NONLINEAR SOLUTIONS OF NEVANLINNA-PICK INTERPOLATION PROBLEMS

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1. Introduction. This article concerns a nonlinear extension of the classical theory of Nevanlinna–Pick interpolation. A matricial form of the classical Nevanlinna–Pick interpolation problem is as follows:

(NP) Given a collection $\{z_1, \dots, z_k\}$ of complex numbers with $|z_j| < 1$, a set of vectors $\{x_1, \dots, x_k\}$ in $\mathbf{C}^n$ and a set of vectors $\{y_1, \dots, y_k\}$ in $\mathbf{C}^m$, find all $m \times n$ matrix functions F analytic on the unit disk such that
(i) $\|F(z)\| \leq 1$ for $|z| < 1$, and
(ii) $F(z_j)x_j = y_j$ for $1 \leq j \leq k$.

The classical result is that solutions F to (NP) exist if and only if the Pick matrix

$$\Lambda(\mathbf{z}, \mathbf{x}, \mathbf{y}) = \left[\frac{x_j^* x_i - y_j^* y_i}{1 - \bar{z}_j z_i}\right]_{1 \leq i, j \leq k}$$

is positive semidefinite (see e.g. [4]). Various recipes exist then for constructing the solutions.

A dual version of (NP) has also been studied.

(NP)$_*$ Given a collection $\{w_1, \dots, w_{k'}\}$ of complex numbers with $|w_j| < 1$, a set of vectors $\{\xi_1, \dots, \xi_{k'}\}$ in $\mathbf{C}^m$ and a set of vectors $\{\eta_1, \dots, \eta_{k'}\}$ in $\mathbf{C}^n$, find all $m \times n$ matrix functions F analytic on the unit disk such that
(i) $\|F(z)\| \leq 1$ for $|z| < 1$, and
(ii)$_*$ $\xi_j^* F(w_j) = \eta_j^*$ for $1 \leq j \leq k'$.

Note that F is a solution of (NP)$_*$ if and only if $F^*(z) = F(\bar{z})^*$ is a solution of a problem of the type (NP). Thus (NP)$_*$ has a solution if and only if the Pick matrix

$$\Lambda_*(\underline{\underline{w}}, \underline{\underline{\xi}}, \underline{\underline{\eta}}) = \left[\frac{\xi_j^* \xi_i - \eta_j^* \eta_i}{1 - \bar{w}_i w_j}\right]_{1 \leq i, j \leq k'}$$

is positive semidefinite. It is also possible to combine these. For simplicity we assume that $z_i \neq w_j$ for $1 < i \leq k$ and $1 \leq j \leq k'$.

(NP)$\cap$(NP)$_*$ Find all matrix functions F which solve (NP) and (NP)$_*$ simultaneously.

The Pick matrix $\Lambda(\underline{\underline{z}}, \underline{\underline{x}}, \underline{\underline{y}}, \underline{\underline{w}}, \underline{\underline{\xi}}, \underline{\underline{\eta}})$ for this problem is more involved but can be computed; it is given in [5].

Received September 12, 1986.
Research of the first two authors was supported in part by the National Science Foundation, Office of Naval Research and the Air Force Office of Scientific Research.
Michigan Math. J. 34 (1987).

One can give an alternative (perhaps more cumbersome) operator-theoretic formulation of (NP) and $(\mathrm{NP})_*$; this will lead to our nonlinear operator generalization. If G is any matrix function in $L^\infty_{m\times n}$ we let $M_G: L^2_n \to L^2_m$ be the multiplication operator $M_G: h \to Gh$ for all $h \in L^2_n$. The operator norm $\|M_G\| = \sup\{\|M_G(f)\|_{L^2_m}: f \in L^2_n, \|f\|_{L^2_n} \le 1\}$ turns out to be identical to the ∞-norm $\|G\|_\infty = \sup\{\|G(z)\|: |z|=1\}$ of G. Operators T of the form $T = M_G$ are characterized among all the operators from L^2_n into L^2_m by the conditions

(1.1) T is *linear*, that is, $T(c_1h_1 + c_2h_2) = c_1T(h_1) + c_2T(h_2)$ for all $h_1, h_2 \in L^2_n$ and $c_1, c_2 \in \mathbf{C}$;

and

(1.2) T is *shift-invariant*, that is, $TM_\chi = M_\chi T$ where $\chi(z) = z$.

If moreover G has bounded analytic continuation to the unit disk, then $T = M_G$ in addition satisfies

(1.3) T is *stable*, that is, T maps $z^kH^2_n$ into $z^kH^2_m$ for k any integer.

[Of course, if T is shift-invariant (1.2), then one need check (1.3) for only one value of k.] We denote the class of all stable mappings from L^2_n to L^2_m as $\mathbf{S}_{m\times n}$. We formulate the interpolation constraints (ii) and $(\mathrm{ii})_*$ as follows.

(1.4) Suppose $f \in L^2_n$ is such that

$$f(z) - \sum_{j=1}^{k} c_j z^k (z - z_j)^{-1} x_j \in z^k H^2_n$$

for some choice of constants $c_j \in \mathbf{C}$ $(1 \le j \le k)$. Then

$$(Tf)(z) - \sum_{j=1}^{k} c_j z^k (z - z_j)^{-1} y_j \in z^k H^2_m$$

and $\xi_p^* T(f)(w_p) = \eta_p^* f(w_p)$ for $1 \le p \le k'$.

Then an operator-theoretic reformulation of $\mathrm{NP} \cap (\mathrm{NP})_*$ is: Find all operators $T: L^2_n \to L^2_m$ which satisfy (1.1)–(1.4). A more general operator Nevanlinna–Pick problem is obtained by dropping the linearity or shift-invariance conditions, for example,

(ONP) Find all operators $T: L^2_n \to L^2_m$ which satisfy conditions (1.3) and (1.4) together with $\|T\| \le 1$, where $\|T\| = \sup\{\|T(h)\|_{L^2_m}: \|h\|_{L^2_n} \le 1\}$.

Let us say that a (possibly nonlinear) mapping $T: L^2_n \to L^2_m$ is a strict contraction provided that $\|T\| < 1$ (so $\|T(h)\|_{L^2_m} \le c\|h\|_{L^2_n}$ for all $h \in L^2_n$ for some $c < 1$); the class of all such mappings we denote by $\mathbf{B}N_{m\times n}$; $\bar{\mathbf{B}}N_{m\times n}$ denotes such mappings with $c \le 1$. We do not demand that such operators T be continuous. A more restrictive condition is to demand that T be a strict Lipschitz contraction (also known as incrementally stable operator in the engineering literature), that is,

$$\|T(h_1) - T(h_2)\|_{L^2_m} \le c\|h_1 - h_2\|_{L^2_n}$$

for all $h_1, h_2 \in L_n^2$ for some $c<1$. The set of all strict Lipschitz contractions we denote by $\mathbf{B}LN_{m\times n}$. The class of Lipschitz contractions ($c\le 1$ in the above) we denote by $\bar{\mathbf{B}}LN_{m\times n}$.

