at the boundary must be equivalent to a linear imbedding if n is at least 3. Here equivalence is up to automorphisms of the two balls. Faran [7] proved that a proper map that extends holomorphically past the boundary must be equivalent to a linear imbedding when m is at most $2n-2$. He also gave a complete analysis [6] of the case $n=2$, $m=3$, assuming three continuous derivatives up to the boundary. He proved that there are exactly four equivalence classes; there is a monomial map in each class. Cima and Suffridge [4] showed that, for some of these results, one needs only two continuous derivatives at the boundary. Forstneric [8] then showed that when m is at most $2n-2$, proper maps with $m-n+1$ continuous derivatives on the sphere extend holomorphically past the sphere. By Faran's result above, such maps are equivalent to linear ones. Forstneric has also some interesting results that apply when the range is any strongly pseudoconvex domain, and Rudin [12] has analyzed the homogeneous case.

The purpose of the present paper is to investigate the proper maps between balls of different dimensions by proving several results about polynomial proper maps, by exhibiting a one-parameter family of inequivalent proper maps from the n-ball to the $2n$-ball, and by formulating a conjecture that unifies many of these results. We also list (omitting the tedious proof) all the proper monomial maps from the 2-ball to the 4-ball, and show how to obtain polynomial proper maps that are not equivalent to any monomial map.

Perhaps the main result of the present paper is the statement that, if two polynomial proper maps are equivalent up to automorphisms of the balls, and they preserve the origin, then they are actually unitarily equivalent. A simple example shows that the hypothesis on preserving the origin is necessary. From this we

Received April 28, 1987. Revision received July 30, 1987.
The author was partially supported by the NSF.
Michigan Math. J. 35 (1988).

derive easily the fact that the maps in our one-parameter family are mutually inequivalent. Another corollary is the existence of polynomial proper mappings that are not equivalent to monomial maps. One of the tools used in the proof is the investigation of the properties of the quotient $(\|f\|^2-1)/(\|z\|^2-1)$ and its complexified form $(\langle f(z), f(w)\rangle -1)/(\langle z, w\rangle -1)$. These expressions satisfy the chain rule, even though no limit has been taken.

The author thanks Joe Cima for his lecture on proper holomorphic maps at the meeting at Penn State in March, 1986, and acknowledges helpful discussions with Jim Faran.

I. Statements and proofs of the theorems. We let $O(n,m)$ denote the set of proper holomorphic maps from the unit ball B_n in $\mathbf{C}^n$ to B_m. We write $P(n,m)$ for the subset of $O(n,m)$ consisting of maps that extend holomorphically past the boundary. We say that f, g are *spherically equivalent* if there are automorphisms of the domain and range balls, φ and ψ, for which $f\varphi=\psi g$. We write $O^*(n,m)$ and $P^*(n,m)$ for the set of equivalence classes. The results of the introduction include the following:

$O^*(1,1)$ is infinite;
$O^*(n,n)$ has one element for $n\geq 2$;
$P^*(n,m)$ has one element for $n\leq m\leq 2n-2$;
$P^*(2,3)$ has exactly four elements;
$O(n,m)$ and hence $O^*(n,m)$ is empty for $m<n$.

We will prove in this paper that $P^*(n,m)$ and hence $O^*(n,m)$ are infinite for $m\geq 2n$.

If f and g are proper holomorphic maps from the same domain ball B_n to (perhaps different) range balls B_m and B_k, and if $e^{i\theta}$ is a complex number of modulus 1, then we define the θ-juxtaposition of f and g, written $J_\theta(f,g)$, to be the proper map from B_n to B_{m+k} given by

1. $$J_\theta(f,g)(z)=(\cos(\theta)f(z), \sin(\theta)g(z)).$$

We define the extend map E from B_n to B_{2n-1} by

2.1 $$E(z_1,\dots,z_n)=(z_1,\dots,z_{n-1}, z_1z_n, z_2z_n,\dots,z_n^2).$$

More generally, if f is any proper map from B_n to B_m, we write Ef for the proper map to B_{n+m-1};

2.2 $$Ef(z_1,\dots,z_n)=(f_1,\dots,f_{m-1}, z_1f_m, z_2f_m,\dots,z_nf_m).$$

Note that E and Ef are proper. This is easiest to see by noting the fact that a map H is proper if and only if $H(z_\nu)$ tends to the boundary whenever z_ν does. Hence a nonconstant map that extends holomorphically past the sphere is proper if and only if its norm is 1 there.

