A METHOD FOR COMPUTING THE KERNEL OF A MAP OF DIVISOR CLASSES OF LOCAL RINGS IN CHARACTERISTIC $p \neq 0$

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Introduction. Let k be an algebraically closed field of characteristic $p \neq 0$ and $G \in k[x, y]$ be a polynomial in two variables. Affine surfaces in A_k^3 defined by equations of the form $z^{p^m} = G$ were introduced by Zariski in [19]. Samuel, in his 1964 Tata notes [18], uses techniques of Galois descent to calculate the divisor class group of several examples of these surfaces for the case $m = 1$. Blass studies their geometry, where he calls them "Zariski surfaces", in [2]. Baba in [1] and Lang in [13] develop techniques for studying their divisors for the case $m > 1$.

The authors together with D. Joyce describe a programmable process for calculating a numerical invariant that completely determines the class group of the surface $z^p = G$ in [14]. Lang then extends this process in [13] to study the kernel of a map from the divisor class group of the surface $X_{m+1}: z^{p^{m+1}} = G$ to the class group of $X_m: z^{p^m} = G$. Thus this global information used to differentiate between non-isomorphic surfaces is easily obtained. In the study of divisors, local information is often more difficult to acquire.

This article presents an algorithm for obtaining data about the class group of the surface $X_m: z^{p^m} = G$ at a singular point.

For each point $(a, b) \in k^2$ such that $G_x(a, b) = G_y(a, b) = 0$ and each integer $m \geq 0$, there is a unique singular point Q_m on X_m. There also exists a map of divisor class groups $\mathrm{Cl}(\mathcal{O}_{X_{m+1}, Q_{m+1}}) \to \mathrm{Cl}(\mathcal{O}_{X_m, Q_m})$ (see Theorem 4.1). Using "Ganong's formula" (Theorem 3.2) and the "nth order Jacobian derivation" (Theorem 3.1), a technique for determining the kernel of this map is described.

After a few brief preliminaries in Section 1, the divisor class group of the ring of fractions of a Krull domain is studied in Section 2. Section 3 uses Ganong's formula to obtain some tools needed in our calculations.

General facts concerning the class group of X_m at a singularity appear in Section 4. Section 5 closes this paper with some examples and a theorem concerning locally factorial rings defined by polynomials of the form $z^{p^m} = G$.

0. Notation.

(0.1) If A is a Krull ring we denote by $\mathrm{Cl}(A)$ the divisor class group of A.
(0.2) Surface-irreducible, reduced, two-dimensional quasi-projective variety over an algebraically closed field.
(0.3) If E is a surface we denote by $\mathrm{Cl}(E)$ the divisor class group of the coordinate ring of E.
(0.4) k—an algebraically closed field of characteristic $p \neq 0$.
(0.5) k^n—set of all n-tuples of elements in k.

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(0.6) A_k^n—affine n-space over k.
(0.7) Given $G \in k[x, y]$ and $m \geq 0$ an integer, let $X_m \subseteq A_k^3$ be the surface defined by $z^{p^m} = G$.
(0.8) If $F \subseteq A_k^3$ is a surface and Q is a point on F, we let F_Q denote the local ring of F at Q.

1. Preliminaries. Samuel developed the technique of Galois descent in his 1964 Tata notes to study the divisor class group of the kernel of a derivation acting on a Krull domain. Lang applied these techniques in [11] to determine the class group of surfaces defined by equations of the form $z^{p^n} = g(x, y)$, where the ground field is of characteristic $p \neq 0$. Below is a summary of results from Samuel's notes and Lang's article that this paper uses. For the definition of a Krull ring the reader is referred to Samuel's notes or to Fossum's book [6].

THEOREM 1.1. *Let $A \subset B$ be Krull rings. If each height-one prime of B contracts to a prime of height less than or equal to one in A then there is a well-defined group homomorphism $\phi: \mathrm{Cl}(A) \to \mathrm{Cl}(B)$. If B is integral over A or if B is A-flat then this condition is satisfied.* (*See* [18, pp. 19–20] *for details.*)

Let B be a Krull ring of characteristic $p \neq 0$ and let Δ be a derivation of the quotient field of B such that $\Delta(B) \subset B$. Let $K = \ker \Delta$ and $A = B \cap K$. Then A is a Krull ring with B integral over A. Thus by Theorem 1.1 there is a well-defined map $\phi: \mathrm{Cl}(A) \to \mathrm{Cl}(B)$. Set $\mathcal{L} = \{t^{-1}\Delta t : t$ belongs to the quotient field of B and $t^{-1}\Delta t \in B\}$. $\mathcal{L}$ is called the additive group of logarithmic derivatives of Δ. Set $\mathcal{L}' = \{u^{-1}\Delta u : u$ is a unit in $B\}$. Then $\mathcal{L}'$ is a subgroup of $\mathcal{L}$.

THEOREM 1.2.
(a) *There exists a canonical homomorphism $\bar{\phi}: \ker \phi \to \mathcal{L}/\mathcal{L}'$.*
(b) *If L is the quotient field of B and $[L : K] = p$ and $\Delta(B)$ is not contained in any height-one prime of B, then $\bar{\phi}$ is an isomorphism* [18, pp. 63–64].

The map $\bar{\phi}$ is described in the following way. If $Q \in \ker \phi$, then $\phi(Q) = tB$ for some $t \in E$. Then $\bar{\phi}(Q) = t^{-1}\Delta t$. (See [18, pp. 62–63] for details.)

THEOREM 1.3. *If $[L : K] = p$, then*
(a) *there exists an $\alpha \in A$ such that $\Delta^p = \alpha\Delta$ and*
(b) *an element $t \in K$ is equal to Dv/v for some $v \in K$ if and only if $\Delta^{p-1}t - \alpha t = -t^p$* [18, pp. 63–64].

THEOREM 1.4. *Let A be a Krull ring and S a multiplicatively closed subset of A. Then $S^{-1}A$ is an A-flat Krull ring and $\phi: \mathrm{Cl}(A) \to \mathrm{Cl}(S^{-1}A)$ is surjective where* $\ker \phi$ *is generated by the divisor classes of height-one primes that intersect S* [18, p. 21].

THEOREM 1.5. *Let A be a Krull ring and m an ideal in A contained in the Jacobson radical of A. Let $\hat{A}$ be the completion of A. Then $\hat{A}$ is A-flat with $A \subset \hat{A}$. If $\hat{A}$ is a Krull ring then so is A, and $\phi: \mathrm{Cl}(A) \to \mathrm{Cl}(\hat{A})$ is an injection* (*see* [18, p. 23]).

2. Again let B be a Krull ring of characteristic $p \neq 0$ with quotient field L, and let Δ be a derivation on L such that $\Delta(B) \subset B$. Let $K = \ker \Delta$ and $A = B \cap K$. Let T be a multiplicatively closed subset of B and $S = T \cap A$. Then S is a multiplicatively closed subset of A. Also, the rings of fractions $S^{-1}A$ and $T^{-1}B$ are Krull rings by Theorem 1.4, with $T^{-1}B$ integral over $S^{-1}A$. It is easy to see that $S^{-1}A = K \cap T^{-1}B$ and that $\Delta(T^{-1}B) \subset T^{-1}B$. By Theorem 1.1 there is a homomorphism $\psi: \mathrm{Cl}(S^{-1}A) \to \mathrm{Cl}(T^{-1}B)$. Let $\mathcal{L}$ be the group of logarithmic derivatives of Δ in B and $\tilde{\mathcal{L}} = \{v^{-1}\Delta v \in B : v \text{ is a unit in } T^{-1}B\}$. Then $\tilde{\mathcal{L}}$ is a subgroup of $\mathcal{L}$. Then by Theorems 1.1 and 1.4 we have the following commutative diagram of exact sequences of group homomorphisms:

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] \\
 & & \ker \alpha_1 \arrow[r] \arrow[d] & \ker \alpha_2 \arrow[d] \\
0 \arrow[r] & \ker \phi \arrow[r] \arrow[d, "\alpha_0"] & \mathrm{Cl}(A) \arrow[r, "\phi"] \arrow[d, "\alpha_1"] & \mathrm{Cl}(B) \arrow[d, "\alpha_2"] \\
0 \arrow[r] & \ker \psi \arrow[r] & \mathrm{Cl}(S^{-1}A) \arrow[r, "\psi"] \arrow[d] & \mathrm{Cl}(T^{-1}B), \arrow[d] \\
 & & 0 & 0
\end{tikzcd}
\tag{2.0}
$$

where $\alpha_0: \ker \phi \to \ker \psi$ is induced by α_1. Clearly $\alpha_1(\ker \phi) \subseteq \ker \psi$ since the rest of the diagram is commutative.

