AN EXTENSION OF WIDDER'S THEOREM

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1. Introduction. In this paper we consider a problem concerning the boundary behavior of solutions of the one-dimensional heat equation on the strip (or the half-plane) $\mathfrak{D}_c = \mathbf{R} \times (0, c)$, where $0 < c \leq +\infty$. By a solution of the heat equation on an open set $\mathfrak{D} \subseteq \mathbf{R}^2$ we understand here a twice continuously differentiable real function $u(x, t)$, $(x, t) \in \mathfrak{D}$, such that $u_{xx} = u_t$ in $\mathfrak{D}$.

It is well known that many properties of such functions are similar to those of harmonic functions (see e.g. [8], [6], [3], [4], and [2]). One of these similarities is that nonnegative harmonic function on $\mathfrak{D}_\infty$ and nonnegative solutions of the heat equation on $\mathfrak{D}_c$ both have Poisson-type integral representations. In the "harmonic" case this fact is attributed to F. Riesz and Herglotz, and in the case of solutions of the heat equation it is a theorem due to Widder [8]. In [5] Hayman and Korenblum obtained "an extension of the Riesz–Herlotz formula" by showing that for a continuous positive nonincreasing function $k(t)$, $t > 0$, the condition

$$\int_0^1 \sqrt{k(t)/t}\, dt < +\infty$$

is equivalent to the property that each harmonic function h defined on $\mathfrak{D}_\infty$, with $h(x, t) \leq k(t)$, $t > 0$, can be represented in the form

$$h(x, t) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{t}{(x-y)^2 + t^2}\, d\left(\lim_{\tau \to 0+} \int_0^y h(z, \tau)\, dz\right) + Ct.$$

The outer integral in the above formula was originally defined by the integration-by-parts formula, but, as shown later in [7], it can be understood as a Riemann–Stieltjes integral (with respect to a function which may not necessarily be of bounded variation). The aim of this paper is to show an analogue of that result for solutions of the heat equation on $\mathfrak{D}_c$.

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2. Main results. Let $K(x, t)$ be the Gauss kernel, that is,

$$K(x, t) = \frac{1}{2\sqrt{\pi t}} \exp\left(-\frac{x^2}{4t}\right), \quad x \in R, \ t > 0.$$

In the sequel k will always denote a positive nonincreasing unbounded continuous function on $(0, +\infty)$.

THEOREM 1. *Let* $\lim_{t \to 0+} \sqrt{t}\, k(t) = 0$ *and*

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$$\int_0^\epsilon k(t)\sqrt{\frac{-\log(\sqrt{t}\,k(t))}{t}}\,dt<+\infty \tag{1}$$

for some $\epsilon>0$. *Let* u *be a solution of the heat equation on* $\mathfrak{D}_c$ $(0<c\le+\infty)$ *with* $u(x,t)\le k(t)$, $t\in(0,c)$. *Then:*

(i) *the limit*

$$\alpha(x)=\lim_{t\to0+}\int_0^x u(z,t)\,dz$$

exists and is finite for each real number x;

(ii) *for each* $t_0\in(0,c)$ *and each* $M>0$ *there is a continuous function* $\kappa\colon[0,1]\to\mathbf{R}$, $\kappa(0)=0$, *such that if* $u(0,t_0)\ge -M$ *then*

$$\alpha(x_2)-\alpha(x_1)\le\kappa(x_2-x_1)(|x_1+x_2|+1)$$

whenever $0<x_2-x_1\le1$ (κ *depends only on* k, t_0, *and* M);

(iii) *for each real number* x *we have* $\alpha(x)=[\alpha(x+)+\alpha(x-)]/2$; *and*

(iv) *for each* $(x,t)\in\mathfrak{D}_c$ *we have*

$$u(x,t)=\int_{-\infty}^{+\infty}K(x-z,t)\,d\alpha(z).$$

Note that (ii) implies that α is locally bounded and that it has one-sided limits at each point. Therefore the integral in (iv) can be understood as an improper Riemann–Stieltjes integral.

Observe that for arbitrary solution u of the heat equation on $\mathfrak{D}_c$ and for arbitrary s $(s>0)$, A $(A>0)$, and B $(B\in\mathbf{R})$, the function $\tilde{u}(x,t)=Au(s^{1/2}x,st)+B$ is a solution of the heat equation on $\mathfrak{D}_{c/s}$. Also, if k satisfies the assumptions of Theorem 1 then so does $\tilde{k}(t)=Ak(st)+B$, if $B\ge0$. Therefore Theorem 1 is a consequence of the following theorem.

THEOREM 2. *Suppose that*

$$k(t)\le\frac{1}{2e\sqrt{\pi t}},\quad 0<t\le1/16$$

and

$$J=\int_0^{1/16}k(t)\sqrt{\frac{-\log(\sqrt{t}\,k(t))}{t}}\,dt<+\infty.$$

Let $c>1$ *and let* u *be a solution of the heat equation in* $\mathfrak{D}_c$ *with* $u(x,t)\le k(t)$, $x\in\mathbf{R}$, $0<t\le1/16$, *and* $u(0,1)=0$. *Then:*

(i) *the limit*

$$\alpha(x)=\lim_{t\to0+}\int_0^x u(z,t)\,dz$$

exists and is finite for each $x\in\mathbf{R}$;

(ii) $\alpha(x_2)-\alpha(x_1)\le(|x_1+x_2|+1)\cdot\kappa(x_2-x_1)$, $0<x_2-x_1\le1$, *where* κ *is some nondecreasing continuous function on* $[0,1]$ *that depends only on* k *and is such that* $\kappa(0)=0$ *and* $\kappa(1)\le C(J+2)\log(J+2)$, *where* C *is some absolute constant*;

(iii) $\operatorname{sgn}(x)\cdot\alpha(x)\geq -D(x^2+1)\exp(x^2/4)$, *with some constant* D *depending only on* k;
(iv) $\alpha(x)=[\alpha(x-)+\alpha(x+)]/2$, $x\in\mathbf{R}$; *and*
(v) $u(t,x)=\int_{-\infty}^{+\infty}K(x-z,t)\,d\alpha(z)$, $x\in\mathbf{R}$, $0<t<1$.