We use this fact to analyze the notion of spherical equivalence. Given any proper map $f\colon B_n$ to B_m that extends holomorphically past the sphere S_{2n-1}, we can find a unique real analytic function Q_f (analytic also in a neighborhood of the closed ball) for which we have

3.1 $$\|f(z)\|^2-1=(\|z\|^2-1)Q_f(z).$$

We will use this formula in several cases, including when f is an automorphism. More generally, we define the complexified form of 3.1 as follows:

3.2 $$Q_f(z,w)=(\langle f(z),f(w)\rangle-1)/(\langle z,w\rangle-1).$$

See Lemmas 6, 7, and 8 below for the properties of Q_f.

We recall the following description of the automorphism group of the ball (see Rudin [11]). Every automorphism is of the form $U\zeta_a$, where U is unitary and ζ_a is a linear fractional transformation of the form $\zeta_a(z)=(L_a z+a)/(1-\langle z,a\rangle)$. Here a is a point in the ball, and L_a has the following particular form:

5. $$L_a(z)=-(sz+(\langle z,a\rangle a/(s+1))),\quad \text{where } s^2=1-\|a\|^2.$$

From formula 5, we derive the following formula. We write ξ for ξ_a.

6. LEMMA. $Q_\xi(z,w)=(1-\|a\|^2)/((1-\langle z,a\rangle)(1-\langle a,w\rangle))$.

Proof. A proof of this formula appears in [11, p. 27], but the reader can easily work out the algebra using formula 5 above. □

Now we study the formal properties of the quotient Q_f. We gather these together in the following lemma and proposition.

7. LEMMA (Chain Rule). *If f,g are proper and $h=g\circ f$, then*

7.1 $$Q_h(z,w)=Q_g(f(z),f(w))(Q_f(z,w)).$$

If all the maps are proper and $f\xi=\psi g$, then

7.2 $$(Q_g(z,w))(Q_\psi(g(z),g(w))=(Q_\xi(z,w))(Q_f(\xi(z),\xi(w))).$$

Conversely, if 7.2 holds then there is a unitary matrix U for which $Uf\xi=\psi g$.

Proof. Calculation using formula 3.2 shows that 7.1 is equivalent to the identity of $\langle gf(z),gf(w)\rangle$ and $\langle h(z),h(w)\rangle$. The statement 7.2 is then clear, while the converse follows (after putting $z=w$) from the often-used fact [5] that norm squared equalities between holomorphic functions imply the existence of such unitary matrices. □

Some other properties of $Q_f(z,w)$ are in the next proposition.

8. PROPOSITION. *Let f be a proper map between balls. The quotient Q_f satisfies the following properties:*

8.1 $0<Q_f(z,z)<\infty$ *for* $\|z\|<1$.

8.2 Q_f *is a polynomial if and only if f is.*

8.3 $Q_{Ef}(z,w)=Q_f(z,w)+f_m(z)f_m(w)$. *(Here Ef is the extend map defined in equation* 2.2.*)*

8.4 Q_f *is* 1 *if and only if f is unitary.*

8.5 $Q_{J\theta(f,g)}(z,w)=\cos^2\theta Q_f(z,w)+\sin^2\theta Q_g(z,w)$.

8.6 $Q_{J\theta}(\mathrm{id}, E.\mathrm{id})(z,w)=1+\sin^2\theta z_n w_n$.

8.7 (Schwarz lemma) *For z, w in the ball, $|Q_f(z,w)|^2 \le Q_f(z,z)Q_f(w,w)$.*

Proof. All the results are simple computations except for 8.7 and 8.2. Since we do not use 8.7, we refer the reader to [11]. To see 8.2, note that if Q_f is a polynomial then so is f. To verify the converse, suppose that f is a polynomial. We use the complexified equation 3.2 that $Q_f(z,w)(1-\langle z,w\rangle)=1-\langle f(z),f(w)\rangle$. We know that Q_f is analytic in z and conjugate analytic in w. For any fixed w, we will find an integer $N=N(w)$ so that $(\partial/\partial z)^\alpha Q_f(z,w)$ vanishes identically in z for $|\alpha|=N$. By the Baire category theorem, it then follows that there is an open set of w's for which $(\partial/\partial z)^\alpha Q_f(z,w)$ vanishes identically with α as above. Since Q_f is also analytic in the w variable, this holds for all w in $\mathbf{C}^n$. By symmetry in z and w, Q_f is then a polynomial in each variable. To establish the existence of $N(w)$, note first that it is clear for $w=0$. Otherwise, we suppose without loss of generality that w_1 does not vanish. We make the substitution that $z_1^*=\langle z,w\rangle-1$ and $z_k^*=z_k$ for $k>1$. This does not change the fact that f is a polynomial in the z^* variables, but makes Q into $(\langle f(z),f(w)\rangle-1)/z_1^*$. Now it is clear that if the quotient of a polynomial by a coordinate is analytic, the quotient is still a polynomial. Thus such an integer $N(w)$ exists for each w, and the result follows. □