We have the following analog of Theorem 1.2.

THEOREM 2.1.

(a) *If $\alpha_0: \ker \phi \to \ker \psi$ is surjective then there is a well-defined homomorphism $\bar{\bar{\psi}}: \ker \psi \to \mathcal{L}/\tilde{\mathcal{L}}$.*

(b) *If $[L:K] = p$ and $\Delta(B)$ is not contained in any height-one prime of B whose intersection with T is empty, then $\bar{\bar{\psi}}$ is an isomorphism.*

Proof. (a) By Theorem 1.2(a) there are canonical homomorphisms $\bar{\phi}: \ker \phi \to \mathcal{L}/\mathcal{L}'$ and $\bar{\psi}: \ker \psi \to \mathcal{L}_0/\mathcal{L}_0'$, where $\mathcal{L} = \{t^{-1}\Delta t \in B : t \in L\}$, $\mathcal{L}' = \{u^{-1}\Delta u : u \text{ is a unit in } B\}$, $\mathcal{L}_0 = \{t^{-1}\Delta t \in T^{-1}B : t \in L\}$, and $\mathcal{L}_0' = \{u^{-1}\Delta u : u \text{ is a unit in } T^{-1}B\}$.

Since $\mathcal{L} \supseteq \tilde{\mathcal{L}} \supseteq \mathcal{L}'$ there is a natural homomorphism $\mathcal{L}/\mathcal{L}' \to \mathcal{L}/\tilde{\mathcal{L}}$.

If $D \in \ker \psi$, then there exists $D_1 \in \ker \phi$ with $\alpha_0(D_1) = D$. Now $\bar{\phi}(D_1) = \bar{t}$ for some $t \in \mathcal{L}$, where $\bar{t}$ represents the image of t in $\mathcal{L}/\mathcal{L}'$ (see Theorem 1.2).

Define $\bar{\bar{\psi}}(D) = \tilde{t}$, where $\tilde{t}$ is the image of $\bar{t}$ in $\mathcal{L}/\tilde{\mathcal{L}}$ under the map $\mathcal{L}/\mathcal{L}' \to \mathcal{L}/\tilde{\mathcal{L}}$. To show that $\bar{\bar{\psi}}: \ker \psi \to \mathcal{L}/\tilde{\mathcal{L}}$ is well defined, first of all note that we have a commutative diagram:

$$
\begin{tikzcd}
\ker \phi \arrow[r, "\bar{\phi}"] \arrow[d, "\alpha_0", "\text{surjection}"'] & \mathcal{L}/\mathcal{L}' \arrow[d] \arrow[dr, "\text{surjection}"] & \\
 & & \mathcal{L}/\tilde{\mathcal{L}}\ . \\
\ker \psi \arrow[r, "\bar{\psi}"] & \mathcal{L}_0/\mathcal{L}_0' \arrow[ur, "\text{injection}"'] &
\end{tikzcd}
\tag{2.1.1}
$$

The map $\mathcal{L}/\mathcal{L}' \to \mathcal{L}_0/\mathcal{L}_0'$ is induced by the inclusions $\mathcal{L} \subset \mathcal{L}_0$ and $\mathcal{L}' \subset \mathcal{L}_0'$. Since the kernel of this homomorphism is $\tilde{\mathcal{L}}/\mathcal{L}'$ we obtain an injection $\mathcal{L}/\tilde{\mathcal{L}} \to \mathcal{L}_0/\mathcal{L}_0'$.

Suppose then that $D_2 \in \ker \phi$ and that $\alpha_0(D_2) = D$ also. Then $\bar{\phi}(D_2) = \bar{s}$ for some $s \in \mathcal{L}$. We will show that $s - t \in \tilde{\mathcal{L}}$.

From (2.1.1) the image of $\bar{s} - \bar{t}$ when mapped to $\mathcal{L}_0/\mathcal{L}_0'$ is equal to

$$\bar{\psi}(\alpha_0(D_2) - \alpha_0(D_1)) = \bar{\psi}(D - D) = 0.$$

Therefore $s - t \in \mathcal{L}_0' \cap \mathcal{L} = \tilde{\mathcal{L}}$.

To prove (b), observe that Theorem 1.2(b) implies that $\bar{\psi}$ is an isomorphism. Then the inclusion $\mathcal{L}/\tilde{\mathcal{L}} \to \mathcal{L}_0/\mathcal{L}_0'$ is surjective also. Therefore $\ker \psi \cong \mathcal{L}_0/\mathcal{L}_0' \cong \mathcal{L}/\tilde{\mathcal{L}}$. □

THEOREM 2.2. *If*

(a) $\alpha_2 : \mathrm{Cl}(B) \to \mathrm{Cl}(T^{-1}B)$ *is an isomorphism, or*

(b) $[L:K] = p$ *and each nonprincipal height-one prime* P *of* B *is unramified over* A,

then α_0 *is surjective.*

Proof. (a) Let $D \in \ker \psi$. Let $D_1 \in \mathrm{Cl}(A)$ be such that $\alpha_1(D_1) = D$. Then $\alpha_2\phi(D_1) = \psi\alpha_1(D_1) = \psi(D) = 0$. Since α_2 is an injection, $\phi(D_1) = 0$, so that $D_1 \in \ker \phi$ and $\alpha_0(D_1) = D$.

(b) With D and D_1 as in (a), $\alpha_2\phi(D_1) = 0$ implies that $\phi(D_1) = E$, where $E \in \mathrm{Cl}(B)$ has the form $E = \Sigma n_i P_i$ where the P_i are nonprincipal height-one primes such that $P_i \cap T \neq \phi$. Let $Q_i = P_i \cap A$ for each i, and let $D_2 \in \mathrm{Cl}(A)$ be given by $D_2 = \Sigma n_i Q_i$. For each height-one prime Q in A, $\phi(Q) = e(P:Q) \cdot P$, where P is the unique height-one prime in B lying over Q, namely, $P = \{b \in B : b^p \in Q\}$, and where $e(P:Q)$ is the ramification index of P over Q.

By (b), $\phi(D_2) = E$, and clearly $Q_i \cap S \neq \phi$ for each i. Then $D_1 - D_2 \in \ker \phi$ and $\alpha_0(D_1 - D_2) = D$. □

This theorem uses (2.0) and an argument employed by Hallier in [9, p. 2]. In this theorem B is a Krull ring of characteristic $p \neq 0$, n is a prime ideal in B, and T is the complement of n in B. We assume that all of the conditions in the introduction of this section hold.

THEOREM 2.3. *Let* $I = \Delta(B) \cdot B$ *be the ideal in* B *generated by* $\Delta(B)$. *Suppose that* $a \in A$ *is such that* $\Delta^p = a\Delta$. *If* $I \subset n$ *and* $a \notin n$ *then*

(1) *an element* $t \in \mathcal{L}$ *is in* $\tilde{\mathcal{L}}$ *if and only if* $t \in n$, *and*

(2) *if the conditions of Theorem* 2.1 *hold then the map* $\psi : \mathrm{Cl}(S^{-1}A) \to \mathrm{Cl}(T^{-1}B)$ *is injective if and only if* $\mathcal{L} \subset n$.