The next theorem states that for k's from a large family of functions the assumptions of Theorem 1 cannot be relaxed.

THEOREM 3. *If* k *does not satisfy the assumptions of Theorem* 1 *and there are positive constants* t_0 *and* C *such that* $k(t/2)\leq Ck(t)$ *for* $0<t\leq t_0$, *then there exists a solution* u *of the heat equation on* $\mathfrak{D}_\infty$ *with* $u(x,t)\leq k(t)$, $t>0$, *such that*

$$\lim_{t\to 0+}\int_0^x u(z,t)\,dz=+\infty \quad \textit{for each } x\neq 0.$$

Theorem 3 will be derived from the following theorem, which is of some interest on its own.

THEOREM 4. *Assume that* k *satisfies the assumptions of Theorem* 3 *and that* $\epsilon>0$, $\delta>0$, *and* $L>0$ *are arbitrary. Then there exists a nonnegative continuous function* f *on* $\mathbf{R}$ *vanishing outside* $[0,\epsilon]$ *such that:*
(i) $|u(x,t)|\leq\epsilon$, $t\geq\epsilon$;
(ii) $u(x,t)\leq\epsilon k(t)$, $0<t\leq 1$; *and*
(iii) $\int_{-\infty}^{+\infty}f(z)\,dz\geq L-\delta$;
where $u(x,t)=\int_{-\infty}^{+\infty}K(x-z,t)f(z)\,dz-LK(x,t)$, $x\in\mathbf{R}$, $t>0$.

REMARK. If we assume that $k(t)=\{t^{1/2}(-\log t)[\log(-\log t)]^{\gamma}\}^{-1}$ for all small t's then k satisfies the conditions of Theorem 1 if $\gamma>3/2$.

3. Auxiliary facts. In this section k will be as specified earlier, with the additional requirement that

$$k(t)\leq\frac{1}{2e\sqrt{\pi t}},\quad 0\leq t<1/16.$$

Let M be a constant greater than or equal to 1. For each such constant and for each $t\in(0,1/16]$ let $x_M(t)$ be the positive solution of the equation

$$k(t)=MK(x,t), \tag{2}$$

that is,

$$x_M(t)=2\sqrt{t[\log M-\log(2\sqrt{\pi t}\,k(t))]}\,.$$

Note that $x_M(t)\geq 2t^{1/2}$. This inequality, together with (2) and the facts that $\partial K/\partial t>0$ if $x>(2t)^{1/2}$ and $\partial K/\partial x<0$ if $x>0$, imply that $x_M(t)$ is an increasing function of t. Clearly, x_M is continuous on $(0,1/16]$. It is also easy to see that $x_M(0+)=0$. Let us denote $t_M=x_M^{-1}$. The domain of t_M is equal to $(0,x_M(1/16)]$ and, since $x_M(1/16)\geq 1/2$, it contains the interval $(0,1/2]$. Let us extend t_M by putting $t_M(0)=0$. Some properties of t_M which follow from just-derived properties of x_M are listed in the following lemma.

LEMMA 1.

(i) *t_M is an increasing continuous function with values in the interval* $[0, 1/16]$;

(ii) $k(t_M(x)) = M \cdot K(x, t_M(x))$, $0 < x \leq 1/2$;

(iii) $t_M(x) \leq x^2/4$, $0 \leq x \leq 1/2$;

(iv) *if* $\tilde{k}(t) \leq k(t)$, $0 < t \leq 1/16$, *then* $\tilde{t}_M(x) \leq t_M(x)$, $0 \leq x \leq 1/2$, *where* $\tilde{t}_M$ *is constructed for* $\tilde{k}$ *in the same manner as* t_M *for* k;

(v) *if* $1 \leq M \leq N$ *then* $t_N(x) \leq t_M(x)$, $0 \leq x \leq 1/2$.

Let

$$I(M, s) = \int_0^s k(t_M(x))\, dx, \quad M \geq 1, \ 0 < s \leq 1/2.$$

Note that $I(M, s)$ may be infinite. By a change of a variable,

$$I(M, s) = \int_0^{t_M(s)} k(t)\, d\left[2\sqrt{t \log \frac{M}{2\sqrt{\pi t}\, k(t)}}\right].$$

LEMMA 2.

(i) *For each fixed* s, $0 < s \leq 1/2$, $I(M, s)$ *is a nondecreasing function of* M;

(ii) $I(M, s) \leq 2(\log M)^{1/2} I(1, s)$, $M \geq e$, $0 < s \leq 1/2$;

(iii) *the following four conditions are mutually equivalent:*

$$\text{(A)} \qquad J = \int_0^{1/16} k(t) \sqrt{\frac{-\log(\sqrt{t}\, k(t))}{t}}\, dt < +\infty,$$

$$\text{(B)} \qquad J_1 = \int_0^{1/16} k(t)\, d\sqrt{-t \log(\sqrt{t}\, k(t))} < +\infty,$$

(C) *there exists* $M \geq 1$ *and* $s \in (0, 1/2)$ *such that* $I(M, s) < +\infty$,

(D) $I(M, s) < +\infty$ *for all* M $(M \geq 1)$ *and* s $(0 < s \leq 1/2)$;

moreover, $J_1 \leq J/2 + C$, *where* C *is an absolute constant.*

Proof. (i) follows from Lemma 1(v).