So far we have discussed only the existence question with regard to (NP)$\cap$ (NP)$_*$. The problem of parameterizing the set of all solutions is settled by the following result, at least for the case where there exist solutions with norm strictly less than one.

PROPOSITION 1.1 (see [4; 5]). *Suppose there exists some solution F of* (NP)$\cap$ (NP)$_*$ *with $\|F\|_\infty<1$. Then there exists a rational $(m+n)\times(m+n)$ function*

$$\Xi(z)=\begin{bmatrix}\Xi_{11}(z) & \Xi_{12}(z)\\ \Xi_{21}(z) & \Xi_{22}(z)\end{bmatrix}$$

such that F is a solution of (NP)$\cap$(NP)$_*$ *if and only if*

$$F=(\Xi_{11}G+\Xi_{12})(\Xi_{21}G+\Xi_{22})^{-1}$$

for some $G\in H^\infty_{m\times n}$ with $\|G\|_\infty\le 1$. The matrix function $\Xi(z)$ satisfies

$$\Xi(\bar{z}^{-1})^*J\Xi(z)=J \tag{1.5}$$

whenever both z and $\bar{z}^{-1}$ are points of analyticity, and

$$\Xi(z)^*J\Xi(z)\le J \tag{1.6}$$

whenever z is a point of analyticity with $|z|<1$, where

$$J=\begin{bmatrix}I_m & 0\\ 0 & -I_n\end{bmatrix}.$$

Moreover, Ξ can be computed explicitly from the data $\underline{\underline{z}}$, $\underline{\underline{x}}$, $\underline{\underline{y}}$, $\underline{\underline{w}}$, $\underline{\underline{\xi}}$, $\underline{\underline{\eta}}$ of the interpolation problem (*see* [6] *for a very general case*).

For

$$A=\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix},$$

any block matrix representing an operator from a space $\mathbf{H}\oplus\mathbf{K}$ (say) to itself, we let $\mathbf{G}_A$ represent the linear fractional map

$$T\to\mathbf{G}_A(T)=(A_{11}T+A_{12})(A_{21}T+A_{22})^{-1}$$

defined on all (possibly nonlinear) mappings from $\mathbf{K}$ into $\mathbf{H}$ for which the inverse mapping $(A_{21}T+A_{22})^{-1}$ exists; note that then the result $\mathbf{G}_A(T)$ is also a mapping from $\mathbf{K}$ to $\mathbf{H}$. In particular, if

$$\Xi(z)=\begin{bmatrix}\Xi_{11}(z) & \Xi_{12}(z)\\ \Xi_{21}(z) & \Xi_{22}(z)\end{bmatrix}$$

is an $(m+n)\times(m+n)$ matrix function as in the proposition, then

$$M_\Xi=\begin{bmatrix}M_{\Xi_{11}} & M_{\Xi_{12}}\\ M_{\Xi_{21}} & M_{\Xi_{22}}\end{bmatrix}$$

is a mapping from $L^2_m \oplus L^2_n$ into itself, and hence the linear fractional map $\mathbf{G}_{M_\Xi}$ is defined on all mappings T from L^2_n into L^2_m for which the inverse $(M_{\Xi_{21}}T+M_{\Xi_{22}})^{-1}$ exists. We can now state the main result of this paper. When the meaning is clear we write $\mathbf{G}_\Xi$ rather than $\mathbf{G}_{M_\Xi}$. We let $\mathbf{B}H^\infty_{m\times n}$ denote the set of all linear shift-invariant operators in $\mathbf{S}_{m\times n}\cap\mathbf{B}N_{m\times n}$; thus this is the class of all operators M_G where $G\in H^\infty_{m\times n}$ and $\|G\|_\infty<1$. Again we often abuse notation and let G stand also for the operator M_G. The closure of $\mathbf{B}H^\infty_{m\times n}$, consisting of $G\in H^\infty_{m\times n}$ with $\|G\|\le 1$, we denote by $\bar{\mathbf{B}}H^\infty_{m\times n}$.

THEOREM 1.2. *Let there be given disjoint sets of complex numbers* $\{z_1,\dots,z_k\}$ *and* $\{w_1,\dots,w_{k'}\}$ *in the open unit disk, sets of vectors* $\{x_1,\dots,x_k\}$ *and* $\{n_1,\dots,n_{k'}\}$ *in* $\mathbf{C}^n$, *and sets of vectors* $\{y_1,\dots,y_k\}$ *and* $\{\xi_1,\dots,\xi_{k'}\}$ *in* $\mathbf{C}^m$.

(1) *Then the following three conditions are equivalent:*
  - (i) *The problem* (NP)$\cap$(NP)$_*$ *has a solution* F *with* $\|F\|_\infty\le 1$, *that is, there exists a* $T\in\bar{\mathbf{B}}H^\infty_{m\times n}$ *which satisfies the interpolation conditions* (1.4).
  - (ii) *There exists a (possibly not shift-invariant and/or nonlinear)* $T\in\bar{\mathbf{B}}\mathbf{L}N_{m\times n}\cap\mathbf{S}_{m\times n}$ *which satisfies the interpolation conditions* (1.4).
  - (iii) *There exists a (possibly not shift-invariant and/or nonlinear)* $T\in\bar{\mathbf{B}}N_{m\times n}\cap\mathbf{S}_{m\times n}$ *which satisfies the interpolation conditions* (1.4).

(2) *Suppose* (NP)$\cap$(NP)$_*$ *has solution* F *with* $\|F\|_\infty<1$, *or (equivalently) that there exists a stable* T *in* $\mathbf{BL}N_{m\times n}$ *or* $\mathbf{B}N_{m\times n}$ *satisfying* (1.4). *Let the matrix function*

$$\Xi=\begin{bmatrix}\Xi_{11} & \Xi_{12}\\ \Xi_{21} & \Xi_{22}\end{bmatrix}$$

be as in Proposition 1.1. *Then:*
  - (i) *The set of all* T *in* $\bar{\mathbf{B}}H^\infty_{m\times n}$ *satisfying* (1.4) *is parameterized as* $\mathbf{G}_\Xi(\bar{\mathbf{B}}H^\infty_{m\times n})$.
  - (ii) *The set of all (possibly nonlinear and/or not shift-invariant)* T *in* $\bar{\mathbf{B}}\mathbf{L}N_{m\times n}\cap\mathbf{S}_{m\times n}$ *satisfying* (1.4) *is parameterized as* $\mathbf{G}_\Xi(\bar{\mathbf{B}}\mathbf{L}N_{m\times n}\cap\mathbf{S}_{m\times n})$.

Moreover, in each of (i) *and* (ii) *in* (2), *the assertions continue to hold with* $\mathbf{B}$ *in place of* $\bar{\mathbf{B}}$ *throughout. Also* $\mathbf{G}_\Xi(T')$ *is shift-invariant if and only if* T' *is shift-invariant.*

While the above theorem says that every interpolation set in $\mathbf{B}H^\infty_{m\times n}$ arises as the range $\mathbf{G}_\Xi(\mathbf{B}H^\infty_{m\times n})$ of a linear fractional map $\mathbf{G}_\Xi$, the converse is also true in the linear case (see [9; 10]). The converse also holds in the nonlinear case. To recover the class of simple first-order interpolation conditions (1.4), we must assume that Ξ has only simple nonintersecting zeros and poles; more general interpolation problems will be considered in the next section.