Proof. (1) If $t \in \mathcal{L}$, then $t = f^{-1}\Delta f$ for some $f \in L$. Replacing f by an element of $B^p f$ we can assume that $f \in B$. If $f \notin n$ then f is a unit in B_n and $t \in \tilde{\mathcal{L}}$. If $f \in n$, then by induction $(af)^{-1}\Delta^j(f) \in nB_n$ for all positive integers j. Let $u = -1 + (af)^{-1}\Delta^{p-1}f$. Then u is a unit in B_n and $\Delta(u^{-1})/u^{-1} = f^{-1}\Delta f = t$. Hence $t \in \tilde{\mathcal{L}}$.

Conversely, $t \in \tilde{\mathcal{L}}$ implies that $t = u^{-1}\Delta u$ for u a unit in B_n. Since $I \subset n$, then $t \in nB_n \cap B = n$. (2) is now an immediate consequence of (1) and (2.0). □

3. Applications of Ganong's formula and the nth order Jacobian derivation. Let k be an algebraically closed field of characteristic $p \neq 0$. Let $G \in k[x,y]$ be such that G_x and G_y have no common factors in $k[x,y]$. Define a derivation D on $k(x,y)$ by

$$D = G_y \frac{\partial}{\partial x} - G_x \frac{\partial}{\partial y}.$$

D is called the *Jacobian derivation* for G on $k(x,y)$.

For each nonnegative integer n, let $A_n = k[x^{p^n}, y^{p^n}, G]$. Note that $A_0 = k[x,y]$. If E_n denotes the quotient field of A_n then E_n is a field extension of E_{n+1} of degree p. In a moment we will show (see Theorem 3.1) that each A_n is Noetherian integrally closed domain and hence a Krull ring. Since $A_n^p \subseteq A_{n+1} \subseteq A_n$ we have that A_n is integral over A_{n+1}. By Theorem 1.1 there exists a well-defined homomorphism $\phi_n : \mathrm{Cl}(A_{n+1}) \to \mathrm{Cl}(A_n)$. Define $D_n : E_n \to E_n$ in the following way: Given $\alpha \in E_n$, it can be written as $\alpha = \sum_{i=0}^{p^n-1} \alpha_i^{p^n} G^i$ for unique $\alpha_i \in k(x,y)$. Then define

$$D_n(\alpha) = \sum_{i=0}^{p^n-1} (D\alpha_i)^{p^n} G^i.$$

In [13] Lang showed that D_n is a derivation on E_n which he called the nth *order Jacobian derivation*. He proved the following result.

THEOREM 3.1. *Let $a \in k[x,y]$ be such that $D^p = aD$. Then*

(a) *D_n is a k-derivation on E_n,*
(b) $\ker D_n \cap A_n = A_{n+1}$,
(c) $\ker \phi_n \cong \mathfrak{L}_n$, *the group of logarithmic derivatives of D_n in A_n, and*
(d) $D_n^p = a^{p^n} D_n$.

(*See* [13, pp. 393, 394, 404].)

The fact that there is an $a \in k[x,y]$ such that $D^p = aD$ is proved in [13, p. 394]. Alternatively, one can prove (a) and (b) of Theorem 3.1 first, the proofs of which do not depend on the existence of a, and then apply Theorem 1.3 to the case $n = 0$.

This next result is known as *Ganong's formula*. It was first conjectured by Ganong [7] and proved by Lang in [13] for the case where $\deg(G_x) = \deg G - 1$. Stohr and Voloch then proved it for arbitrary $G \in k[x,y]$ in [17], where it was used to study the Cartier operator.

THEOREM 3.2 (Ganong's formula). *Let D be the Jacobian derivation for G on $k(x,y)$. Then for all $\alpha \in k(x,y)$,*

$$D^{p-1}\alpha - a\alpha = -\sum_{i=0}^{p-1} G^i \nabla (G^{p-i-1}\alpha),$$

where

$$D^p = aD \quad \text{and} \quad \nabla = \frac{\partial^{2p-2}}{\partial x^{p-1} \partial y^{p-1}}.$$

(*See* [13, p. 395].)

The next two theorems, which will be used extensively, utilize Ganong's formula.

THEOREM 3.3. *Let $A_n = k[x^{p^n}, y^{p^n}, G]$ and $\mathfrak{L}_n$ be the group of logarithmic derivatives of D_n in A_n. Let $t = \alpha_0^{p^n} + \alpha_1^{p^n} G + \cdots + \alpha_{p^n-1}^{p^n} G^{p^n-1} \in A_n$. Then $t \in \mathfrak{L}_n$ if and only if*

(1) $\nabla(G^i \alpha_j) = 0$ *for* $0 \le i \le p-1$, $0 \le j \le p^n - 1$, *and* $j \not\equiv 0 \pmod p$, *and*
(2) $\nabla(G^i \alpha_{sp}) = \alpha^p_{s+(p-i-1)p^{(n-1)}}$ *for* $0 \le s \le p^{(n-1)} - 1$ *and* $0 \le i \le p-1$.

Proof. By Theorems 1.3 and 3.1(d), $t \in \mathfrak{L}_n$ if and only if $D_n^{p-1}t - a^{p^n}t = -t^p$. The definition of D_n implies that $t \in \mathfrak{L}_n$ if and only if

$$\sum_{j=0}^{p^n-1} (D^{p-1}\alpha_j - a\alpha_j)^{p^n} G^j = -\sum_{j=0}^{p^n-1} \alpha_j^{p^{(n+1)}} G^{jp}. \tag{3.3.1}$$

Comparing coefficients in (3.3.1) we obtain $t \in \mathfrak{L}_n$ if and only if

$$\begin{aligned} &(1) \quad D^{p-1}\alpha_j - a\alpha_j = 0 \text{ for } j \not\equiv 0 \pmod p,\ 0 \le j \le p^n - 1, \text{ and} \\ &(2) \quad \sum_{s=0}^{p^{(n-1)}-1} (D^{p-1}\alpha_{sp} - a\alpha_{sp})^{p^n} G^{sp} = -\sum_{r=0}^{p^n-1} \alpha_r^{p^{(n+1)}} G^{rp}. \end{aligned} \tag{3.3.2}$$

Taking pth roots, (2) of (3.3.2) becomes

$$\sum_{s=0}^{p^{(n-1)}-1} (D^{p-1}\alpha_{sp} - a\alpha_{sp})^{p^{(n-1)}} G^s = -\sum_{r=0}^{p^n-1} \alpha_r^{p^n} G^r. \tag{3.3.3}$$

Compare both sides of (3.3.3) and we have that (2) is equivalent to

$$D^{p-1}\alpha_{sp} - a\alpha_{sp} = -\sum_{i=0}^{p-1} \alpha^p_{s+ip^{(n-1)}} G^i \quad \text{for } 0 \le s \le p^{(n-1)} - 1. \tag{3.3.4}$$

Now apply Ganong's formula to the left sides of (1) of (3.3.2) and (3.3.4) and compare coefficients of G to obtain the desired result. □

THEOREM 3.4. *Let*

$$D = G_y \frac{\partial}{\partial x} - G_x \frac{\partial}{\partial y}$$

be the Jacobian derivation and β be such that $D^p = \beta D$. If $(a,b) \in k^2$ is such that $G_x(a,b) = G_y(a,b) = 0$ then $\beta(a,b) = (\sqrt{H(a,b)}\,)^{p-1}$, where $H = G_{xy}^2 - G_{xx}G_{yy}$.

Proof. If $p = 2$, then $Dx = G_y$, $D^2x = G_{xy}G_y$, and hence $\beta = G_{xy}$. Since $G_{xx} = G_{yy} = 0$, $H = G_{xy}^2$ so that in this case $\beta = \sqrt{H}$ and the formula holds for all $(a,b) \in k^2$.

If $p > 2$ then for each $\alpha \in k(x,y)$, $D^{p-1}\alpha - \beta\alpha = -\sum_{i=0}^{p-1} G^i \nabla(G^{p-i-1}\alpha)$ by Ganong's formula (Theorem 3.2).