(ii) Let $M \geq e$ and let $p = \log M + 1$. Then

$$I(M, s) = I(e^{p-1}, s) = \int_0^{t_M(s)} k(t)\, d\left[2\sqrt{t \log \frac{e^p}{2e\sqrt{\pi t}\, k(t)}}\right]$$

$$\leq \int_0^{t_M(s)} k(t)\, d\left[2\sqrt{t \log \frac{e^p}{(2e\sqrt{\pi t}\, k(t))^p}}\right] \leq 2\sqrt{\log M}\, I(1, s).$$

(iii) To prove the equivalence of (A) and (B) let us note first that the limits

$$\lim_{\delta \to 0+} \int_\delta^{1/16} k(t)\, d\sqrt{-t \log(\sqrt{t}\, k(t))}$$

and

$$\lim_{\delta \to 0+} \int_\delta^{1/16} k(t) \sqrt{\frac{-\log(\sqrt{t}\, k(t))}{t}}\, dt$$

always exist, although they may be infinite. Therefore it is enough to prove that

$$2\int_\delta^{1/16} k(t)\, d\sqrt{-t\log(\sqrt{t}\,k(t))} - \int_\delta^{1/16} k(t)\sqrt{\frac{-\log(\sqrt{t}\,k(t))}{t}}\,dt$$

remains bounded as δ approaches 0. But

$$\left|\int_\delta^{1/16} k(t)\left[2d\sqrt{-t\log(\sqrt{t}\,k(t))}\right] - \sqrt{\frac{-\log(\sqrt{t}\,k(t))}{t}}\,dt\right|$$
$$\le \left|\int_\delta^{1/16} 2k(t)\sqrt{t}\,d\sqrt{-\log(\sqrt{t}\,k(t))}\right| \le 2\int_0^{1/2e\sqrt{\pi}} u\,d\sqrt{-\log u} < +\infty.$$

The equivalence of (B), (C), and (D) follows from (i) and (ii). □

4. Proof of Theorem 2. Let us assume that k, c, and u are as in the assumption of Theorem 2. In the beginning of this proof we assume in addition that u is bounded from above and extends continuously to $\{(x,t): 0\le t<c\}$. This extension will be denoted also by u. By Widder's theorem [8], for each $(x,t)\in\mathfrak{D}_c$ we have

$$u(x,t)=\int_{-\infty}^{+\infty} K(x-z,t)\,d\alpha(z),$$

where

$$\alpha(z)=\int_0^z u(w,0)\,dw.$$

Let

$$L=\sup_{0\le z_2-z_1\le 1}\frac{\alpha(z_2)-\alpha(z_1)}{|z_1+z_2|+1}. \tag{3}$$

This supremum is finite since u is bounded from above. We can assume that $L>0$ since otherwise $u\equiv 0$, in which case the theorem is trivial. Note that (3) implies that

$$\alpha(z_2)-\alpha(z_1)\le L[(|z_1|\vee|z_2|)^2+2(|z_1|\vee|z_2|)+1] \tag{4}$$

whenever $z_1<z_2$ and $z_1\cdot z_2\ge 0$. Here and everywhere $a\vee b$ denotes the larger of the two numbers a and b. By (4), if $0\le z_1<z_2\le+\infty$ then

$$\begin{aligned}\int_{z_1}^{z_2} K(z,1)\,d\alpha(z) &= K(z_2,1)[\alpha(z_2)-\alpha(z_1)]-\int_{z_1}^{z_2}[\alpha(z)-\alpha(z_1)]\,dK(z,1)\\ &\le L\left[K(z_2,1)\cdot(z_2^2+2z_2-1)+\int_0^\infty (z^2+2z+1)\left(-\frac{\partial K(z,1)}{\partial z}\right)dz\right]\\ &\le L\left\{\sup_{z\ge 0}[K(z,1)(z^2+2z+1)]\right.\\ &\qquad\left.+\int_0^\infty (z^2+2z+1)\left(-\frac{\partial K(z,1)}{\partial z}\right)dz\right\}\\ &= C_1 L.\end{aligned}$$

The same estimate holds in the case when $-\infty\le z_1<z_2\le 0$, so that

$$\int_{z_1}^{z_2} K(z,1)\,d\alpha(z) \le C_1 L$$

whenever $-\infty \le z_1 < z_2 \le +\infty$ and both z_1 and z_2 have the same sign. Since

$$\int_{-\infty}^{+\infty} K(z,1)\,d\alpha(z) = u(0,1) = 0,$$

we have

$$\int_{z_1}^{z_2} K(z,1)\,d\alpha(z) \ge -3C_1 L \quad \text{for any } z_1, z_2,\ z_1 < z_2. \tag{5}$$

On the other hand, if $0 \le z_1 < z_2$ then, by the second mean value theorem,

$$\begin{aligned}\int_{z_1}^{z_2} K(z,1)\,d\alpha(z) = {}&[K(z_1,1) - K(z_2,1)] \cdot [\alpha(z') - \alpha(z_1)]\\ &+ K(z_2,1)[\alpha(z_2) - \alpha(z_1)],\end{aligned} \tag{6}$$

with some $z' \in (z_1, z_2)$. By (4), the first summand on the right-hand side of this equality does not exceed $L(z_2^2 + 2z_2 + 1)/2\sqrt{\pi}$. Therefore, comparing (5) and (6) we obtain

$$\alpha(z_2) - \alpha(z_1) \ge \frac{-L}{2\sqrt{\pi}K(z_2,1)}(z_2^2 + 2z_2 + 1 + 6C_1\sqrt{\pi}). \tag{7}$$