THEOREM 1.3. *Suppose* $\Xi(z)$ *is a rational* $(m+n)\times(m+n)$ *matrix function satisfying* (1.5) *and* (1.6) *such that the zeros and poles of* Ξ *are all simple and nonintersecting. Suppose also that* $\Xi_{22}^{-1}\Xi_{21}\in H^\infty_{m\times n}$. *Then there is a set of interpolating conditions* (1.4) *such that the range of* $\mathbf{G}_\Xi$ *on* $\bar{\mathbf{B}}\mathbf{L}N_{m\times n}$ *equals the set of all Lipschitz contractions satisfying* (1.4).

The reader should note that we do not obtain a linear fractional map parameterization of the class of all nonlinear T in $\bar{\mathbf{B}}N_{m\times n}$ satisfying the interpolation conditions (1.4). The difficulty is that graphs of general nonlinear contractions do not have a coordinate-free Krein space description as do graphs of Lipschitz-1 functions and of linear contractions (see Section 3). A similar remark applies to the parameterization results in Theorem 2.2 for the more general interpolation problem discussed there.

The basic principle that nonlinear solutions exist if and only if the linear ones exist [(i) $\Leftrightarrow$ (iii) in Theorem 1.2(1)] already appears in the article of Khargonekar and Poola [11] as a small piece in the solution of a more involved control theory problem. Also, linear fractional maps have appeared before in a nonlinear context in the work of Desoer and Liu [8] and Anantharam and Desoer [1] to parameterize stabilizing nonlinear controllers of nonlinear plants; here, however, there is no a priori bound on a norm or a Lipschitz constant. Our Grassmannian formulation and linear fractional parameterization of all solutions with Lipschitz constant at most 1 appears to be new. Also we feel that without an article such as this the fine ideas of [11], [8], and [1] might well be lost to the mathematical Nevanlinna–Pick community.

2. The general interpolation problem. In this section we rewrite the interpolation problem (NP)$\cap$(NP)$_*$ in an equivalent form which then leads to a more general problem with no extra burden in notation. This second form of the problem is also more convenient for our methods of proof to be presented in the next section.

Let $\phi(z)$ and $\psi(z)$ be rational $m\times m$ and $n\times n$ phase functions respectively; thus $\phi(z)$ and $\psi(z)$ are respectively $m\times m$ and $n\times n$ unitary valued for $|z|=1$. Let $K(z)$ be a rational $m\times n$ matrix function. Then a more general matrix Nevanlinna–Pick interpolation problem is:

(GNP) Given ϕ, ψ, K as above, find $F\in K+\phi H^\infty_{m\times n}\psi$ with $\|F\|_\infty\leq 1$.

To recover (NP)$\cap$(NP)$_*$ from this more general problem (GNP), one first chooses $\phi(z)$ to be a finite matrix Blaschke product having simple zeros at the points $\{w_1,\dots,w_{k'}\}$ with left zero vector ξ_j^* at w_j:

$$\xi_j^*\phi(w_j)=0 \quad \text{for } 1\leq j\leq k'.$$

Then choose $\psi(z)$ to be the finite matrix Blaschke product with simple zeros at $\{z_1,\dots,z_k\}$ and with right zero vector x_j at z_j:

$$\psi(z_j)x_j=0 \quad \text{for } 1\leq j\leq k.$$

Finally choose K to be any $H^\infty_{m\times n}$ function (not necessarily with $\|K\|_\infty\leq 1$) which satisfies the interpolation conditions

$$K(z_j)x_j=y_j,\quad 1\leq j\leq k;$$
$$\xi_j^*K(w_j)=\eta_j^* \quad 1\leq j\leq k'.$$

With this choice of ϕ, ψ, K, (GNP) reduces to the problem (NP)$\cap$(NP)$_*$ discussed in Section 1. By taking ϕ and ψ to be more general matrix Blaschke products in

(GNP), we have a convenient way to discuss higher-order Nevanlinna–Pick interpolation problems involving interpolation of derivatives as well as functional values. By choosing $\phi = I_m$, $\psi = I_n$, and $K =$ a general function in $L^\infty_{m\times n}$, we obtain the matrix Nehari problem.

It is easy to rewrite (GNP) in operator-theoretic form: Find a linear shift-invariant operator $T \in M_K + M_\phi \cdot \mathbf{S}_{m\times n} \cdot M_\psi$ with $\|T\| \le 1$. This leads us to the nonlinear version of (GNP):

(NGNP) Find $T \in M_K + M_\phi \mathbf{S}_{m\times n} \cdot M_\psi$ with $\|T\| \le 1$.

We can also demand that T be shift-invariant and/or a (strict) Lipschitz contraction. With the special choice of ϕ, ψ, K mentioned above it is easy to check that the condition $T \in M_K + M_\phi \cdot \mathbf{S}_{m\times n} \cdot M_\psi$ is equivalent to T satisfying the interpolation conditions (1.4).

Results for the more general linear Nevanlinna–Pick interpolation problem (GNP) parallel those mentioned in Section 1 for (NP)$\cap$(NP)$_*$; for a fuller discussion see [4]. The following summarizes the situation.

PROPOSITION 2.1. *Let K, ϕ, ψ be as in* (GNP). *Let $\Gamma : \psi^{-1}H^2_n \to \phi H^{2\perp}_m$ be defined by*

$$\Gamma = P_{\phi H^{2\perp}_m} M_K \mid \psi^{-1} H^2_n. \tag{2.1}$$

Then there exist solutions F of (GNP) *with $\|F\|_\infty < 1$ if and only if $\|\Gamma\| < 1$. Moreover, in this case there exists a rational $(m+n)\times(m+n)$ matrix function*

$$\Xi(z) = \begin{bmatrix} \Xi_{11}(z) & \Xi_{12}(z) \\ \Xi_{21}(z) & \Xi_{22}(z) \end{bmatrix}$$

satisfying (1.5) *such that F is a solution of* (GNP) *if and only if $F = \mathbf{G}_\Xi(G)$ for some $G \in H^\infty_{m\times n}$ with $\|G\|_\infty \le 1$.*

The main goal of this paper is to obtain the analogues of Theorems 1.2 and 1.3 for the more general nonlinear interpolation problem (NGNP). Precisely they are the following.

THEOREM 2.2. *Let K, ϕ, ψ be as in* (GNP).