Set $\alpha = 1$; then $\beta = \sum_{i=0}^{p-1} G^i \nabla(G^{p-i-1})$. Let

$$\bar{G} = G(x+a, y+b) \quad \text{and} \quad \bar{\beta} = \sum_{i=0}^{p-1} \bar{G}^i \nabla(\bar{G}^{p-i-1}).$$

Then $\bar{\beta}(0,0) = \sum_{i=0}^{p-1} G(a,b)^i \nabla(G^{p-i-1})(a,b) = \beta(a,b)$. By Taylor's formula,

$$\begin{aligned} G(x,y) = G(a,b) + G_{xx}(a,b)\frac{(x-a)^2}{2} + G_{xy}(a,b)(x-a)(y-b) \\ + G_{yy}(a,b)\frac{(y-b)^2}{2} + (\text{higher degree terms}). \end{aligned}$$

Thus

$$\bar{G}(x,y) = G(a,b) + G_{xx}(a,b)\frac{x^2}{2} + G_{xy}(a,b)xy + G_{yy}(a,b)\frac{y^2}{2} + (\text{higher degree terms}).$$

Let $\bar{\bar{G}} = \bar{G} - G(a,b)$ and $\bar{\bar{\beta}} = -\sum_{i=0}^{p-1}\bar{G}\nabla(\bar{\bar{G}}^{p-i-1})$. Since $(\bar{\bar{G}})_x = (\bar{G})_x$ and $(\bar{\bar{G}})_y = (\bar{G})_y$ it follows that $\bar{\bar{\beta}}(x,y) = \bar{\beta}(x,y)$ and $\bar{\bar{\beta}}(0,0) = \beta(a,b)$. Since $\bar{\bar{G}}(0,0) = 0$ it follows that $\bar{\bar{\beta}}(0,0) = \nabla(\bar{\bar{G}}^{p-1})(0,0)$. A simple calculation yields that the lowest-degree term in $\bar{\bar{G}}^{p-1}$ is

$$\left\{\sum_{t=0}^{(p-1)/2}\binom{p-1}{2i}\binom{2i}{i}G_{xy}^{p-2i-1}\left(\frac{G_{yy}}{2}\right)^i\left(\frac{G_{yy}}{2}\right)^i\right\}(a,b)\cdot x^{p-1}y^{p-1} + (\text{other monomials of degree } 2p-2).$$

Thus the lowest-degree term of $\nabla(\bar{\bar{G}}^{p-1})$ is the constant term

$$\left\{\sum_{i=0}^{(p-1)/2}(-1)^i\binom{(p-1)/2}{i}G_{xy}^{p-2i-1}(G_{xx}G_{yy})^i\right\}(a,b).$$

In this step a combinatorial identity is used (see [8, p. 90, identity 2.40]). Thus the constant term in $\nabla(\bar{\bar{G}}^{p-1})$ is $(H(a,b))^{(p-1)/2}$. Therefore $\nabla(\bar{\bar{G}}^{p-1})(0,0) = (\sqrt{H(a,b)}\,)^{p-1}$. □

COROLLARY 3.5. *Let $t = \sum_{j=0}^{p^n-1}\alpha_j^{p^n}G^j \in A_n$. If $t \in \mathfrak{L}_n$ then the degree of each α_j is less than or equal to* $\deg G - 2$. *Furthermore, $t = 0$ if and only if $\alpha_0 = 0$.*

Proof. If $j = 0$, then by Theorem 3.3 $\nabla(G^{p-1}\alpha_0) = \alpha_0^p$, where

$$\nabla = \frac{\partial^{2p-2}}{\partial x^{p-1}\partial y^{p-1}}.$$

Compare degrees on both sides of the equality to obtain

$$p\deg\alpha_0 \le (p-1)\deg G + \deg\alpha_0 - 2(p-1).$$

This implies that $(p-1)\deg\alpha_0 \le (p-1)(\deg G - 2)$ and therefore that $\deg\alpha_0 \le \deg G - 2$.

Now proceed by reverse induction on $v(j)$, where $v(j)$ is the highest power of p that divides j.

If $v(j) \ge n$ then $j = 0$. Assume then that $v(j) = m < n$. We can write $j = s + (p-(i+1))p^{n-1}$ for unique $s = 0, 1, \ldots, p^{(n-1)} - 1$ and $i = 0, 1, \ldots, p-1$. Since $v(j) = m$, it must be that $s = rp^m$ for some $r = 1, \ldots, p-1$. By Theorem 3.3, $\alpha_{s+(p-(i+1))p^{(n-1)}}^p = \nabla(G^i\alpha_{sp}) = \nabla(G^i\alpha_{rp^{(m+1)}})$. Comparing degrees we see that

$$\begin{aligned} p\deg(\alpha_j) &\le \deg(\alpha_{rp^{(m+1)}}) + i\deg G - 2(p-1) \\ &\le \deg(\alpha_{rp^{(m+1)}}) + (p-1)\deg G - 2(p-1). \end{aligned}$$

From the induction hypothesis it follows that

$$p\deg(\alpha_j) \le \deg G - 2 + (p-1)\deg G - 2(p-1).$$

Thus $p\deg\alpha_j \le p\deg G - 2p$ and $\deg\alpha_j \le \deg G - 2$.

The argument for the last statement of this corollary is again by reverse induction on $v(j)$ and is almost identical to the above. (See also the discussion of the algorithm in Section 5.) □

COROLLARY 3.6. *Let* $t=\sum_{j=0}^{p^n-1}\alpha_j^{p^n}G^j\in\mathfrak{L}_n$. *Let* $Q\in k^2$ *be such that* $G_x(Q)=G_y(Q)=0$ *and* $H(Q)\neq 0$, *where* $H=G_{xy}^2-G_{xx}G_{yy}$. *Then* $t(Q)=0$ *if and only if* $\alpha_j(Q)=0$ *for each* $j=0,1,\dots,p^n-1$.

Proof by induction on $v(j)$, *the highest power of* p *that divides* j. If $v(j)=0$, then $j\not\equiv 0 \pmod p$. By Theorem 3.3, $\nabla(G^i\alpha_j)=0$ for each $i=0,1,\dots,p-1$. By Ganong's formula this implies that $D^{p-1}\alpha_j-a\alpha_j=0$, where D is the Jacobian derivation on $k[x,y]$ and $D^p=aD$. Substitute the coordinates of Q for x and y. By Theorem 3.4, $(\sqrt{H(Q)})^{p-1}\alpha_j(Q)=0$. Therefore $\alpha_j(Q)=0$.

Now assume that $v(j)=u+1>0$. Then $j=rp^{u+1}$ for some $r=1,\dots,p-1$. Let $s=rp^u$. By Theorem 3.3, $\nabla(G^i\alpha_j)=\nabla(G^i\alpha_{sp})=\alpha^p_{s+(p-(i+1))p^{(n-1)}}$. By induction, $\alpha_{s+(p-(i+1))p^{(n-1)}}(Q)=0$ for $0\le i\le p-1$. Thus $\nabla(G^i\alpha_j)(Q)=0$ for $0\le i\le p-1$. By Theorems 3.2 and 3.4 we obtain $\alpha_j(Q)=0$ as above.

If $v(j)=\infty$, then $j=0$. Since $\alpha_j(Q)=t(Q)=0$ for $j\neq 0$, it follows that $\alpha_0(Q)=0$. □

4. The class group of $Z^{p^n}=G$ at a singular point. For each nonnegative integer n, let $X_n\subseteq A_k^3$ be the surface defined by the equation $z^{p^n}=G(x,y)$, where $G\in k[x,y]$ and G_x and G_y have no common factor in $k[x,y]$. By the Jacobian criterion, X_n is a normal affine surface (see [16, p. 125]). Therefore the coordinate ring of X_n and the local ring of X_n at a point of X_n are Noetherian integrally closed domains and are thus Krull rings (see [18, p. 5]). For the definition of the local ring of a variety at a point, see [10, p. 16]. For an affine surface such as X_n and a point Q on X_n, the local ring of X_n at Q is isomorphic to the coordinate ring of X_n localized at the prime ideal corresponding to Q [10, p. 17].