Similarly, when $z_1 < z_2 \le 0$ we have

$$\alpha(z_2) - \alpha(z_1) \ge \frac{-L}{2\sqrt{\pi}K(z_1,1)}(z_1^2 + 2|z_1| + 1 + 6C_1\sqrt{\pi}). \tag{8}$$

Note that (7) and (8) imply

$$\operatorname{sgn}(z) \cdot \alpha(z) \ge \frac{-L}{2\sqrt{\pi}K(z,1)}(z^2 + 2|z| + 1 + 6C_1\sqrt{\pi}). \tag{9}$$

Let us fix arbitrary x_1, x_2, and x with $x_1 < x < x_2$ and $x_2 - x_1 \le 1$. Let us assume that $0 < t \le 1/16$. By (7) and (8) we have

$$\begin{aligned}&\int_{x_2}^{4|x_2|+4} K(x-z,t)\,d\alpha(z)\\ &\quad \ge \frac{-LK(x-x_2,t)}{\sqrt{\pi}K(4|x_2|+4,1)}[(4|x_2|+4)^2 + 2(4|x_2|+4) + 1 + 6C_1\sqrt{\pi}]\\ &\quad \ge -C_2 L \exp(5x_2^2) K(x-x_2,t),\end{aligned} \tag{10}$$

with some absolute constant C_2. On the other hand, by (7), we have

$$\begin{aligned}&\int_{4|x_2|+4}^{\infty} K(x-z,t)\,d\alpha(z)\\ &\quad = -\int_{4|x_2|+4}^{\infty} [\alpha(z) - \alpha(4|x_2|+4)]\left[\frac{\partial}{\partial z}K(x-z,t)\right] dz \ge\end{aligned}$$

$$\geq \frac{L}{2\sqrt{\pi}} \int_{4|x_2|+4}^{\infty} \frac{z^2+2z+1+6C_1\sqrt{\pi}}{K(z,1)} \left[\frac{\partial}{\partial z} K(x-z,t)\right] dz$$
$$= \frac{-L}{4\sqrt{\pi}\, t^{3/2}} \exp\left(\frac{x^2}{4(1-t)}\right) \int_{4|x_2|+4}^{\infty} P(z) \cdot \exp[-\gamma(z-z_0)^2]\, dz,$$

where $\gamma=(1-t)/4t$, $z_0=x/(1-t)$, and $P(z)=(z-x)(z^2+2z+1+6C_1\sqrt{\pi})$. Note that there is an absolute constant C_3 such that $P(z)\leq C_3(z-z_0)^3$ whenever $z\geq 4|x_2|+4$. Hence

$$\begin{aligned} &\int_{4|x_2|+4}^{\infty} K(x-z,t)\, d\alpha(z) \\ &\geq \frac{-C_3 L}{4\sqrt{\pi}\, t^{3/2}} \exp\left(\frac{x^2}{4(1-t)}\right) \int_{4|x_2|+4}^{\infty} (z-z_0)^3 \exp[-\gamma(z-z_0)^2]\, dz \\ &= \frac{-C_3 L}{4\sqrt{\pi}\, t^{3/2}} \exp\left(\frac{x^2}{4(1-t)}\right) \frac{[\gamma(4|x_2|+4-z_0)^2+1]}{2\gamma^2} \exp[-\gamma(4|x_2|+4-z_0)^2] \\ &\geq C_4 L \exp(5x_2^2) K(x_2-x,t) \end{aligned} \tag{11}$$

with an absolute constant C_4, and where the last inequality is justified by the fact that $\exp[-\gamma(4|x_2|+4-x_0)^2]\leq 2\sqrt{\pi t}K(x_2-x,t)$. Combining (10) and (11) we obtain

$$\int_{x_2}^{\infty} K(x-z,t)\, d\alpha(z) \geq -(C_2+C_4)L\exp(5x_2^2)K(x_2-x,t). \tag{12}$$

We can prove similarly that

$$\int_{-\infty}^{x_1} K(x-z,t)\, d\alpha(z) \geq -(C_2+C_4)L\exp(5x_1^2)K(x_1-x,t). \tag{13}$$

For x_1, x_2 as before and for $M\geq 1$ let

$$T_M(x)=t_M\left(\frac{x_2-x_1}{2}-\left|x-\frac{x_1+x_2}{2}\right|\right), \quad x\in[x_1,x_2].$$

Note that $T_M(x)\leq 1/16$, $x\in[x_1,x_2]$. Let

$$u_1(x,t)=\int_{x_1}^{x_2} K(x-z,t)\, d\alpha(z), \quad t>0,\ x\in\mathbf{R}$$

and let $u_2=u-u_1$. If we set $M=e\vee 2L(C_2+C_4)\exp[5(|x_1+x_2|+1)^2]$ then, by (12), (13), and Lemma 1(ii), we have $-u_2(x, T_M(x))\leq k(T_M(x))$, $x\in(x_1,x_2)$. Hence

$$u_1(x,T_M(x))\leq 2k(T_M(x)), \quad x\in(x_1,x_2). \tag{14}$$

Let us introduce an auxiliary function,

$$w_M(x,t)=2\sqrt{2\pi e}\int_{x_1}^{x_2} K(x-z,t)k(T_M(z))\, dz,$$

where the integral is convergent in virtue of Lemma 2. If $x \in [(x_1+x_2)/2, x_2)$ then by Lemma 1(iii) we have

$$w_M(x, T_M(x)) \geq 2\sqrt{2\pi e} \int_x^{x+\sqrt{2T_M(x)}} K(x-z, T_M(x))k(T_M(z))\, dz$$
$$\geq 2\sqrt{2\pi e}\, k(T_M(x))K(\sqrt{2T_M(x)}, T_M(x))\sqrt{2T_M(x)} \geq 2k(T_M(x)).$$