(1) *The following three conditions are equivalent.*
   (i) *There exists $F \in K + \phi H^\infty_{m\times n}\psi$ with $\|F\|_\infty \le 1$, that is, $\|\Gamma\| \le 1$ where Γ is as in* (2.1).
   (ii) *There exists a* (*possibly not shift-invariant and/or nonlinear*) *$T \in [M_K + M_\phi \mathbf{S}_{m\times n} M_\psi] \cap \bar{\mathbf{B}}\mathbf{L}N_{m\times n}$.*
   (iii) *There exists a* (*possibly not shift-invariant and/or nonlinear*) *$T \in [M_K + M_\phi \mathbf{S}_{m\times n} M_\psi] \cap \bar{\mathbf{B}}N_{m\times n}$.*

   Suppose there exists a solution F of (GNP) *with $\|F\|_\infty < 1$, or* (*equivalently*) *that there is a solution T of* (NGNP) *in either $\mathbf{B}\mathbf{L}N_{m\times n}$ or $\mathbf{B}N_{m\times n}$, so $\|\Gamma\| < 1$.*

(2) *Let the matrix function*

$$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} \\ \Xi_{21} & \Xi_{22} \end{bmatrix}$$

be as in Proposition 2.1. *Then:*

(i) $[M_K + M_\phi \mathbf{S}_{m\times n} M_\psi] \cap \bar{\mathbf{B}}L^\infty_{m\times n} = \mathbf{G}_\Xi(\bar{\mathbf{B}}\mathbf{L}H^\infty_{m\times n})$.

(ii) $[M_K + M_\phi \mathbf{S}_{m\times n} M_\psi] \cap \bar{\mathbf{B}}\mathbf{L}N_{n\times m} = G_\Xi(\bar{\mathbf{B}}\mathbf{L}N_{m\times n} \cap \mathbf{S}_{m\times n})$.

Shift-invariant solutions of (NGNP) in $\bar{\mathbf{B}}\mathbf{L}N_{m\times n}$ are obtained by plugging the shift-invariant operators in $\bar{\mathbf{B}}\mathbf{L}N_{m\times n} \cap \mathbf{S}_{m\times n}$ into $\mathbf{G}_\Xi$.

THEOREM 2.3. *Suppose* $\Xi(z)$ *is a rational* $(m+n)\times(m+n)$ *matrix function satisfying* (1.5) *which is regular on the unit circle. Suppose also that* $\Xi_{22}^{-1}\Xi_{21} \in H^\infty_{n\times m}$. *Then there exist a rational* $m\times n$ *matrix function* K, *an* $m\times m$ *matrix Blaschke product* ϕ, *and an* $n\times n$ *matrix Blaschke product* ψ *such that*

$$\mathbf{G}_\Xi(\bar{\mathbf{B}}\mathbf{L}N_{m\times n} \cap \mathbf{S}_{m\times n}) = [M_K + M_\phi \mathbf{S}_{m\times n} M_\psi] \cap \bar{\mathbf{B}}\mathbf{L}N_{m\times n}.$$

The first part of Theorem 2.2 is straightforward, so we give its proof now.

Proof of Theorem 2.2(1). The idea of the proof is essentially the same as that of Khargonekar and Poola [11] for the analogous result in their context. Clearly it suffices to prove only (iii) $\Rightarrow$ (i) in (1). Thus suppose there exists a

$$T = M_K + M_\phi S M_\psi \in [M_K + M_\phi \mathbf{S} M_\psi] \cap \bar{\mathbf{B}}N_{m\times n}.$$

Then for any $h \in \phi^{-1}H^2_n$,

$$\begin{aligned} \|T\|\,\|h\| \geq \|T(h)\| &= \|(M_K + M_\phi S M_\psi)(h)\| \\ &\geq \|P_{\phi H^{2\perp}_m}(M_K + M_\phi S M_\psi)(h)\| \\ &= \|P_{\phi H^{2\perp}_m}(Kh)\| = \|\Gamma h\|, \end{aligned}$$

so $\|\Gamma\| \leq \|T\| \leq 1$ and (i) follows. □

3. Krein space preliminaries. A Krein space $\mathbf{K}$ by definition is the direct sum $\mathbf{K} = \mathbf{K}_+ \oplus \mathbf{K}_-$ of two Hilbert spaces $\mathbf{K}_\pm$ with indefinite inner product [,] given by

$$[k_+ \oplus k_-, k'_+ \oplus k'_-] = \langle k_+, k'_+\rangle_{\mathbf{K}_+} - \langle k_-, k'_-\rangle_{\mathbf{K}_-}.$$

A general reference for such spaces is Bognar's book [7]. The invariant object of interest is only the pair $(\mathbf{K}, [\cdot,\cdot])$, that is, a linear space $\mathbf{K}$ with an indefinite inner product. A decomposition of $\mathbf{K}$ of the form $\mathbf{K}_+ \oplus \mathbf{K}_-$ is called a *fundamental* decomposition and in general is not unique. In applications one may want to be free to pick a fundamental decomposition to work with. The best *intrinsic* characterization of a space is the following (see [7]): An indefinite inner product space $(\mathbf{K}, [\ ,\ ])$ is a Krein space if and only if there is a Hilbert space inner product $\langle\cdot,\cdot\rangle$ on $\mathbf{K}$ of the form $\langle x, y\rangle = [Wx, y]$ with W invertible.

We say that a subset $\mathbf{P}$ of the Krein space $(\mathbf{K}, [\cdot,\cdot])$ is *positive* if $[x,x] \geq 0$ for all $x \in \mathbf{P}$. A (linear) subspace $\mathbf{P}$ is said to be *maximal positive* if it is positive and not contained in any larger positive subspace. We say that the linear subspace is *uniformly positive* if the restriction of $[\cdot,\cdot]$ to $\mathbf{P}$ makes $\mathbf{P}$ a Hilbert space. A subset $\mathbf{P}$ is said to be *incrementally positive* if the difference set $\Delta\mathbf{P} = \{x_1 - x_2 : x_1, x_2 \in \mathbf{P}\}$ is a positive set. We say that $\mathbf{P}$ is *maximal incrementally positive* if it is incrementally positive and not contained in any larger incrementally positive subspace.

When we replace $[\cdot,\cdot]$ by $-[\cdot,\cdot]$ in the above, we get the notions of *negative set* or *subspace, maximal negative subspace, uniformly negative subspace, incrementally negative subset* and *maximal incrementally negative.* We shall be primarily interested in "negative" rather than "positive." For any subset $\mathbf{S}$, we let $\mathbf{S}'$ denote its $[\ ,\ ]$-orthogonal complement

$$\mathbf{S}' = \{x \in \mathbf{K}: [y,x] = 0 \text{ for all } y \in \mathbf{S}\}.$$

A subspace $\mathbf{G}$ is said to be *regular* (or *ortho-complemented*) if $\mathbf{G}+\mathbf{G}'=\mathbf{K}$; equivalently $\mathbf{G}\dot{+}\mathbf{G}'=\mathbf{K}$ (i.e., $\mathbf{G}+\mathbf{G}'=\mathbf{K}$ and $\mathbf{G}\cap\mathbf{G}'=(0)$). This is the precise situation when $[\ ,\ ]$ restricted to $\mathbf{G}$ makes $\mathbf{G}$ a Krein space in its own right. The uniformly positive (resp., uniformly negative) subspaces are precisely the positive (resp., negative) subspaces which are regular. If $\mathbf{P}$ is uniformly maximal positive then its orthogonal complement $\mathbf{N}=\mathbf{P}'$ is uniformly maximal negative and $\mathbf{K}=\mathbf{P}\dot{+}\mathbf{N}$ is a fundamental decomposition of $\mathbf{K}$; moreover every fundamental decomposition arises in this way. Equivalently, one could start with a uniformly maximal negative subspace $\mathbf{N}$ and set $\mathbf{P}=\mathbf{N}'$.