For each pair $(a,b)\in k^2$ and integer $n\ge 0$, there is a unique point Q_n on X_n with x coordinate a and y coordinate b. In this section we will study a group homomorphism $\mathrm{Cl}(\mathcal{O}_{Q_{n+1}})\to\mathrm{Cl}(\mathcal{O}_{Q_n})$, where $\mathcal{O}_{Q_n}$ is the local ring of X_n at Q_n.

THEOREM 4.1. *Let* $(a,b)\in k^2$, $n\ge 0$ *an integer, and* X_n *and* Q_n *be as above. Let* $A_n=k[x^{p^n},y^{p^n},G]$. *Then, for each* n,

(a) *the coordinate ring of* X_n *is isomorphic to* A_n, *and*

(b) *there is a well-defined group homomorphism* $\theta_n\colon\mathrm{Cl}(\mathcal{O}_{Q_{n+1}})\to\mathrm{Cl}(\mathcal{O}_{Q_n})$.

Proof. (a) The coordinate ring of X_n is $R_n=k[x,y,z]/I$, where I is the ideal in $k[x,y,z]$ generated by z^p-G. Let $\Phi\colon k[x,y,z]\to A_n$ be the map that sends each $\alpha\in k$ to α^{p^n}, x to x^{p^n}, y to y^{p^n}, and z to G. Φ is a surjective ring homomorphism since k is perfect. Then the kernel of Φ is a height-one prime containing I. Since I is height one, $I=\ker\Phi$. Therefore R_n is isomorphic to A_n.

To prove (b), note that we have the following commutative diagram for every n:

$$\begin{array}{ccc}\mathcal{O}_{Q_{n+1}} & \longrightarrow & \mathcal{O}_{Q_n}\\ \downarrow\cong & & \downarrow\cong\\ (A_{n+1})_{J_{n+1}} & \longrightarrow & (A_n)_{J_n},\end{array}$$

where for each n, J_n is the maximal ideal in A_n generated by $x^{p^n}-a^{p^n}$, $y^{p^n}-b^{p^n}$, and G.

Clearly, $(A_n)_{J_n}$ is integral over $(A_{n+1})_{J_{n+1}}$ and both rings are Krull rings. By Theorem 1.1 there is a well-defined group homomorphism $\bar{\theta}_n\colon \mathrm{Cl}(A_{n+1})_{J_{n+1}} \to \mathrm{Cl}(A_n)_{J_n}$ inducing a homomorphism $\theta_n\colon \mathrm{Cl}(\mathcal{O}_{Q_{n+1}}) \to \mathrm{Cl}(\mathcal{O}_{Q_n})$.

A point $(a,b,c)\in X_n$ is a singular point of X_n if and only if

$$G_x(a,b)=G_y(a,b)=0.$$

Thus, if Q_n is not a singular point of X_n then Q_{n+1} is not a singular point of X_{n+1}, and in this case $\mathrm{Cl}(\mathcal{O}_{Q_{n+1}})=\mathrm{Cl}(\mathcal{O}_{Q_n})=0$. Thus the only interesting case for analyzing θ_n in Theorem 4.1 is when $G_x(a,b)=G_y(a,b)=0$.

In [14] the writers described an algorithm for computing the group of logarithmic derivatives of the Jacobian derivation acting on a polynomial ring. With the assistance of Joyce [5] this algorithm was converted to a computer program when the ground field k is an algebraic closure of a finite field. Using Theorems 2.1-2.3 we will show how this algorithm can be modified to produce an algorithm for calculating the kernel of the homomorphism $\theta_n\colon \mathrm{Cl}(\mathcal{O}_{Q_{n+1}}) \to \mathrm{Cl}(\mathcal{O}_{Q_n})$ under suitable conditions.

In addition to assuming that $G_x(a,b)=G_y(a,b)=0$, we will hereafter assume (unless stated otherwise) that $H(a,b)\neq 0$, where $H=G_{xy}^2-G_{xx}G_{yy}$.

Let Q_n and Q_{n+1} be the singular points of X_n and X_{n+1} corresponding to (a,b). Then we have a commutative diagram of group homomorphisms,

$$\begin{array}{ccccccc} 0 \longrightarrow & \ker\phi_n & \longrightarrow & \mathrm{Cl}(X_{n+1}) & \xrightarrow{\phi_n} & \mathrm{Cl}(X_n) \\ & \downarrow{\scriptstyle\beta_n} & & \downarrow{\scriptstyle\alpha_n} & & \downarrow{\scriptstyle\alpha_{n+1}} \\ 0 \longrightarrow & \ker\theta_n & \longrightarrow & \mathrm{Cl}(\mathcal{O}_{Q_{n+1}}) & \xrightarrow{\theta_n} & \mathrm{Cl}(\mathcal{O}_{Q_n}), \end{array} \tag{4.2}$$

where $\alpha_j\colon \mathrm{Cl}(X_j)\to \mathrm{Cl}(\mathcal{O}_{Q_j})$ is the surjection of Theorem 1.4 and $\beta_n\colon \ker\phi_n \to \ker\theta_n$ is induced by α_n. Our final assumption will be that β_n is a surjection.

One easily checks that all of the conditions of Theorem 2.1 are met. Therefore the kernel of θ_n is isomorphic to $\mathcal{L}_n/\tilde{\mathcal{L}}_n$, where $\mathcal{L}_n$ is the group of logarithmic derivatives of the nth order Jacobian derivation D_n on A_n (see Theorem 3.1), and $\tilde{\mathcal{L}}_n=\{u^{-1}D_n u\colon u \text{ is a unit in } \mathcal{O}_{Q_n}\}$.

By Theorems 3.1, 2.1, and 2.3, $H(a,b)\neq 0$ implies that $\tilde{\mathcal{L}}_n=\mathcal{L}_n\cap m$, where m is the maximal ideal of A_n corresponding to $Q_n\in X_n$. Thus if we could calculate $\mathcal{L}_n$ and $\mathcal{L}_n\cap m$ then we would be done. But first some facts about $\mathrm{Cl}(\mathcal{O}_{Q_n})$ and $\ker\theta_n$ need to be collected.

THEOREM 4.3. *If* $H(a,b)\neq 0$, *then* $\mathrm{Cl}(\mathcal{O}_{Q_n})\cong \mathbf{Z}/p^r\mathbf{Z}$ *for some* $r\le n$.

Proof. Assume first that $p>2$. Let $\hat{\mathcal{O}}_{Q_n}$ be the completion of $\mathcal{O}_{Q_n}$. After a linear change of coordinates we can assume that $a=0$ and $b=0$. By Taylor's formula,

$$\begin{aligned} G(x,y) &= G(0,0)+G_{xx}(0,0)\frac{x^2}{2}+G_{xy}(0,0)xy \\ &\quad + G_{yy}(0,0)\frac{y^2}{2}+(\text{higher degree terms}). \end{aligned}$$

Since $H(0,0) \neq 0$, we can make another linear change of coordinates and assume that $G(x,y) = G(0,0) + xy +$ (higher degree terms). Finally, if z is replaced by $z + (G(0,0))^{(1/p^n)}$, then the surface $z^{p^n} = G(x,y)$ is seen to be isomorphic to $z^{p^n} = xy +$ (higher degree terms). Therefore we may assume that G has this form and that $Q_n = (0,0,0)$ on X_n.