The same estimation holds for $x \in (x_1, (x_1+x_2)]$. Hence

$$(15) \qquad w_M(x, T_M(x)) \geq 2k(T_M(x)), \quad x \in (x_1, x_2).$$

The function $u_1 - w_M$ is a solution of the heat equation on $\mathfrak{D}_\infty$. It is bounded from above and extends continuously to $\bar{\mathfrak{D}}_\infty \setminus \{(x_1, 0), (x_2, 0)\}$. It vanishes on $\{(x, 0) : x \notin [x_1, x_2]\}$ and, by (14) and (15), it is nonpositive on $\{(x, T_M(x) : x \in (x_1, x_2)\}$. Hence, by the maximum principle it is nonpositive on $\{(x, t) : t > 0$ if $x \notin (x_1, x_2)$; $t > T_M(x)$ if $x \in (x_1, x_2)\}$. Therefore

$$\int_{x_1}^{x_2} [u_1(x, 0) - w_M(x, 0)]\, dx \leq 0.$$

Thus

$$(16) \qquad \begin{aligned} \alpha(x_2) - \alpha(x_1) &\leq 2\sqrt{2\pi e} \int_{x_1}^{x_2} k(T_M(z))\, dz \\ &= 4\sqrt{2\pi e} \int_0^{(x_2-x_1)/2} k(t_M(z))\, dz = 4\sqrt{2\pi e}\, I(M, (x_2-x_1)/2). \end{aligned}$$

Since x_1 and x_2 were arbitrary (with $0 < x_2 - x_1 \leq 1$), (16) implies, by Lemma 2(i) and by (3), that

$$L(|x_1+x_2|+1) \leq 8\sqrt{2\pi e} \log\{e \vee 2L(C_2+C_4) \exp[5(|x_1+x_2|+1)^2]\} I(1, 1/2).$$

This implies that there is a constant C_5 such that

$$(17) \qquad L \leq C_5[I(1, 1/2)+2] \log[I(1, 1/2)+2].$$

By (16) and Lemma 2(ii), it is easy to see that there exists a continuous function κ depending only on $I(1, s/2)$ (as a function of s) such that $\alpha(x_2) - \alpha(x_1) \leq (|x_1+x_2|+1)\kappa(x_2-x_1)$ whenever $0 < x_2 - x_1 \leq 1$. The estimation of $\kappa(1)$ follows by (3), (17) and Lemma 2(ii).

Part (iii) of the theorem follows by (9) and (17). Parts (i) and (iv) are trivial, and (v) is a consequence of Widder's theorem.

Let us come back to the general case; that is, let u be as in the assumption of Theorem 2. For $\theta \in (0, 1)$ let

$$u_\theta(x, t) = u(\sqrt{\theta}x, (t-1)\theta + 1), \quad (x, t) \in \mathfrak{D}_c.$$

The function u_θ satisfies the assumptions of the first part of the proof. Hence, if we denote

$$\alpha_\theta(x) = \int_0^x u_\theta(z, 0)\, dz$$

then

$$\alpha(x_2)-\alpha(x_1)\le(|x_1+x_2|+1)\kappa(x_2-x_1) \tag{18}$$

if $0<x_2-x_1\le 1$, and

$$\operatorname{sgn}(x)\alpha_\theta(x)\ge -D(x^2+1)\exp(x^2/4). \tag{19}$$

Note that κ and D depend only on k. But

$$\alpha_\theta(x)=\frac{1}{\sqrt{\theta}}\int_0^{\sqrt{\theta}x}u(z,1-\theta)\,dz.$$

By (18) and (19), similarly as in the Helly selection theorem, we can find an increasing sequence (θ_n) converging to 1, as well as a function α on $\mathbf{R}$ such that α satisfies (ii), (iii), and (iv) of the theorem (hence its discontinuities are of the first kind only) and such that

$$\lim_{n\to\infty}\alpha_{\theta_n}(x)=\alpha(x)$$

if x is a point of continuity of α. By (18), (19), and the Lebesgue dominated convergence theorem, for each $(x,t)\in\mathfrak{D}_1$ we have

$$\begin{aligned}\int_{-\infty}^{+\infty}K(x-z,t)\,d\alpha(z)&=-\int_{-\infty}^{+\infty}\alpha(z)\left(\frac{\partial}{\partial z}K(x-z,t)\right)dz\\ &=\lim_{n\to\infty}\left[-\int_{-\infty}^{+\infty}\alpha_{\theta_n}(z)\left(\frac{\partial}{\partial z}K(x-z,t)\right)dz\right]\\ &=\lim_{n\to\infty}u_{\theta_n}(x,t)=u(x,t),\end{aligned}$$

which proves (v).

For arbitrary x_1 and x_2 we have

$$\begin{aligned}\lim_{t\to0+}\int_{x_1}^{x_2}u(x,t)\,dx&=\lim_{t\to0+}\int_{x_1}^{x_2}\left(\int_{-\infty}^{+\infty}K(x-z,t)\,d\alpha(z)\right)dx\\ &=-\lim_{t\to0+}\int_{-\infty}^{+\infty}\left[\int_{x_1}^{x_2}\alpha(z)\left(\frac{\partial}{\partial z}K(x-z,t)\right)dx\right]dz\\ &=\lim_{t\to0+}\int_{-\infty}^{+\infty}\alpha(z)[K(x_2-z,t)-K(x_1-z,t)]\,dz\\ &=\frac{\alpha(x_2-)+\alpha(x_2+)}{2}-\frac{\alpha(x_1-)+\alpha(x_1+)}{2}.\end{aligned}$$