We now combine these lemmas to prove our first theorem.

9. THEOREM. *Suppose that $f, g\colon B_n$ to B_m are proper polynomial maps. Suppose that $f(0)=g(0)=0$. If f and g are spherically equivalent, then f and g are actually unitarily equivalent. Thus, if there are automorphisms ξ and ψ for which $g\xi=\psi f$, then ξ and ψ must be unitary maps.*

Proof. From the description of the automorphisms preceding formula 5, we may assume that $\xi=U\xi_a$ and that $\psi=V\xi_b$ for unitary maps U and V, and ξ_a and ξ_b are as in that description. Next we use formula 7.1, 8.4, and apply Lemma 6. We obtain the formula

9.1
$$\begin{aligned}(Q_f(z,w))&(1-\|b\|^2)(1-\langle z,a\rangle)(1-\langle a,w\rangle)\\ &=(Q_g(U\xi_a(z),U\xi_a(w)))(1-\|a\|^2)(1-\langle f(z),b\rangle)(1-\langle b,f(w)).\end{aligned}$$

We set $w=0$ in 9.1, and note that $\langle \xi_a(z),a\rangle-1=(\|a\|^2-1)/(1-\langle z,a\rangle)$. This gives

9.2
$$\begin{aligned}\langle f(z),&f(0)\rangle-1\\ &=(\langle g(U\xi_a(z)),g(Ua)\rangle-1)(1-\langle f(z),b\rangle)(1-\langle b,f(0)\rangle)/(1-\|b\|^2).\end{aligned}$$

In the special case that $f(0)=0$, we see that 9.2 shows that a constant is the product of a polynomial and a rational function. If $g(Ua)$ does not vanish, the inner product $\langle g(U\xi_a(z)),g(Ua)\rangle$ does not vanish at $z=0$, and hence nearby. The denominator of this rational function is $1-\langle z,a\rangle$ to some power, but this does not divide the numerator, because $1-\langle z,a\rangle$ does not divide $a+L_az$. It is impossible for the product of this with a polynomial to be a constant. This shows that $g(Ua)$ must vanish. Putting this into 9.2, and using $f(0)=0$, we get that $\langle f(z),b\rangle$ must be a constant. Since $f(0)=0$, this constant is 0. Put into 9.1 the fact that

$\langle f(z), b\rangle = \langle b, f(w)\rangle = 0$. The left side is then a polynomial in z, while the right side is singular along the variety $1 = \langle z, a\rangle$. This is possible only if this variety is empty, or that $a = 0$. Now we can write $g = U\xi_b f V^*$. From here we cannot conclude that $b = 0$, as example 15.2 below shows. However, if also $g(0) = 0$, we obtain that $0 = U\xi_b f V^*(0) = Ub$. Thus $b = 0$ and $g = UfV^*$, which is the unitary relationship. □

As a corollary of this theorem, we obtain the following result. In formula 10 below, we are taking the θ juxtaposition of the identity map and the extend map defined in equation 2, and composing with a linear transformation so as to map into the smallest possible ball.

10. THEOREM. *$P^*(n, 2n)$ is infinite. Hence $P^*(n, m)$ is infinite for $m \geq 2n$. In fact, the proper maps F_θ, defined by* 11, *are inequivalent for all θ with* $0 \leq \theta \leq \pi/2$.