Then $\hat{\mathcal{O}}_{Q_n} = k[[x^{p^n}, y^{p^n}, G]]$. In $k[[x,y]]$, G factors into $G = uv$, where $u = x +$ (higher degree terms) and $v = y +$ (higher degree terms). Then $k[[x,y]] = k[[u,v]]$ and $\hat{\mathcal{O}}_{Q_n} \cong k[[u^{p^n}, v^{p^n}, uv]]$. It is known (see [12, p. 630]) that the class group of $k[[u^{p^n}, v^{p^n}, uv]]$ is $\mathbf{Z}/p^n\mathbf{Z}$ generated by the height-one prime (u^{p^n}, uv). By Theorem 1.5, $\mathrm{Cl}(\mathcal{O}_{Q_n})$ injects into $\mathrm{Cl}(\hat{\mathcal{O}}_{Q_n})$. Therefore $\mathrm{Cl}(\mathcal{O}_{Q_n})$ is cyclic of order p^r with $r \leq n$. □

The case $p = 2$ is left as an exercise for the reader.

THEOREM 4.4. *The kernel of* $\theta_n : \mathrm{Cl}(\mathcal{O}_{Q_{n+1}}) \to \mathrm{Cl}(\mathcal{O}_{Q_n})$ *is* 0 *or* $\mathbf{Z}/p\mathbf{Z}$ *if* $H(a,b) \neq 0$.

Proof. As in the proof of Theorem 4.3, we can assume that Q_n and Q_{n+1} both have coordinates $(0,0,0)$ on X_n and X_{n+1} and that $G = xy +$ (higher degree terms). Let $\hat{\mathcal{O}}_{Q_j}$ be the completion of $\mathcal{O}_{Q_j}$ for $j = n, n+1$. Then we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Cl}(\mathcal{O}_{Q_{n+1}}) & \xrightarrow{\theta_n} & \mathrm{Cl}(\mathcal{O}_{Q_n}) \\ \downarrow & & \downarrow \\ \mathrm{Cl}(\hat{\mathcal{O}}_{Q_{n+1}}) & \xrightarrow{\bar{\theta}_n} & \mathrm{Cl}(\hat{\mathcal{O}}_{Q_n}), \end{array}$$

where the maps $\mathrm{Cl}(\mathcal{O}_{Q_j}) \to \mathrm{Cl}(\hat{\mathcal{O}}_{Q_j})$ are injections for $j = n, n+1$. Therefore the kernel of θ_n injects into the kernel of $\bar{\theta}_n$. In the proof of Theorem 4.3 we noted that $\hat{\mathcal{O}}_{Q_n} \cong k[[u^{p^n}, v^{p^n}, uv]]$ and that $\mathrm{Cl}(\hat{\mathcal{O}}_{Q_n})$ is isomorphic to $\mathbf{Z}/p^n\mathbf{Z}$ generated by (u^{p^n}, uv). This prime clearly does not ramify over $\mathcal{O}_{Q_{n+1}}$. It follows that $\bar{\theta}_n$ is surjective with kernel isomorphic to $\mathbf{Z}/p\mathbf{Z}$. Therefore the kernel of θ_n is 0 or $\mathbf{Z}/p\mathbf{Z}$. □

REMARK 4.5. Note that in the proof of Theorems 4.3 and 4.4 only the conditions that G_x and G_y have no common factors and $H(a,b) \neq 0$ are needed. The hypothesis that α_n restricts to $\ker \phi_n$ to give a surjection in diagram (4.2) was not used. The proof of the next theorem also assumes only the first two conditions.

So far we have used the fact that the coordinate ring of X_n is isomorphic to $A_n = k[x^{p^n}, y^{p^n}, G]$ and that A_n is integral over A_{n+1} to study the homomorphism of class groups of local rings $\theta_n : \mathrm{Cl}(\mathcal{O}_{Q_{n+1}}) \to \mathrm{Cl}(\mathcal{O}_{Q_n})$.

We have that $A_n^p \subseteq A_{n+1} \subseteq A_n$ and A_{n+1} is integral over A_n^p. It is also easy to show that the quotient field of A_{n+1} is of degree p over the quotient field of A_n^p. Since k is perfect, A_n is clearly isomorphic to A_n^p. Thus by Theorem 1.1, the inclusion $A_n^p \subset A_{n+1}$ induces another homomorphism $\mathrm{Cl}(X_n) \to \mathrm{Cl}(X_{n+1})$. In [13], Lang showed that this global mapping is an injection for all n. We now will prove the local version of this fact.

THEOREM 4.6. *For each n, let Q_n be the singular point on X_n corresponding to (a, b), where $G_x(a, b) = G_y(a, b) = 0$. If $H(a, b) \neq 0$ then for each n there is an injection $\omega_n: \mathrm{Cl}(\mathcal{O}_{Q_n}) \to \mathrm{Cl}(\mathcal{O}_{Q_{n+1}})$.*

Proof. As in Theorem 4.3 we can assume that, for each n, Q_n is the origin on X_n. If we think of $\mathcal{O}_{Q_n}$ as the localization of $A_n^p = k[x^{p^{n+1}}, y^{p^{n+1}}, G^p]$ at the maximal ideal generated by $x^{p^{(n+1)}}, y^{p^{(n+1)}}, G^p$, and of $\mathcal{O}_{Q_{n+1}}$ as the localization of A_{n+1} at the maximal ideal generated by $x^{p^{n+1}}, y^{p^{n+1}}, G$, then we have that $\mathcal{O}_{Q_n} \subset \mathcal{O}_{Q_{n+1}}$ with $\mathcal{O}_{Q_{n+1}}$ integral over $\mathcal{O}_{Q_n}$. By Theorem 1.1 there is a mapping $\omega_n: \mathrm{Cl}(\mathcal{O}_{Q_n}) \to \mathrm{Cl}(\mathcal{O}_{Q_{n+1}})$. Passing to completion we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Cl}(\mathcal{O}_{Q_n}) & \xrightarrow{\omega_n} & \mathrm{Cl}(\mathcal{O}_{Q_{n+1}}) \\ \downarrow & & \downarrow \\ \mathrm{Cl}(\hat{\mathcal{O}}_{Q_n}) & \xrightarrow{\bar{\omega}_n} & \mathrm{Cl}(\hat{\mathcal{O}}_{Q_{n+1}}), \end{array} \tag{4.7}$$

where the maps $\mathrm{Cl}(\mathcal{O}_{Q_j}) \to \mathrm{Cl}(\hat{\mathcal{O}}_{Q_j})$ are injections for $j = n, n+1$. As in the proof of Theorem 4.4 we have that

$$\hat{\mathcal{O}}_{Q_n} = k[[u^{p^{n+1}}, v^{p^{n+1}}, u^p v^p]] \quad \text{and} \quad \hat{\mathcal{O}}_{Q_{n+1}} = k[[u^{p^{n+1}}, v^{p^{n+1}}, uv]],$$

where $G = uv$ in $k[[x, y]]$. Also, $\mathrm{Cl}(\hat{\mathcal{O}}_{Q_n}) \cong \mathbf{Z}/p^n\mathbf{Z}$ is generated by the height-one prime $(u^{p^{n+1}}, u^p v^p)$ and $\mathrm{Cl}(\hat{\mathcal{O}}_{Q_{n+1}}) \cong \mathbf{Z}/p^{(n+1)}\mathbf{Z}$ is generated by $(u^{p^{n+1}}, uv)$. Since $(u^{p^{n+1}}, u^p v^p)$ ramifies in $\hat{\mathcal{O}}_{Q_{n+1}}$, the map $\bar{\omega}_n$ corresponds to multiplication by p from $\mathbf{Z}/p^n\mathbf{Z}$ to $\mathbf{Z}/p^{(n+1)}\mathbf{Z}$. Thus $\bar{\omega}_n$ is clearly injective. From the diagram, so is ω_n. □

COROLLARY 4.8. *If $\mathrm{Cl}(\mathcal{O}_{Q_n}) = 0$, then $\mathrm{Cl}(\mathcal{O}_{Q_r}) = 0$ for all $r \leq n$.*

COROLLARY 4.9. *If the order of $\mathrm{Cl}(\mathcal{O}_{Q_n})$ is p^n, then the order of $\mathrm{Cl}(\mathcal{O}_{Q_{n+1}})$ is p^n or p^{n+1}.*

Proof. Use Theorems 4.4 and 4.6. □

5. In this section we provide an algorithm for computing the kernel of θ_n: $\mathrm{Cl}(\mathcal{O}_{Q_{n+1}}) \to \mathrm{Cl}(\mathcal{O}_{Q_n})$ when $H(a, b) \neq 0$ and when β_n in diagram (4.2) is a surjection. By Theorems 2.1 and 2.3 we have that $\ker \theta_n \cong \mathcal{L}_n / \mathcal{L}_n \cap m$, where $\mathcal{L}_n$ is the group of logarithmic derivatives of D_n in A_n and m is the maximal ideal in A_n corresponding to Q_n.