Let us fix a fundamental decomposition $\mathbf{K}=\mathbf{K}_+\oplus\mathbf{K}_-$ and write $\mathbf{K}$ in column notation

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_+ \\ \mathbf{K}_- \end{bmatrix}.$$

Then it is well known that negative subspaces $\mathbf{G}$ are those of the form

$$\mathbf{G} = \begin{bmatrix} T \\ I \end{bmatrix} \mathbf{D}$$

for some subspace $\mathbf{D}\subset\mathbf{K}_-$ and linear contraction operator $T: \mathbf{D}\to\mathbf{K}_+$ (with respect to the Hilbert space norms on $\mathbf{D}\subset\mathbf{H}_-$ and $\mathbf{K}_+$). Also $\mathbf{G}$ is maximal negative if and only if $\mathbf{D}=\mathbf{K}_-$. Equivalently $\mathbf{G}$ is a maximal negative subspace if and only if $\mathbf{G}$ is a negative subspace and

$$\mathbf{G} + \begin{bmatrix} \mathbf{K}_+ \\ 0 \end{bmatrix} = \mathbf{K}.$$

By the remarks above concerning fundamental decompositions we may rephrase this in yet another way: $\mathbf{G}$ *is a maximal negative subspace if and only if* $\mathbf{G}$ *is a negative subspace such that* $\mathbf{G}+\mathbf{P}=\mathbf{K}$ *for some* (*or equivalently any*) *maximal uniformly positive subspace* $\mathbf{P}$. All these facts have been reviewed in greater detail elsewhere (see [4; 3; 10]).

Less standard are the corresponding facts for incrementally negative subsets. We state the result formally as follows.

LEMMA 3.1. (a) *A subset* $\mathbf{G}\subset\mathbf{K}$ *is incrementally negative if and only if*

$$\mathbf{G} = \begin{bmatrix} T \\ I \end{bmatrix} \mathbf{D}$$

for some $\mathbf{D}\subset\mathbf{K}_-$ *and* (*possibly nonlinear*) *Lipschitz*-1 *mapping* $T: \mathbf{D}\to\mathbf{K}_+$ (*i.e.,* $\|T(x_1)-T(x_2)\|_{\mathbf{K}_+} \le \|x_1-x_2\|_{\mathbf{K}_-}$ *for all* $x_1, x_2 \in \mathbf{D}$).

(b) **G** *is maximal incrementally negative if and only if* $\mathbf{D}=\mathbf{K}_-$, *or equivalently,* $\mathbf{G}+\mathbf{P}=\mathbf{K}$ *for some* (*or equivalently any*) *maximal uniformly positive subspace* **P**.

Thus Lipschitz-1 mappings play the same role with respect to incrementally negative subsets as linear contraction operators play with respect to negative subspaces.

Proof. The only nontrivial part is the assertion in (b) that $\mathbf{D}=\mathbf{K}_-$ if **G** is maximal incrementally negative. Equivalently any Lipschitz-1 mapping T from a subset $\mathbf{D}\subset\mathbf{K}_-$ into $\mathbf{K}_+$ can be extended to a Lipschitz mapping defined on all of $\mathbf{K}_-$ into $\mathbf{K}_+$. The validity of this last statement is one of the main results of [13] (see also [12]). □

An indefinite inner product space (**K**, [,]) we say is a *pseudo-Krein space* if the quotient space $\mathbf{K}/(\mathbf{K}\cap\mathbf{K}')$ is a Krein space in the indefinite inner product induced by [,]. A more intrinsic condition is to say that (**K**, [,]) is a pseudo-Krein space if there is a Hilbert space inner product $\langle\ ,\ \rangle$ on **K** of the form $\langle x,y\rangle=[Wx,y]$, where W is a bounded operator with closed range. The following extension of Lemma 3.1 is obtained simply by applying Lemma 3.1 to the Krein space $\mathbf{K}/(\mathbf{K}\cap\mathbf{K}')$. For pseudo-Krein spaces there is a distinction between the notion of "maximal uniformly positive subspace" (i.e., a uniformly positive subspace not contained in any larger uniformly positive subspace) and "uniformly positive and maximal positive subspace." The correct condition in Lemma 3.2 is "maximal uniformly positive."

LEMMA 3.2. *Let* **K** *be a pseudo-Krein space. Then a subset* $\mathbf{G}\subset\mathbf{K}$ *is maximal incrementally negative if and only if* $\mathbf{G}+\mathbf{P}=\mathbf{K}$ *for some* (*or equivalently any*) *maximal uniformly positive subspace* **P**.

4. The Grassmannian approach. The goal of this section is to derive the remaining portions of Theorem 2.2 concerning the linear fractional map parameterizations of solutions of the interpolation problem (NGNP) in various classes. The method used here to analyze (NGNP) is that developed by Ball and Helton in [4]; the basic idea is to translate the operator-theoretic problem to a problem about the geometry of subspaces of a Krein space which in turn is easier to analyze. Here we simply adapt the development there to the present nonlinear setting.

The key observation is that the error class $M_\phi \mathbf{S}_{m\times n} M_\psi$ can be characterized as those operators E for which

$$E(\psi^{-1}\chi^k H_n^2)\subset \phi\chi^k H_m^2 \tag{4.1}$$

for $k=0, \pm 1, \pm 2, \ldots$; this is a simple consequence of the definition of the class $\mathbf{S}_{m\times n}$ of stable operators. This suggests that for T any operator in **N** we let $\mathbf{G}_T^k$ be its graph space with domain $\psi^{-1}\chi^k H_m^2$:

$$\mathbf{G}_T^k=\left\{\begin{bmatrix} T(h)\\ h\end{bmatrix} : h\in\psi^{-1}\chi^k H_n^2\right\}. \tag{4.2}$$

Note that $\mathbf{G}_T^k$ is a (in general nonlinear) manifold in $L_m^2 \oplus \psi^{-1}\chi^k H_n^2$. For each $k=0, \pm 1, \ldots$, we form the auxiliary subspace $\mathbf{M}^k$ of $L_m^2 \oplus \psi^{-1}\chi^k H_n^2$ given by

$$\mathbf{M}^k = \begin{bmatrix} M_K \\ I \end{bmatrix} \psi^{-1}\chi^k H_n^2 + \begin{bmatrix} \phi\chi^k H_m^2 \\ 0 \end{bmatrix}. \tag{4.3}$$

Note that $\mathbf{M}^k = M_{\chi^k}\mathbf{M}^0$ due to the shift-invariance of the various operators in its definition. From the observation (4.2) we deduce that $T \in M_K + M_\phi \mathbf{S}_{m\times n} M_\chi$ if and only if

$$\mathbf{G}_T^k \subset \mathbf{M}^k \tag{4.4}$$

for $k=0, \pm 1, \ldots$. To express the condition $T \in \bar{\mathbf{B}}\mathbf{L}N_{m\times n}$ in terms of the sequence of graph spaces $\{\mathbf{G}_T^k;\ k=0, \pm 1, \ldots\}$ we must introduce a sequence of Krein spaces $\{(\mathbf{K}^k, [\ ,\ ]_J);\ k=0, \pm 1, \ldots\}$, where

$$\mathbf{K}^k = L_m^2 \oplus \psi^{-1}\chi^k H_n^2. \tag{4.5}$$

By Lemma 3.1 we see that if $T \in \bar{\mathbf{B}}\mathbf{L}N_{m\times n}$ then

$$\mathbf{G}_T^k \text{ is } \mathbf{K}^k\text{-maximal incrementally negative} \tag{4.6}$$

for $k=0, \pm 1, \ldots$. If all the graph spaces $\mathbf{G}_T^k$ are coming from the fixed mapping T, then we necessarily have the nesting condition

$$\mathbf{G}_T^{k+1} \subset \mathbf{G}_T^k \tag{4.7}$$

for $k=0, \pm 1, \ldots$. We are now ready to state the converse.