11. Put $F_\theta(z) = (z_1, \ldots, z_{n-1}, \cos(\theta)z_n, \sin(\theta)z_1 z_n, \ldots, \sin(\theta)z_n^2)$.

Proof. Suppose that F_θ and F_η are equivalent. By Theorem 9, they are related by unitary matrices. In particular, there is a unitary matrix $U = (U_{ij})$ for which $\|F_\theta(z)\|^2 = \|F_\eta(Uz)\|^2$. From this it follows from Lemma 7 that $Q_\theta(z, z) = Q_\eta(Uz, Uz)$. Here we have written θ for F_θ to avoid an extra subscript. Calculation or 8.6 shows that

$$12. \qquad Q_\theta(z, z) = |z_n|^2 \sin^2(\theta) + 1 = |Uz_n|^2 \sin^2(\eta) + 1 = Q_\eta(Uz, Uz).$$

Equating coefficients of each variable shows us that $|U_{kn}| = 0$ for $k < n$, therefore that $|U_{nn}| = 1$ and hence that $\sin^2(\theta) = \sin^2(\eta)$. Because of our restriction on θ and η, we obtain the desired result that $\theta = \eta$. □

13. COROLLARY. *$P^*(n, m)$ is infinite for $m \geq 2n$.*

Proof. Compose the maps F_θ with linear isometric imbeddings into the larger ball. □

14. REMARK. We noted above that the maps F_θ can be obtained by taking the θ juxtaposition of the identity map and the extend map. We compose with a linear map that amounts to identifying the result with a map into the ball of smallest possible dimension. This description amounts to a factorization in the sense of the conjecture of the next section. More generally we can derive other infinite families by other juxtapositions.

15. EXAMPLES. There is exactly one other one-parameter family of proper monomial maps (preserving 0) from the 2-ball to the 4-ball. This is also a θ juxtaposition and can be written as follows. Write (z, w) for (z_1, z_2).

$$15.1 \qquad G_\theta(z, w) = (z^2, (1 + \cos^2(\theta))^{1/2} zw, \cos^2(\theta) w^2, \sin(\theta) w).$$

In the statement of Theorem 9, it would not be enough to assume only that $f(0) = 0$. Suppose, for example, that $g(z) = J_\theta(f, 1)$, where 1 denotes the constant function. Then g will be a proper monomial map that does not preserve the origin, but is equivalent to the map $(f, 0)$. To be more concrete, put

15.2 $$g(z) = -(1-|b|^2)^{1/2}(z_1, z_1 z_2, z_2^2, 0) + (0,0,0,b).$$

Then g is proper, $g(0)$ does not vanish, and g is equivalent to $(z_1, z_1z_2, z_2^2, 0)$ via the map ξ_b, where $b = (0,0,0,b)$.

We now give an explicit example of a polynomial map that preserves the origin, yet is not equivalent to a monomial.

16. THEOREM. *For a generic unitary map L, the proper map f obtained by $f = ELE$ is a polynomial map that is not equivalent to a monomial.*

Note. By $f = ELE$, we mean the following: begin with the map E in definition 2, apply a unitary map L, and apply E to the result. For example, put

16.1 $$f_\theta(z,w) = \\ (z, \cos(\theta)z^2w + \sin(\theta)zw^2, \cos(\theta)zw^2 + \sin(\theta)w^3, -\sin(\theta)zw + \cos(\theta)w^2).$$

Then, for $0 < \theta < \pi/2$, f_θ is not equivalent to a monomial.

Proof. By Theorem 9, if f is equivalent to a monomial map then it is unitarily equivalent to one. It is then a trivial matter of linear algebra to see that this is impossible. We omit the details. □

17. REMARK. Note that 16.1 gives an example of a nonmonomial map from the 2-ball to the 4-ball. In general, for $n > 1$, this idea gives an example from the n-ball to the $(3n-2)$-ball. When $n = 1$, however, E and L commute, so we cannot have a proper polynomial map that is not a monomial in the unit disk.

II. A conjecture. In this section we formulate a conjecture that casts all the results of the introduction into one framework. We try to "factorize" each proper map into simpler ones. We consider, in addition to the extend map and linear maps, the inverse of the extend map. We call this the collapse map. It is not proper, but seems to be necessary for the factorization result. More precisely, we consider the following operations.

1. OPERATIONS. Let f be a proper map between balls. Let φ be an automorphism of the domain, ψ an automorphism of the range. Let U be an isometric linear imbedding of $\mathbf{C}^n$ into some higher-dimensional $\mathbf{C}^m$, and let P denote an orthogonal projection from $\mathbf{C}^m$ to $\mathbf{C}^n$. We define new proper maps as follows:

1.1 $L_\varphi f = \varphi \circ f$;

1.2 $R_\psi f = f \circ \psi$;

1.3 $E_\varphi f = (f_1, \ldots, f_{m-1}, \varphi_1 f_m, \varphi_2 f_m, \ldots, \varphi_n f_m)$;

1.4 $C_\varphi g = h$ if $g = E_\varphi h$;

1.5 $L_U f = U \circ f$;

1.6 $L_P g = P \circ g$.

