THE ASYMPTOTIC BOUNDARY OF A SURFACE IMBEDDED IN $\mathbf{H}^3$ WITH NONNEGATIVE CURVATURE

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Introduction. From a function-theoretic standpoint, a noncompact complete Riemann surface M with nonnegative curvature has only one point "at infinity." If M is imbedded isometrically in hyperbolic space then one can identify an asymptotic boundary $\partial_\infty M$ as the limit points of M on the ideal boundary of hyperbolic space. We will usually work in the ball model, $\mathbf{B}^3$. The ideal boundary of $\mathbf{H}^3$ is naturally identified with the unit sphere. The asymptotic boundary of M is the set of limit points of M on the unit sphere with respect to the Euclidean topology of $\mathbf{B}^3$. We will prove the following theorem.

THEOREM. *If M is a C^5 complete imbedding of $\mathbf{R}^2$ into $\mathbf{H}^3$ with nonnegative Gauss curvature then the asymptotic boundary of M is a single point.*

The proof uses the hyperbolic Gauss map defined in [4] and draws heavily on results obtained there on surfaces represented as envelopes of horospheres. To apply the machinery of [4] we will prove several propositions on the Gauss map of convex surfaces in $\mathbf{H}^3$ which generalize known results from Euclidean space. By a convex surface M we shall mean a surface which bounds a geodesically convex region D. This is equivalent to the condition that every point of M have a supporting plane, [7, p. 8.10].

It is an easy consequence of Cohn-Vossen's inequality (see [5]),

$$\int_M K\,dA \le 2\pi\chi,$$

which always holds for complete surfaces with nonnegative curvature, that M is topologically equivalent to a sphere, plane, or cylinder. If K is nonzero at any point then M must be a plane or a sphere. The horospheres are examples of imbeddings of $\mathbf{R}^2$ into $\mathbf{H}^3$ with nonnegative curvature. It is reasonable to inquire if there are any nontrivial examples. In the third section we construct a family of deformations of the horosphere through embedded surfaces with strictly positive curvature.

It would be interesting to know if the hypotheses:

(a) M is complete
(b) M is immersed
(c) M has nonnegative curvature

imply that M is imbedded. If one appends the hypothesis that $\partial_\infty M$ is a single point then it follows that M is imbedded. In fact a stronger result is true.

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COROLLARY 4.4. *If ψ is a complete immersion of a topological surface M into $\mathbf{H}^3$ such that*
(a) *every point p of M has a neighborhood N such that $\psi(N)$ is part of the boundary of a strictly convex body, and*
(b) *$\partial_\infty \psi(M)$ is a single point,*
then $\psi(M)$ is the boundary of a convex region and homeomorphic to $\mathbf{R}^2$.

This result is a consequence of an easy extension to hyperbolic space of Van Heijenoort's theorem on locally convex sets in Euclidean space. For $\partial_\infty \psi(M)$ consisting of two points, a counterexample is described.

In this paper the words line, plane, arc, etc. will refer to a hyperbolic geodesic, a hyperbolic plane, an arc of a hyperbolic geodesic, etc. If p and q are two distinct points in $\mathbf{H}^3$ or its ideal boundary, then γ_{pq} is the unique arc between them.

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1. Convex surfaces and the Gauss map. We will establish a well-known fact that an imbedded surface with everywhere positive extrinsic curvature bounds a convex set. We will also show that the Gauss map for a convex surface is injective. We orient M so that both principal curvatures (k_1, k_2) are positive.

PROPOSITION 1.1. *If M is a complete, smooth, properly imbedded, connected surface in $\mathbf{H}^3$ with both principal curvatures positive then the inner component of $\mathbf{H}^3 \backslash M$ is a convex set.*

REMARKS. (1) The fact that M is properly imbedded implies that $\mathbf{H}^3 \backslash M$ has two components.

(2) The inner component of $\mathbf{H}^3 \backslash M$ is the one into which the oriented unit normal field points.

Proof. Let the inner component of $\mathbf{H}^3 \backslash M$ be denoted by D. The hypothesis on the principal curvatures implies that for each point p on M there is a neighborhood $\tilde{N}_p$ such that $\tilde{N}_p$ lies on one side of the tangent plane to M at p. To see this, one considers the intersections of M and its tangent plane with planes containing the normal line to M at p. From the existence of a local supporting plane it follows easily that each point p on M has a neighborhood N_p such that for every $q \in N_p$ the line γ_{pq} lies in $\bar{D}$. If N_p equals M for every p in M then D must be a convex set. Otherwise we could find a pair of points m and n in D such that γ_{mn} is not contained in D. As M is complete and properly embedded, $\gamma_{mn} \cap M$ must contain at least two points. Therefore we could find two points p, q in M such that γ_{pq} lies in $\bar{D}^c$.

To prove that N_p equals M we will show that N_p is both open and closed. That N_p is closed is obvious; let $\{q_n\} \subset N_p$ converge to q. As each arc γ_{pq_n} lies inside $\bar{D}$, γ_{pq} must as well.

If N_p is not open then there is a sequence of points q_n in N_p^c tending to a point q in N_p. Therefore each arc γ_{pq_n} has points in $\bar{D}^c$. As above we can find pairs of points (r_n, s_n) on $\gamma_{pq_n} \cap M$ such that the arcs $\gamma_{r_n s_n}$ lie in $\bar{D}^c$. Either the distance between r_n and s_n tends to zero or it does not. In the latter case we can choose subsequences r_n and s_n tending to distinct points r and s. The arc γ_{rs} must lie in M, for γ_{rs} is a subset of γ_{pq} which lies in $\bar{D}$ but it is the limit of $\{\gamma_{r_n s_n}\}$ which lie in $\bar{D}^c$. This is not possible, since the principal curvatures are both positive, and therefore the second fundamental form of M is positive definite. If M contained a geodesic arc the second fundamental form would be indefinite on this set. Thus r_n and s_n must tend to a common limit.