PROPOSITION 4.1. *Suppose* $\{\mathbf{G}^k : k=0, \pm 1, \ldots\}$ *is a sequence of subsets of* L_{m+n}^2. *Then* $\mathbf{G}^k = \mathbf{G}_T^k$ *(as defined in* (4.2)) *for some* $T \in [M_K + M_\phi \mathbf{S}_{m\times n} M_\psi] \cap \bar{\mathbf{B}}\mathbf{L}N_{m\times n}$ *if and only if, for* $k=0, \pm 1, \ldots$,

(i) $\mathbf{G}^k \subset \mathbf{M}^k$, *where* $\mathbf{M}^k$ *is as in* (4.3);
(ii) $\mathbf{G}^k$ *is* $\mathbf{K}^k$*-maximal incrementally negative, where* $\mathbf{K}^k$ *is as in* (4.5); *and*
(iii) $\mathbf{G}^{k+1} \subset \mathbf{G}^k$.

Moreover, T is uniquely determined by the sequence $\{\mathbf{G}^k : k=0, \pm 1, \ldots\}$. *We also have the equivalences:*

(1) *T is shift-invariant* $\Leftrightarrow \mathbf{G}^k = M_{\chi^k}\mathbf{G}^0$ *for all k.*
(2) *T is linear* $\Leftrightarrow \mathbf{G}^k$ *is a linear subspace for all k.*

Proof. The necessity of (i), (ii), and (iii) was established above. Conversely, suppose $\mathbf{G}^k$ satisfies (i), (ii), and (iii). By Lemma 3.1, condition (ii) implies that $\mathbf{G}^k$ has the form

$$\mathbf{G}^k = \begin{bmatrix} T^k \\ I \end{bmatrix} \psi^{-1}\chi^k H_n^2$$

for a unique Lipschitz-1 mapping $T^k : \psi^{-1}\chi^k H_n^2 \to L_m^2$. By the nesting property (iii), T^k is an extension of T^{k+1}, so there is a single Lipschitz-1 mapping T defined on $\bigcup_{k=-\infty}^{\infty} \psi^{-1}\chi^k H_n^2$ such that each T_k is a restriction of T. Since any Lipschitz-1 mapping is continuous and $\bigcup_{k=-\infty}^{\infty} \psi^{-1}\chi^k H_n^2$ is dense in L_n^2, we see that T has a unique Lipschitz-1 extension to the whole space L_n^2. Finally condition (i) in turn guarantees that $T \in M_K + M_\phi \mathbf{S}_{m\times n} M_\psi$. The equivalences (1) and (2) are now routine to verify. □

We are now ready to analyze when $[M_K + M_\phi \mathbf{S}_{m\times n} M_\psi] \cap \bar{\mathbf{B}}\mathbf{L}N_{m\times n}$ is nonempty. We first analyze the easier problem where the nesting condition (iii) in Proposition 4.1 is dropped.

LEMMA 4.2. *Let K, ϕ, ψ be given as above and let k be any integer. Then there exists a subset $\mathbf{G}^k$ satisfying* (i) *and* (ii) *of Lemma* 4.1 *if and only if* $\|\Gamma\| \le 1$, *where* $\Gamma: \psi^{-1} H_n^2 \to \phi H_m^{2\perp}$ *is given by* (2.1). [*In particular, the condition is independent of k.*]

Proof. Suppose $\mathbf{G}^k \subset \mathbf{M}^k$ and $\mathbf{G}^k$ is $\mathbf{K}^k$-maximal incrementally negative. By Lemma 3.1,

$$\mathbf{G}^k + \begin{bmatrix} L_m^2 \\ 0 \end{bmatrix} = \mathbf{K}^k = \begin{bmatrix} L_m^2 \\ \psi^{-1}\chi^k H_n^2 \end{bmatrix}. \tag{4.8}$$

When we intersect each side of this equation with $\mathbf{M}^k$, since $\mathbf{G}^k \subset \mathbf{M}^k$ we obtain

$$\mathbf{G}^k + \begin{bmatrix} \phi\chi^k H_m^2 \\ 0 \end{bmatrix} = \mathbf{M}^k. \tag{4.9}$$

Since K, ψ, and ϕ are rational, one can show that at worst $\mathbf{M}^k$ is a pseudo-Krein space in the inner product $[\ ,\ ]_J$. Since $\mathbf{G}^k \subset \mathbf{M}^k \subset \mathbf{K}^k$, $\mathbf{G}^k$ being $\mathbf{K}^k$-maximal incrementally negative forces $\mathbf{G}^k$ to be $\mathbf{M}^k$-maximal incrementally negative. It is also clear that

$$\mathbf{P}^k = \begin{bmatrix} \phi\chi^k H_m^2 \\ 0 \end{bmatrix}$$

is a uniformly positive subspace of $\mathbf{M}^k$. Now by Lemma 3.2, (4.9) implies that $\mathbf{P}^k$ is a $\mathbf{M}^k$-maximal uniformly positive subspace.

Conversely, suppose that $\mathbf{P}^k$ is a $\mathbf{M}^k$-maximal uniformly positive subspace and let $\mathbf{G}^k$ be any $\mathbf{M}^k$-maximal incrementally negative subspace. Then, by the converse side of Lemma 3.2, equation (4.9) holds. But (4.9) clearly implies (4.8). Since $\begin{bmatrix} L_m^2 \\ 0 \end{bmatrix}$ is a $\mathbf{K}^k$-maximal uniformly positive subspace, Lemma 3.1 implies that $\mathbf{G}^k$ is $\mathbf{K}^k$-maximal incrementally negative. We have established: There exist $\mathbf{K}^k$-maximal incrementally negative subsets contained in $\mathbf{M}^k$ if and only if $\mathbf{P}^k$ is $\mathbf{M}^k$-maximal uniformly positive.