A typical logarithmic derivative of D_n in A_n will have the form $t = \sum_{j=0}^{p^n - 1} \alpha_j^{p^n} G^j$, where $\alpha_j \in k[x, y]$ is of degree less than or equal to $N = \deg G - 2$ by Corollary 3.5, and where the α_j satisfy the equations of Theorem 3.3:

$$\text{(5.1)} \quad \begin{array}{ll} (1) & \nabla(G^i \alpha_j) = 0 \text{ for } 0 \leq i \leq p-1,\ 0 \leq j \leq p^n - 1, \text{ and } j \not\equiv 0 \pmod{p} \text{ and} \\ (2) & \nabla(G^i \alpha_{sp}) = \alpha_{s+(p-i-1)p^{n-1}}^p \text{ for } 0 \leq s \leq p^{(n-1)} - 1, \text{ and } 0 \leq i \leq p-1. \end{array}$$

Beginning with α_0 we let $\alpha_0 = \sum_{0 \leq r+m \leq N} T_{rm} x^r y^m$, where the T_{rm} are indeterminants over k.

Let $j = 0$ in (1) of (5.1) and $s = 0$ and $i = p-1$ in (2); we see that

$$\nabla(G^{p-1}\alpha_0) = \alpha_0^p \quad \text{and} \quad \nabla(G^i \alpha_0) = 0 \quad \text{for } 0 \leq i \leq p-2. \tag{5.2}$$

Substitute α_0 into the equations of (5.2) and compare coefficients to obtain a system of p-linear and linear equations of the form

$$(5.3)\qquad \begin{aligned} l_{rm} &= T_{rm}^p, \quad 0 \le r+m \le N \quad \text{and} \\ l_q &= 0, \quad 0 \le q \le M \quad \text{for some positive integer } M. \end{aligned}$$

It is easy to show (see [13, p. 397]) that there are only a finite number of solutions to (5.3).

Let α_0 now correspond to a solution of (5.3). Using (2) of (5.1) we obtain

$$\alpha_{(p-i-1)p^{n-1}} = [\nabla(G^i \alpha_0)]^{1/p} \quad \text{for } 0 \le i \le p-2.$$

Continue to use (2) of (5.1) to obtain the rest of the α_j in t. It is easy to see, by the same argument used in the proof of Corollary 3.5, that all of the α_j are determined in this way. Now substitute the α_j into (1) of (5.1). If any of the equations in (1) is not satisfied then $t \notin \mathcal{L}_n$.

Thus for each α_0 determined above we find the corresponding candidate for t and finally test it against the equations in (1) to determine if t is a genuine element of $\mathcal{L}_n$. This gives an algorithm for determining $\mathcal{L}_n$.

We then evaluate $t(a,b)$ for each $t \in \mathcal{L}_n$. If for each $t \in \mathcal{L}_n$, $t(a,b)=0$, then $\mathcal{L}_n = \mathcal{L}_n \cap m = \tilde{\mathcal{L}}_n$ and $\ker \theta_n = 0$. Otherwise $\mathcal{L}_n \cong \mathbf{Z}/p\mathbf{Z}$ by Theorem 4.4.

EXAMPLE 5.4. Let k be an algebraically closed field of characteristic 3. Let $G = x(1+x-y+xy+y^2+x^3+x^2y-xy^2+y^3)$ and let X_1 be the surface defined by $z^3 = G$. It is easy to show that G_x, G_y, and H do not meet. If $(a,b) \in k^2$ is such that $G_x(a,b) = G_y(a,b) = 0$, then in diagram (4.2) for the case $n=0$ we have the commutative diagram

$$\begin{array}{ccccc} \ker\phi_0 & \longrightarrow & \mathrm{Cl}(X_1) & \xrightarrow{\phi_0} & 0 = \mathrm{Cl}(X_0) \\ \downarrow{\scriptstyle \beta_0} & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha_1} \\ \ker\theta_0 & \longrightarrow & \mathrm{Cl}(\mathcal{O}_{Q_1}) & \xrightarrow{\theta_0} & 0, \end{array}$$

where Q_1 is the corresponding singularity on X_1. By Theorem 2.2, $\ker\phi_0 \to \ker\theta_0$ is a surjection since $A_0 = k[x,y]$ has no nonprincipal height-one primes. Therefore we can use the algorithm to find $\mathrm{Cl}(\mathcal{O}_{Q_1})$ at each singular point. We then find that $\mathcal{L}_0$, which by Theorem 3.1 is isomorphic to $\mathrm{Cl}(X_1)$, is $\mathbf{Z}/p\mathbf{Z}$ generated by $t = -1+x-y+x^2+xy$. X_1 has nine singularities and $G_y = xt$. When $x=0$ there are three singular points whose y coordinates are the roots of $y^3+y^2-y+1=0$. Also $t=-1-y$ when $x=0$, and since $y=-1$ is not a root of y^3+y^2-y+1 it follows that $t \neq 0$ at the three singular points with x coordinate 0. We conclude that $\mathrm{Cl}(\mathcal{O}_{Q_1}) \cong \mathbf{Z}/p\mathbf{Z}$ at these singularities. When $x \neq 0$, X_1 has six singular points and $t=0$ at these since $G_y = xt$. Thus at these points $\mathrm{Cl}(\mathcal{O}_{Q_1}) = 0$.

EXAMPLE 5.5. Continuing Example 5.4: Let X_2 be the surface defined by $z^{3^2} = x(1+x-y+xy+y^2+x^3+x^2y-xy^2+y^3)$. Then one calculates $\mathcal{L}_2$ and finds that $\mathcal{L}_2 \cong \mathbf{Z}/p\mathbf{Z}$ generated by $t = (-1+x-y+x^2+xy)^3$. We then have an exact sequence

$$0 \longrightarrow \mathbf{Z}/p\mathbf{Z} \longrightarrow \mathrm{Cl}(X_2) \xrightarrow{\phi_1} \mathrm{Cl}(X_1).$$

In Example 5.4 we saw that $\mathrm{Cl}(X_1) \cong \mathbf{Z}/p\mathbf{Z}$ and is generated by the height-one prime $(z,x) = P_1$. Clearly the height-one prime in $\mathrm{Cl}(X_2)$ given by $(z,x) = P_2$ is not principal. Also, P_2 does not ramify so that ϕ_1 is surjective. Therefore $\mathrm{Cl}(X_2) \cong \mathbf{Z}/p^2\mathbf{Z}$ is generated by P_2 and $\ker \phi_1$ is generated by pP_2.

Let Q_1 be a nonprincipal height-one prime of X_1. Then, from Remark 4.5 and the above discussion, we have a commutative diagram:

(5.5.1)
$$\begin{array}{ccc} \mathbf{Z}/p\mathbf{Z} & \xrightarrow[\text{injection}]{} & \mathbf{Z}/p^2\mathbf{Z} \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{Cl}(X_1) & \xrightarrow[\text{injection}]{} & \mathrm{Cl}(X_2). \end{array}$$

Since P_2 does not ramify, the map $\mathbf{Z}/p\mathbf{Z} \to \mathbf{Z}/p^2\mathbf{Z}$ must be multiplication by p. This implies that if $Q_2 = Q_1 \cap A_2$ in $\mathrm{Cl}(X_2)$ then $pQ_2 \neq 0$ and hence $\phi_1(Q_2) = Q_1$. So by Theorem 2.2, $\ker \phi_1 \to \ker \theta_1$ is surjective. Repeating the argument used in Example 5.4 we have that $\mathrm{Cl}(\mathcal{O}_{Q_2}) = 0$ at six singularities and $\mathrm{Cl}(\mathcal{O}_{Q_1}) \cong \mathbf{Z}/p^2\mathbf{Z}$ at the three others.