This, together with the preceding remarks, completes the proof of Theorem 2. □

5. Proofs of Theorems 3 and 4.

LEMMA 3. *If k does not satisfy the assumptions of Theorem* 1 *and* $k_1(t)=\min\{k(t),(2e\sqrt{\pi t})^{-1}\}$ *then*

$$\int_0^{1/16}k_1(t)\,d\left[2\sqrt{t\log\frac{M}{2\sqrt{\pi t}\,k_1(t)}}\right]=+\infty,\quad M\ge 1.$$

Proof. In virtue of Lemma 2(iii) it is enough to show that

$$\int_0^{1/16} k_1(t)\, d[\sqrt{-t\log(t^{1/2}k_1(t))}\,] = +\infty. \tag{20}$$

If $\lim_{t\to 0+} t^{1/2}k(t) = 0$, then $k_1(t) = k(t)$ for all sufficiently small t's and therefore (20) follows in this case. If $\limsup_{t\to 0+} t^{1/2}k(t) > 0$ then there is a $\theta \in (0,1]$ and a decreasing sequence (t_n) converging to 0, with $t_1 \le 1/16$ and such that $k(t_n) \ge \theta(2e\sqrt{\pi t_n})^{-1}$, $n \ge 1$. Let $k_2(t) = \min\{k(t), \theta(2e\sqrt{\pi t})^{-1}\}$. Then

$$\begin{aligned}
&\int_0^{1/16} k_2(t)\, d[\sqrt{-t\log(\sqrt{t}\,k_2(t))}\,] \\
&\quad \ge \sum_{n=1}^{\infty} k_2(t_n)[\sqrt{-t_n\log(\sqrt{t_n}\,k_2(t_n))} - \sqrt{t_{n+1}\log(\sqrt{t_{n+1}}\,k_2(t_{n+1}))}\,] \\
&\quad = \frac{\theta}{2e\sqrt{\pi}}\sqrt{\log\frac{2e\sqrt{\pi}}{\theta}}\sum_{n=1}^{\infty}\left(1 - \sqrt{\frac{t_{n+1}}{t_n}}\right) = +\infty,
\end{aligned}$$

where the last series is divergent because $\lim_{n\to\infty}(t_n)^{1/2} = 0$. But, since $k_2 \le k_1$, (20) follows by Lemma 1(iv) and (ii). □

LEMMA 4. *Suppose that $k(t/2) \le Ck(t)$ and $k(t) \le (2e\sqrt{\pi t})^{-1}$ for $t \in (0, t_0]$ with some positive constants C and t_0. Then, for each $M \ge 1$ and each $x \in (0, x_M(t_0)]$, we have*
(i) *$k(t_M(x/2)) \le C^4 k(t_M(x))$ and*
(ii) *$k(t_M(\theta x)) \le C^4\theta^{-\alpha}k(t_M(x))$, $0 < \theta < 1$,*
where $\alpha = 4\log_2 C$.

Proof. We will prove first that

$$t_M(x/2) \ge 2^{-4}t_M(x), \quad 0 < x \le x_M(t_0). \tag{21}$$

Suppose that this is not the case for some $x \in (0, x_M(t_0)]$. Since $t_M(x/2) \le t_M(x)$, we have $k(t_M(x/2)) \ge k(t_M(x))$. By Lemma 1(ii), this implies that

$$\sqrt{\frac{t_M(x)}{t_M(x/2)}} \ge \exp\left[\frac{x^2}{4t_M(x)}\left(\frac{t_M(x)}{4t_M(x/2)} - 1\right)\right].$$

Since $x^2/(4t_M(x)) \ge 1$ by Lemma 1(iii) and since $t_M(x)/(4t_M(x/2)) > 1$ by our supposition, we have

$$\sqrt{\frac{t_M(x)}{t_M(x/2)}} \ge \exp\left(\frac{t_M(x)}{4t_M(x/2)} - 1\right),$$

which is false for $t_M(x/2) \le 2^{-4}t_M(x)$.

(i) is a consequence of (21) and of the assumption of the lemma; (ii) follows easily by (i). □

Proof of Theorem 4. Without any loss of generality we can and do assume that $\epsilon \le 1/2$. Let k satisfy the assumptions of Theorem 3. Note that if k does not satisfy the assumptions of Theorem 1 then neither does ϵk. Let then $\tilde{k} = \min\{(2e\sqrt{\pi t})^{-1}, \epsilon k(t)\}$, $t > 0$. By Lemma 3,

$$\int_0^{1/16} \tilde{k}(t)\, d\left[2\sqrt{t \log \frac{M}{2\sqrt{\pi t}\,\tilde{k}(t)}}\right] = \int_0^{\tilde{x}_M(1/16)} \tilde{k}(\tilde{t}_M(x))\, dx = +\infty, \quad M \geq 1,$$

where $\tilde{x}_M$ and $\tilde{t}_M$ correspond to $\tilde{k}$ via the construction from Section 3. Since $C \geq \sqrt{2}$, we have $\tilde{k}(t/2) \leq C\tilde{k}(t)$, $0 < t \leq t_0$.

Let σ, $\sigma \in (0, \epsilon)$, be such that

$$|K(x,t) - K(x-z,t)| \leq \frac{1}{2L} \min(\epsilon, \tilde{k}(1)), \quad 0 < z \leq \sigma,$$

if either $t \geq \epsilon$ or $|x| \geq 1/2$. Let η, $0 < \eta \leq \delta$, be such that

$$\eta K(x,t) \leq (1/2) \min\{\epsilon, \tilde{k}(1)\} \quad \text{if either } t \geq \epsilon \text{ or } |x| \geq 1/2.$$

Let $M \geq L \vee 1$ be so large that $\tilde{t}_M(1/2) \leq t_0$ and $x^{-2}\tilde{t}_M(x) \leq (8\alpha)^{-1}$, $0 < x \leq 1/2$, where $\alpha = 4 \log_2 C$. Note that then $x^{-2}\tilde{t}_M(x) \leq 1/8$, since $\alpha \geq 2$.