It remains only to see that this last condition is equivalent to $\|\Gamma\| \le 1$. Let us introduce the operator $\Gamma_k: \psi^{-1}\chi^k H_n^2 \to \phi\chi^k H_m^2$ by $\Gamma_k = M_{\chi^k} \Gamma M_{\chi^{-k}}$. Since M_χ is unitary, $\|\Gamma_k\| = \|\Gamma\|$ for all k. By using Γ_k we get a $[\ ,\ ]_J$-orthogonal decomposition of $\mathbf{M}^k$:

$$\mathbf{M}^k = \begin{bmatrix} \Gamma_k \\ I \end{bmatrix} \psi^{-1}\chi^k H_n^2 \oplus_J \mathbf{P}^k.$$

From this decomposition it is clear that $\mathbf{P}^k$ is $\mathbf{M}^k$-maximal uniformly positive if and only if $\begin{bmatrix} \Gamma_k \\ I \end{bmatrix} \psi^{-1}\chi^k H_n^2$ is a negative subspace, or (equivalently) if and only if $\|\Gamma\| = \|\Gamma_k\| \le 1$. The lemma follows. □

By Lemma 4.2 combined with 4.1, we know that $\|\Gamma\| \le 1$ is necessary for $[M_K + M_\phi \mathbf{S} M_\psi] \cap \bar{\mathbf{B}}\mathbf{L}N_{m\times n}$ to be nonempty. To impose additional structure on the interpolation problem in its Grassmannian formulation (such as the nesting property (iii) in Lemma 4.1) it is convenient to assume $\|\Gamma\| < 1$; the case $\|\Gamma\| \le 1$ can then be handled by approximation. Then by the results of [4] as summarized in Proposition 2.1, there is a rational $(m+n) \times (m+n)$ matrix function $\Xi(z)$ such that (among other things)

$$\mathbf{M}^k = \Xi \chi^k H^2_{m+n} \tag{4.10}$$

and

$$\Xi^* J \Xi = J \tag{4.11}$$

(where $\Xi^*(z) = \Xi(\bar{z}^{-1})^*$) for all k. We now can prove the parameterization results in Theorem 2.2.

Proof of Theorem 2.2(2). By Proposition 4.1, the angle operator-graph space correspondence $T \leftrightarrow \{\mathbf{G}^k_T\colon k=0, \pm 1, \dots\}$ gives a one-to-one correspondence between mappings

$$T \in [M_K + M_\phi \mathbf{S}_{m\times n} M_\psi] \cap \bar{\mathbf{B}}\mathbf{L}N_{m\times n}$$

and sequences $\{\mathbf{G}^k\colon k=0, \pm 1, \dots\}$ of subsets of L^2_{m+n} satisfying (i), (ii), and (iii). Thus, characterizing the set of such mappings T is equivalent to characterizing such sequences $\{\mathbf{G}^k\colon k=0, \pm 1, \dots\}$. By Lemma 4.2, since $\|\Gamma\| < 1$, conditions (i) and (ii) simplify to

$$\mathbf{G}^k \text{ is } \mathbf{M}^k\text{-maximal incrementally negative.} \tag{4.12}$$

By properties (4.10) and (4.11) of Ξ, the operator M_Ξ of multiplication by Ξ is a $[\ ,\ ]_J$-unitary transformation of $\chi^k H^2_{m+n}$ onto $\mathbf{M}^k$ for all k. Hence $\mathbf{G}^k$ satisfies (4.12) if and only if $\mathbf{G}^k = M_\Xi \cdot \mathbf{G}^k_1$, where

$$\mathbf{G}^k_1 \text{ is } (\chi^k H^2_{m+n})\text{-maximal incrementally negative.} \tag{4.13}$$

The nesting property (iii) on $\mathbf{G}^k$ translates to a nesting condition on the pullbacks $\mathbf{G}^k_1$:

$$\mathbf{G}^{k+1}_1 \subset \mathbf{G}^k_1 \tag{4.14}$$

for all k. A fundamental decomposition for the Krein space $(\chi^k H^2_{m+n}, [\ ,\ ]_J)$ is

$$\chi^k H^2_{m+n} = \begin{bmatrix} \chi^k H^2_m \\ 0 \end{bmatrix} \oplus_J \begin{bmatrix} 0 \\ \chi^k H^2_n \end{bmatrix}.$$

Thus by Lemma 3.1 the subspace $\mathbf{G}^k_1$ is $(\chi^k H^2_{m+n})$-maximal incrementally negative if and only if

$$\mathbf{G}^k_1 = \begin{bmatrix} T^k_1 \\ I \end{bmatrix} \chi^k H^2_n$$

for some Lipschitz-1 mapping T^k_1 from $\chi^k H^2_n$ into $\chi^k H^2_m$. Now the nesting property (4.14) means that there must be a single Lipschitz-1 mapping T_1 from L^2_n into

L_m^2 such that each T_1^k is a restriction of T_1. Since each T_1^k maps $\chi^k H_n^2$ into $\chi^k H_m^2$ we conclude that T_1 maps $\chi^k H_n^2$ into $\chi^k H_m^2$ for every k, that is, $T_1 \in \mathbf{S}_{m\times n}$. Conversely, if

$$\mathbf{G}_1^k = \begin{bmatrix} T_1 \\ I \end{bmatrix} \chi^k H_n^2$$

for some $T_1 \in \bar{\mathbf{B}}\mathbf{LN}_{m\times n} \cap \mathbf{S}_{m\times n}$, then $\mathbf{G}_1^k$ satisfies (4.13) and (4.14) for every k. Putting all this together, we conclude that $T \in [M_K + M_\phi \mathbf{S}_{m\times n} M_\psi] \cap \bar{\mathbf{B}}\mathbf{LN}_{m\times n}$ if and only if, for some $T_1 \in \bar{\mathbf{B}}\mathbf{LN}_{m\times n} \cap \mathbf{S}_{m\times n}$,

$$\begin{bmatrix} T \\ I \end{bmatrix} \psi^{-1} \chi^k H_n^2 = M_\Xi \cdot \begin{bmatrix} T_1 \\ I \end{bmatrix} \chi^k H_n^2 \tag{4.15}$$

for all k. Since both $\bigcup_k \psi^{-1}\chi^k H_n^2$ and $\bigcup_k \chi^k H_n^2$ are dense in L_n^2, we conclude that (4.15) holding for every k is equivalent to

$$\begin{aligned} \begin{bmatrix} T \\ I \end{bmatrix} L_n^2 &= M_\Xi \begin{bmatrix} T_1 \\ I \end{bmatrix} L_n^2 \\ &= \begin{bmatrix} \mathbf{G}_\Xi(T_1) \\ I \end{bmatrix} L_n^2; \end{aligned}$$

that is, $T = \mathbf{G}_\Xi(T_1)$. We have thus established

$$[M_K + M_\phi \mathbf{S}_{m\times n} M_\psi] \cap \bar{\mathbf{B}}\mathbf{LN}_{m\times n} = \mathbf{G}_\Xi(\bar{\mathbf{B}}\mathbf{LN}_{m\times n} \cap \mathbf{S}_{m\times n}).$$

This proves (2.ii) in Theorem 2.2.

Note that the operator M_Ξ is linear and shift-invariant, and hence if $\mathbf{G}^k = M_\Xi \mathbf{G}_1^k$ as above then

$$\mathbf{G}^k = M_{\chi^k} \mathbf{G}^0 \Leftrightarrow \mathbf{G}_1^k = M_{\chi^k} \mathbf{G}_1^0$$

and

$$\mathbf{G}^k \text{ is linear} \Leftrightarrow \mathbf{G}_1^k \text{ is linear.}$$

Combining these observations with statements (1), (2), and (3) in Proposition 4.1 completes the proof of the remaining statements in Theorem 2.2. □

With the machinery now all in place, the proof of Theorem 2.3 is also easy. The first part of the proof is a simplification of the proof of Theorem 5.10 from [4].