EXAMPLE 5.6. It is not always necessary to know that $\ker \phi_n \to \ker \theta_n$ is surjective in order to determine the $\ker \theta_n$. In the proof of Theorem 2.1(a) we saw that there is a natural injection from $\mathcal{L}_n/\tilde{\mathcal{L}}_n \to (\mathcal{L}_n)_0/(\mathcal{L}_n')_0$, where

$$(\mathcal{L}_n)_0 = \{t^{-1}D_n t \in \mathcal{O}_{Q_n} : t \text{ is in the quotient field of } \mathcal{O}_{Q_n}\}$$

and

$$(\mathcal{L}_n')_0 = \{u^{-1}D_n u : u \text{ is a unit in } \mathcal{O}_{Q_n}\}.$$

Thus if $H(a,b) \neq 0$ and if there is a $t \in \mathcal{L}_n$ such that $t(a,b) \neq 0$, then $\ker \theta_n$ is isomorphic to $\mathbf{Z}/p\mathbf{Z}$. Consider for example the surface X_n defined by $Z^{p^n} = G$, where $G = xy + x^{p+1} + y^{p+1}$. For each n, $D(G_y)^{p^n}/(G_y)^{p^n} = 1 \in \mathcal{L}_n$. Then if Q_n is any singular point of X_n, $\ker \theta_n : \mathrm{Cl}(\mathcal{O}_{Q_{n+1}}) \to \mathrm{Cl}(\mathcal{O}_{Q_n})$ is isomorphic to $\mathbf{Z}/p\mathbf{Z}$.

We have an exact sequence for each n,

(5.6.1)
$$0 \to \mathbf{Z}/p\mathbf{Z} \to \mathrm{Cl}(\mathcal{O}_{Q_{n+1}}) \to \mathrm{Cl}(\mathcal{O}_{Q_n}).$$

In case $n=0$, $\mathrm{Cl}(\mathcal{O}_{Q_0}) = 0$ and $\mathrm{Cl}(\mathcal{O}_{Q_1}) \cong \mathbf{Z}/p\mathbf{Z}$, generated by $P_1 = (z+xy, x+y^p)$.

For each n, let $P_n \in \mathrm{Cl}(\mathcal{O}_{Q_n})$ be the contraction of P_1 to $\mathcal{O}_{Q_n}$. One sees immediately that $P_n = (x+y^p, z^{p^{n-1}} + xy)$.

Note that

$$\text{in } \mathcal{O}_{Q_n}\left[\frac{1}{x+y^p}\right], \quad y = \frac{z^{p^n} - x^{p+1}}{x+y^p}.$$

Therefore $\mathcal{O}_{Q_n}[1/(x+y^p)]$ is a localization of $k[x,z]$. This implies, by Theorem 1.4, that $\mathrm{Cl}(\mathcal{O}_{Q_n}[1/(x+y^p)]) = 0$ and that $\mathrm{Cl}(\mathcal{O}_{Q_n})$ is generated by P_n.

The homomorphism $\mathcal{O}_{Q_n} \to \mathcal{O}_{Q_{n-1}}$ maps z to z, x to x^p, and y to y^p; $z^{p^n} + xy$ is a parameter for P_{n+1} and its value in $\mathcal{O}_{P_n}$ is p. Therefore the map $\mathrm{Cl}(\mathcal{O}_{Q_{n+1}}) \to \mathrm{Cl}(\mathcal{O}_{Q_n})$ is just multiplication by p. By induction we conclude that for each n the sequence $0 \to \mathbf{Z}/p\mathbf{Z} \to \mathrm{Cl}(\mathcal{O}_{Q_n}) \to 0$ is exact. That is, $\mathrm{Cl}(\mathcal{O}_{Q_n}) \cong \mathbf{Z}/p\mathbf{Z}$ for all n and at all singularities of X_n.

Another application of this technique is the following theorem.

THEOREM 5.7. *Let $G \in k[x,y]$ be of degree d and assume that G_x and G_y intersect in k^2 at the maximum possible number of distinct points. Thus number is $(d-1)^2$ if $d \not\equiv 0 \pmod p$ and is d^2-3d+3 otherwise* (*see* [4, p. 284]). *Then the coordinate ring of $X_n: z^{p^n} = G$ is factorial if and only if it is locally factorial.*

Proof. If the coordinate ring of X_n is factorial then certainly $\mathrm{Cl}(\mathcal{O}_{Q_n}) = 0$ at each singular point $Q_n \in X_n$, by Theorem 1.4.

We prove the converse by induction. So assume that $\mathrm{Cl}(\mathcal{O}_{Q_n}) = 0$ at each singularity of X_n. If $n=0$, then X_0 is isomorphic to the affine plane A_k^2, and thus $\mathrm{Cl}(X_0) = 0$.

Now assume that $n > 0$ and that the theorem holds for all integers r such that $0 \leq r < n$. Since X_n is locally factorial, so is X_r for all $r < n$ by Theorem 4.6. Thus $\mathrm{Cl}(X_r) = 0$ for $r < n$. By Theorem 3.1, $\mathrm{Cl}(X_n) \cong \mathfrak{L}_{n-1}$.

Let $t \in \mathfrak{L}_{n-1}$. Since $\mathrm{Cl}(X_{n-1}) = 0$, Theorems 2.1 and 2.2 imply that $\mathrm{Cl}(\mathcal{O}_{Q_n}) \cong \mathfrak{L}_{n-1}/\tilde{\mathfrak{L}}_{n-1}$ at each singularity $Q_n \in X_n$. Therefore $\mathfrak{L}_{n-1} = \tilde{\mathfrak{L}}_{n-1}$, so it must be that $t(a,b) = 0$ at each point $(a,b) \in k^2$ such that $G_x(a,b) = G_y(a,b) = 0$. We have that $t = \sum_{i=0}^{p(n-1)-1} \alpha^{p(n-1)} G^i$ for some $\alpha_i \in k[x,y]$. By Corollary 3.6, $\alpha_0(a,b) = 0$ for each (a,b) as above. Since G_x and G_y have no common factors, if $\alpha_0 \not\equiv 0$ then we can factor α_0 in $k[x,y]$ into $\alpha_0 = uv$, where u is relatively prime to G_x and v is relatively prime to G_y. Then u and G_x intersect in at most $\deg(u)\cdot(d-1)$ points and v and G_y intersect in at most $\deg(v)\cdot(d-1)$ points.

By Corollary 3.5, the total number of these intersection points is at most

$$(d-1)(\deg u + \deg v) = (d-1)\deg \alpha_0 \leq (d-1)(d-2).$$

Since G_x and G_y intersect in at least $(d-1)(d-2)+1$ points, α_0 must be identically 0. By Corollary 3.5 this implies that $t = 0$.

We conclude that $\mathrm{Cl}(X_n) \cong \mathfrak{L}_{n-1} = 0$. □

REMARK 5.8. The condition that G_x and G_y intersect in the maximum possible number of distinct points is a generic one (see [4, p. 284]).

QUESTION 5.9. A significant question to ask is, "What conditions on G will guarantee that the map $\ker \phi_n \to \ker \theta_n$ is surjective?" Certainly the condition that each nonprincipal height-one prime of A_n is unramified over A_{n+1} does not always hold (see, e.g., Example 5.6). Of course if $\mathrm{Cl}(X_n) = 0$ then $\ker \phi_n \to \ker \theta_n$ is surjective by Theorem 2.2, and we have that $\mathfrak{L}_n/\tilde{\mathfrak{L}}_n \cong \mathrm{Cl}(\mathcal{O}_{Q_{n+1}})$.

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