Let us choose two decreasing sequences (z_n) and (z_n') of positive real numbers so that

(a) $z_1 = \sigma$;

(b) $z_n' < z_n$ and

$$\int_{z_n'}^{z_n} \tilde{k}(t_M(z))\, dz > MC^4 e^{-2\alpha} 2^{\alpha};$$

(c) $z_{n+1} < z_n'/2$; and

(d) $(L-\eta)K(x - z_{n+1}, \tilde{t}_M(z)) \leq [(L-\eta) + \eta/4] K(x, \tilde{t}_M(x))$ whenever $z_n' \leq x \leq 1/2$.

Let f be a nonnegative continuous function on $\mathbf{R}$ which vanishes outside $\bigcup_{1 \leq n \leq N}(z_n', z_n)$ for some finite N and is such that

(a′) $f(z) \leq \eta 2^{-\alpha-2} C^{-4} M^{-1} \tilde{k}(\tilde{t}_M(z))$, $0 < z \leq 1/2$;

(b′)
$$\int_{z_n'}^{z_n} f(z)\, dz \leq \frac{\eta}{4e^{2\alpha}};$$

and

(c′)
$$\int_{-\infty}^{+\infty} f(z)\, dz = L - \eta.$$

Suppose that $t \geq \epsilon$ or $|x| \geq 1/2$. Then

$$|u(x,t)| \leq \int_0^{\sigma} |K(x-z,t) - K(x,t)| f(z)\, dz + \eta K(x,t)$$

$$\leq \frac{\min\{\epsilon, \tilde{k}(1)\}}{2L} \cdot (L-\eta) + \frac{\min\{\epsilon, \tilde{k}(1)\}}{2} < \min\{\epsilon, \tilde{k}(1)\},$$

which, in particular, gives (i).

Now, let $0 < x \leq 1/2$ and $0 < t \leq \tilde{t}_M(x)$. Let n be the least positive integer with $z_n' \leq x$. Then

$$\int_{-\infty}^{+\infty} K(x-z,t)f(z)\,dz = \left(\int_0^{z_{n+1}} + \int_{z_{n+1}}^{4\alpha t/x} + \int_{4\alpha t/x}^{x/2} + \int_{x/2}^{+\infty}\right) K(x-z,t)f(z)\,dz$$
$$= A_1 + A_2 + A_3 + A_4.$$

In estimations of A_1, A_2, A_3 we will use the fact that, since $x^{-2}\tilde{t}_M(x) \le 1/8$, the function $K(x-z,t)$ is increasing in t, $0 < t \le \tilde{t}_M(x)$, for each $z \in (0, x/2)$. By this fact and by (c′), we have

$$A_1 \le \int_0^{z_{n+1}} K(x-z, \tilde{t}_M(x)) f(z)\,dz$$
$$\le K(x - z_{n+1}, \tilde{t}_M(x)) \int_0^{z_{n+1}} f(z)\,dz \le (L-\eta) K(x - z_{n+1}, \tilde{t}_M(x)).$$

Applying (d) we obtain that

$$A_1 \le [(L-\eta) + \eta/4] K(x, \tilde{t}_M(x)). \tag{23}$$

If $z_{n+1} \ge 4\alpha t/x$ then $A_2 \le 0$. If $z_{n+1} < 4\alpha t/x$ then, since $2\alpha t/x < x/2$ and by (b′), we have

$$\begin{aligned} A_2 &\le \int_{z_{n+1}}^{4\alpha t/x} K(x-z, \tilde{t}_M(x)) f(z)\,dz \\ &\le \int_{z_n'}^{z_n} f(z)\,dz\, K\left(x - \frac{4\alpha \tilde{t}_M(x)}{x}, \tilde{t}_M(x)\right) \\ &\le \frac{\eta}{4e^{2\alpha}} \exp\left(2\alpha - \frac{4\alpha^2 \tilde{t}_M(x)}{x^2}\right) K(x, \tilde{t}_M(x)) \le \frac{\eta}{4} K(x, \tilde{t}_M(x)). \end{aligned} \tag{24}$$

Next, by (a′) and Lemma 4(ii) applied to $\tilde{k}$, we have

$$A_3 \le \int_{4\alpha t_M(x)/x}^{x/2} K(x-z,t) \cdot \frac{\eta}{2^{\alpha+2} C^4 M} \tilde{k}(\tilde{t}_M(z))\,dz$$
$$\le \frac{\eta C^4 x^\alpha}{2^{\alpha+2} C^4 M} \tilde{k}(\tilde{t}_M(x)) \int_{4\alpha t/x}^{x/2} z^{-\alpha} K(x-z,t)\,dz.$$

Since $(\partial/\partial z)(z^\alpha K(x-z,t)) > 0$ on $(4\alpha t/x, x - 4\alpha t/x)$, we have

$$A_3 \le \frac{\eta}{4M} \tilde{k}(\tilde{t}_M(x)) \int_{x/2}^{x - 4\alpha t/x} K(x-z,t)\,dz \le \frac{\eta}{4M} \tilde{k}(\tilde{t}_M(x)). \tag{25}$$

Finally, by Lemma 4(i) and by (a′),

$$\begin{aligned} A_4 &\le \int_{x/2}^{1/2} K(x-z,t) \cdot \frac{\eta}{4^{\alpha+2} C^4 M} \tilde{k}(\tilde{t}_M(z))\,dz \\ &\le \frac{\eta}{4C^4 M} \tilde{k}(\tilde{t}_M(x/2)) \le \frac{\eta}{4M} \tilde{k}(\tilde{t}_M(x)). \end{aligned} \tag{26}$$