Proofs of Theorems 1.3 *and* 2.3. Let $\Xi(z)$ be a rational $(m+n)\times(m+n)$ matrix function satisfying (1.5) for which $\Xi_{22}^{-1}\Xi_{21} \in H_{n\times m}^\infty$, and consider the M_χ-invariant subspace $\mathbf{M} = M_\Xi H_{m\times n}^2$. Define auxiliary invariant subspaces $\mathbf{P} \subset L_m^2$ and $\mathbf{N} \subset L_n^2$ by

$$\begin{bmatrix} \mathbf{P} \\ 0 \end{bmatrix} = \mathbf{M} \cap \begin{bmatrix} L_m^2 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ \mathbf{N} \end{bmatrix} = P_{\left[\begin{smallmatrix} 0 \\ L_n^2 \end{smallmatrix}\right]} \mathbf{M}.$$

Since Ξ is regular on the unit circle $\mathbf{P}$ and $\mathbf{N}$ are both full range simply invariant, so by the Beurling–Lax–Halmos theorem there are phase functions ϕ and ψ such that

$$\mathbf{P} = \phi H_m^2 \quad \text{and} \quad \mathbf{N} = \psi^{-1} H_n^2.$$

Since Ξ is rational one can show that ϕ and ψ must be rational as well. Choose K to be any rational $m\times n$ matrix function such that

$$\begin{bmatrix} K \\ I \end{bmatrix}\psi^{-1}x\in \mathbf{M}$$

for each vector $x\in \mathbf{C}^n$. Then one can show that $\mathbf{M}$ has the form

$$\mathbf{M}=\begin{bmatrix} K \\ I \end{bmatrix}\psi^{-1}H_n^2+\begin{bmatrix} \phi H_m^2 \\ 0 \end{bmatrix}.$$

From the definition of ψ we see that

$$\Xi_{21}H_m^2+\Xi_{22}H_n^2=\psi^{-1}H_n^2.$$

From the hypothesis that $\Xi_{22}^{-1}\Xi_{21}\in H_{n\times m}^\infty$ we get in turn

$$\psi^{-1}H_n^2=\Xi_{21}H_m^2+\Xi_{22}H_n^2=\Xi_{22}H_n^2. \tag{4.16}$$

As $\Xi^*J\Xi=J$, $\mathbf{M}=\Xi H_{m+n}^2$, and $0\oplus H_n^2$ is H_{m+n}^2-maximal $[\,,\,]_J$-negative, we see that

$$\begin{bmatrix} \Xi_{12} \\ \Xi_{22} \end{bmatrix}H_n^2=\Xi\begin{bmatrix} 0 \\ H_n^2 \end{bmatrix}$$

is $\mathbf{M}$-maximal $[\,,\,]_J$-negative. From (4.16) we read off that this space is $\mathbf{K}$-maximal $[\,,\,]_J$-negative, where $\mathbf{K}=L_m^2\oplus\psi^{-1}H_n^2$, and hence

$$\begin{bmatrix} \Xi_{12} \\ \Xi_{22} \end{bmatrix}H_n^2+\begin{bmatrix} L_m^2 \\ 0 \end{bmatrix}=\begin{bmatrix} L_m^2 \\ \psi^{-1}H_n^2 \end{bmatrix}.$$

When we intersect with $\mathbf{M}$ we obtain

$$\begin{bmatrix} \Xi_{12} \\ \Xi_{22} \end{bmatrix}H_n^2+\begin{bmatrix} \phi H_m^2 \\ 0 \end{bmatrix}=\mathbf{M}.$$

From this it follows that $\begin{bmatrix} \phi H_m^2 \\ 0 \end{bmatrix}$ is $\mathbf{M}$-maximal uniformly positive. As in the proof of Lemma 4.2, we get $\|\Gamma\|\le 1$ where Γ is as in (2.1). Since $\mathbf{M}$ is a $[\,,\,]_J$-unitary copy of the Krein space H_{m+n}^2 under M_Ξ, $\mathbf{M}$ is a Krein space and we must have $\|\Gamma\|<1$. We have thus constructed rational matrix functions K, ϕ, ψ such that Ξ arises from these K, ϕ, ψ exactly as does the Ξ from the prescribed K, ϕ, ψ in the proof of Theorem 2.2. We conclude that

$$\mathbf{G}_\Xi(\bar{\mathbf{B}}\mathbf{LN}_{m\times n}\cap \mathbf{S}_{m\times n})=[M_K+M_\phi \mathbf{S}_{m\times n}M_\psi]\cap \bar{\mathbf{B}}\mathbf{LN}_{m\times n},$$

and Theorem 2.3 follows.

Moreover, it is easy to see that $\Xi(z)^*J\Xi(z)\le J$ on the unit disk implies that

$$\mathbf{G}_\Xi(\bar{\mathbf{B}}H_{m\times n}^\infty)\subset \bar{\mathbf{B}}H_{m\times n}^\infty.$$

In particular, $K=\mathbf{G}_\Xi(0)\in H_{m\times n}^\infty$. The error class $\phi H_{m\times n}^\infty\psi$ is generated as a linear space by the set of all differences $\mathbf{G}_\Xi(G_1)-\mathbf{G}_\Xi(G_2)$, with $G_1, G_2\in\bar{\mathbf{B}}H_\infty$; hence $\phi H_{m\times n}^\infty\psi\subset H_{m\times n}^\infty$ and we have $\phi\in H_{m\times n}^\infty$, $\psi\in H_{n\times n}^\infty$. From

$$\Xi H^2_{m+n} = \mathbf{M} = \begin{bmatrix} \phi & K\psi^{-1} \\ 0 & \psi^{-1} \end{bmatrix} H^2_{m+n}$$

we see that Ξ has the same set of zeros and poles inside the unit disk $\mathbf{D}$ as $\left[\begin{smallmatrix} \phi & K\psi^{-1} \\ 0 & \psi^{-1} \end{smallmatrix}\right]$. Clearly the zeros of $\left[\begin{smallmatrix} \phi & K\psi^{-1} \\ 0 & \psi^{-1} \end{smallmatrix}\right]$ inside $\mathbf{D}$ coincide with the zeros of ϕ inside $\mathbf{D}$ and the poles of $\left[\begin{smallmatrix} \phi & K\psi^{-1} \\ 0 & \psi^{-1} \end{smallmatrix}\right]$ inside $\mathbf{D}$ coincide with the zeros of ψ inside $\mathbf{D}$, since K, ϕ, ψ are all analytic on $\mathbf{D}$. If Ξ has simple nonintersecting zeros and poles on $\mathbf{D}$, then ϕ and ψ have simple nonintersecting zeros on $\mathbf{D}$. We are now finally in the situation where (NGNP) specializes to the nonlinear version of interpolation problem (ONP) discussed in Section 1, and Theorem 1.3 follows as well. □

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