Note that the operations 1.1, 1.2, 1.3, and 1.5 send f to a new proper map for every f, while 1.4 and 1.6 preserve properness only for certain g. Of course, in 1.3, 1.4, 1.5, and 1.6, the image ball may lie in a different dimension.

2. CONJECTURE. Suppose that f is a proper map from B_n to B_m that is sufficiently regular at the boundary. Then there is a finite list of intermediate balls and operations A_j (each in the above list) so that

2.1 $$f = \Pi A_j(\mathrm{id}).$$

3. EXAMPLE. The proper map $f(z, w) = (z^3, \sqrt{3}zw, w^3)$ can be factorized as follows: Each E is of type 1.3, while each L is of type 1.5 or 1.6, and C is of type 1.4.

$$\begin{array}{rll}
(z,w) \mapsto & (z, zw, w^2) & E \\
& (z, zw, zw^2, w^3) & E \\
& (zw, zw^2, w^3, z) & L \\
& (zw, zw^2, w^3, z^2, zw) & E \\
& (zw, zw^2, w^3, zw, z^2) & L \\
& (zw, zw^2, w^3, zw, z^3, z^2w) & E \\
& (z^3, zw, zw, w^3, z^2w, zw^2) & L \\
& (z^3, zw, zw, w^3, zw) & C \\
& (z^3, \sqrt{3}zw, w^3) & L.
\end{array}$$

Similar factorizations apply to all the maps listed in this paper. In fact the author has proved and will show in a future paper that all monomial proper maps admit such a factorization, and, if one allows slightly more general linear transformations, that all polynomial proper maps do also. It is also easy to see that if f and g have factorizations, then so does $J_\theta(f, g)$.

III. Monomial maps from the 2-ball to the 4-ball. There is a simple algorithm that generates all the monomial maps from B_n to B_k that preserve the origin. This algorithm amounts to recognizing that such monomial maps are in one-to-one correspondence with polynomial maps with at most k terms, with nonnegative coefficients in the n real variables $x_j = |z_j|^2$, that equal 1 on the hyperplane in $\mathbf{R}^n$ defined by $\Sigma x_j = 1$. One can use elementary methods to find all such maps.

We have seen above, however, that there are infinitely many polynomial (nonmonomial) examples that result from intertwining linear maps and the extend map. Nevertheless, the reader might be interested in the list of fifteen maps below. We omit the calculations.

1. THEOREM. *If f is a monomial proper map from B_2 to B_4, then f is equivalent to one of the following:*

1.1 $(z, w, 0, 0)$;

1.2 $(z^2, zw, w, 0)$;

1.3 $(z^2, \sqrt{2}zw, w^2, 0)$;

1.4 $(z^3, \sqrt{3}zw, w^3, 0)$;

1.5 $(z^3, \sqrt{3}z^2w, \sqrt{3}zw^2, w^3)$;

1.6 (z^3, z^2w, zw, w);

1.7 (z^2, z^2w, zw^2, w);
1.8 $(z^2, \sqrt{2}z^2w, \sqrt{2}zw^2, w^2)$;
1.9 $(z^3, \sqrt{3}z^2w, \sqrt{2}zw^2, w^2)$;
1.10 $(z, z^2w, \sqrt{2}zw^2, w^3)$;
1.11 $(z^4, z^3w, \sqrt{3}zw, w^3)$;
1.12 $(z^4, \sqrt{3}z^2w, zw^3, w)$;
1.13 $(z^5, \sqrt{5}z^3w, \sqrt{5}zw^2, w^5)$;
1.14 $(z, \cos(\theta)w, \sin(\theta)zw, \sin(\theta)w^2)$;
1.15 $(z^2, \sqrt{(1+\cos^2(\theta))}zw, \cos(\theta)w^2, \sin(\theta)w)$.

Note that the families of maps in 1.14 and 1.15 are inequivalent as θ runs between 0 and $\pi/2$. Thus there are the four examples that map to the 3-ball, nine new discrete examples, and two one-parameter families.

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Department of Mathematics
University of Illinois
Urbana, IL 61801