By (23), (24), (25), and (26), we have

$$\int_{-\infty}^{+\infty} K(x-z,t) f(z)\,dz \le \left(L - \frac{\eta}{2}\right) K(x, \tilde{t}_M(x)) + \frac{\eta}{2M} \tilde{k}(\tilde{t}_M(x)), \tag{27}$$
$$0 < x \le 1/2,\ 0 < t \le \tilde{t}_M(x).$$

By Lemma 1(ii), this implies that

$$u(x,t)\le \tilde{k}(\tilde{t}_M(x))\le \tilde{k}(t)\le \epsilon k(t),\quad 0<x\le 1/2,\ 0<t\le \tilde{t}_M(x). \tag{28}$$

The inequality (27) implies also that $u(x,\tilde{t}_M(x))\le 0$, $0<x\le 1/2$. Since by (c′) $u(0,t)\le 0$ $(t<0)$ and by (22) $u(1/2,t)\le \tilde{k}(1)$ $(t\ge \tilde{t}_M(1/2))$, an application of the maximum principle gives that $u(x,t)\le \tilde{k}(1)$ if $0\le x\le 1/2$ and $t\ge \tilde{t}_M(x)$. In particular,

$$u(x,t)\le \tilde{k}(1)\le \tilde{k}(t)\le \epsilon k(t)\quad \text{if } 0\le x\le 1/2 \text{ and } \tilde{t}_M(x)\le t\le 1. \tag{29}$$

If $x\ge 1/2$ then, by (22),

$$u(x,t)\le \tilde{k}(1)\le \epsilon k(t),\quad 0<t\le 1. \tag{30}$$

Finally, if $x\le 0$ then, by (c′),

$$\begin{aligned} u(x,t)&=\int_0^\infty K(x-z,t)f(z)\,dz-LK(x,t)\\ &=\int_0^\infty [K(x-z,t)-K(x,t)]f(z)\,dz-\eta K(x,t)<0<\epsilon k(t),\quad t>0. \end{aligned} \tag{31}$$

Combining (28), (29), (30), and (31), we obtain (ii). □

Proof of Theorem 3. Let us construct sequences (ϵ_j) of real positive numbers and(f_j) of nonnegative functions on **R** inductively as follows.

Let $\epsilon_1=1/2$ and let f_1 be the function from Theorem 4 corresponding to $L=6$ and $\delta=1$. If $\epsilon_1,\epsilon_2,\dots,\epsilon_j$ and $f_1,f_2,\dots,f_j$ are already chosen then let ϵ_{j+1} be a positive number less than or equal to $\epsilon_j/2$, and such that for $0<t<\epsilon_{j+1}$ and $1\le i\le j$ we have

$$\int_x^0 u_i(z,t)\,dz\le -1,\quad x<0 \tag{32}$$

and

$$\int_0^x u_i(z,t)\,dz\ge \frac{4}{5}\int_0^x f_i(z)\,dz-3,\quad \epsilon_j<x,$$

where

$$u_i(x,t)=\int_{-\infty}^{+\infty} f_i(z)K(x-z,t)\,dz-LK(x,t).$$

Also, let f_{j+1} be the function from Theorem 4 corresponding to $\epsilon=\epsilon_{j+1}$, $L=6$, and $\delta=1$. This step is possible by Theorem 4 and by the fact that whenever μ is a finite measure on **R** and $w(x,t)=\int_{-\infty}^{+\infty} K(x-z,t)\,d\mu(z)$ then

$$\lim_{t\to 0+}\int_a^b w(z,t)\,dz=\frac{\mu([a,b])+\mu((a,b))}{2}$$

uniformly in a and b, $a<b$.

Let $\tilde{u}(x,t)=\sum_{i=1}^\infty u_i(x,t)$. If $x>0$ and $j_0=\min\{j:\epsilon_j<x\}$ then for arbitrary j $(j>j_0)$ and t $(\epsilon_{j+1}\le t<\epsilon_j)$ we have

$$\int_0^x \tilde{u}(z,t)\,dz = \sum_{i=1}^{j_0-1}\int_0^x u_i(z,t)\,dz + \sum_{i=j_0}^{j-1}\int_0^x u_i(z,t)\,dz$$
$$+\int_0^x u_j(z,t)\,dz + \sum_{i=j+1}^{\infty}\int_0^x u_i(z,t)\,dz$$
$$\geq -3j_0 + (j-j_0)\left(\frac{4}{5}\cdot 5 - 3\right) - 3 - \sum_{i=j+1}^{\infty}\frac{1}{2^i} \geq j - 4(j_0+1).$$

Hence $\lim_{t\to 0+}\int_0^x \tilde{u}(z,t)\,dz = +\infty$ if $x>0$. If $x<0$ and $0<t<\epsilon_{j+1}$ then, by (32),

$$\int_0^x \tilde{u}(z,t)\,dz \geq j.$$

Hence $\lim_{t\to 0+}\int_0^x \tilde{u}(z,t)\,dz = +\infty$, $x>0$. On the other hand,

$$\tilde{u}(x,t) \leq \sum_{i=1}^{\infty}\frac{1}{2^i}k(t) = k(t), \quad 0<t\leq 1.$$

Hence $u = \tilde{u} - k(1)$ satisfies the assertion of Theorem 3. □

REMARK. The above proof can be easily modified to give (under assumptions of Theorem 3) a solution u of the heat equation on $\mathfrak{D}_\infty$ with $u(x,t)\leq k(t)$, $t>0$, and such that for each $x\neq 0$:

$$\limsup_{t\to 0+}\int_0^x u(z,t)\,dz = +\infty \quad \text{and} \quad \liminf_{t\to 0+}\int_0^x u(z,t)\,dz = -\infty.$$

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