The arcs γ_{pq_n} lie in a ball of radius $R > 0$ about p, $B(p, R)$. M is smooth and therefore the neighborhoods N_m (for $m \in B(p, R) \cap M$) each contain a ball of a fixed size. The sequences $\{r_n\}$ and $\{s_n\}$ are contained in $B(p, R) \cap M$ and the distance from r_n to s_n tends to zero. Thus $r_n \in N_{s_n}$ for large enough n, an obvious contradiction to the fact that $\gamma_{r_n s_n} \subset \bar{D}^c$. Therefore N_p is both open and closed; as p was an arbitrary point in M, the proposition is proved. □

REMARK. The surface need not be smooth, as three derivatives suffice for the argument.

An immediate consequence of Proposition 1.1 is the following.

LEMMA 1.2. *Under the hypotheses of Proposition* 1.1, *the arc between any two points in M lies in D.*

To apply the methods of [4] we need to study the Gauss map of M with respect to the *outer* normal. We will denote this map by $G(p)$.

PROPOSITION 1.3. *If M is an imbedded surface which bounds a convex region D then the outer Gauss map of M is injective into* $S^2 \setminus \partial_\infty M$.

Proof. The injectivity is an easy consequence of hyperbolic geometry: suppose there are two points p and q in M with $G(p) = G(q) = g$. The points p, q, and g determine a hyperbolic plane h; the convex curve $h \cap M$ bounds the convex region $h \cap D$. Let ℓ_p and ℓ_q denote the supporting lines to $M \cap h$ at p and q respectively. It follows easily that the triangle pgq has angle sum at least π, for ℓ_p and ℓ_q are orthogonal to γ_{pg} and γ_{qg} (respectively) while γ_{pq} lies in the interior of $D \cap h$ (see Figure 1).

To prove that $G(M) \subset S^2 \setminus \partial_\infty M$, we observe that supporting plane H at P lies exterior to D. From this it is apparent that $\partial_\infty M$ cannot have a point in the interior of the region of $S^2 \setminus \partial_\infty M$ determined by the outer normal to M at p. $G(p)$ lies in this region and therefore in $S^2 \setminus \partial_\infty M$. □

To use the formulae derived in [4] we must invert the Gauss map and represent M as an envelope of horospheres. Let $\theta = G(p)$ and define $\rho(\theta)$ so that $H(\theta, \rho(\theta))$ is the horosphere through θ and p. $H(\theta, \rho(\theta))$ is contained in the exterior half space determined by the support plane to M at p. From Lemma 1.2 it follows that $H(\theta, \rho(\theta))$ meets M only at p. The technical hypothesis which we must check is that $\rho(\theta)$ is at least in $C^4(G(M))$. As this is a local question it is

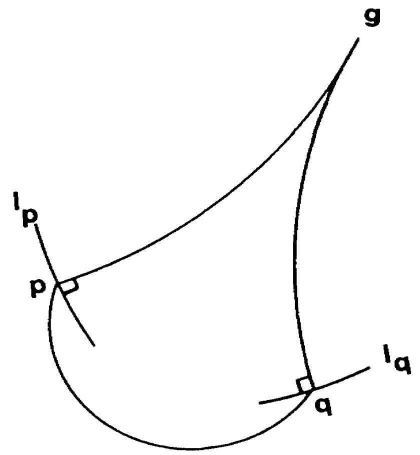


Figure 1

convenient to represent M as an immersion $i: U \to \mathbf{H}^3$. Assume that i is a C^k map from an open set in $\mathbf{R}^2$ into $\mathbf{H}^3$. At each point $i(x)$ there is a unit normal vector $N_{i(x)}$; N is a C^{k-1} vector field.

The Gauss map of M can be represented as a real analytic function in the coordinate functions of $i(x)$ and N, induced from the representation of $\mathbf{H}^3$ in $\mathbf{B}^3 \subset \mathbf{R}^3$. From this is it evident that $G \circ i(x)$ is a C^{k-1} map from U into $V = G(i(U))$.

A point $p \in \mathbf{B}^3$ and a point $\phi \in S^2$ uniquely determine a horosphere in $\mathbf{H}^3$, $\mathcal{H}_{p,\phi}$. The horocyclic distance (p, ϕ) is defined by:

$$|(p,\phi)| = \inf_{q \in \mathcal{H}_{p,\phi}} d(0,q), \quad 0 = (0,0,0).$$

(p, ϕ) is positive if 0 is in the exterior of $\mathcal{H}_{p,\phi}$ and negative otherwise; (p, ϕ) is a real analytic function in the coordinates of p and ϕ. Thus $\tilde{\rho}(x) = (i(x), G \circ i(x))$ is in $C^{k-1}(U)$. Suppose $G \circ i$ is invertible; let $F(\theta) = (G \circ i)^{-1}(\theta)$. The function $\rho(\theta)$ defined above is given by $\rho(\theta) = \tilde{\rho}(F(\theta))$. If we show that $F(\theta)$ is a C^{k-1} mapping then it follows that $\rho(\theta)$ is a C^{k-1} function. By the inverse function theorem it suffices to show that the Jacobian of $G \circ i$ is invertible. As i is an immersion, its Jacobian is everywhere of rank two; thus we only need to show that the Jacobian of G is everywhere invertible.

To study the Jacobian of G we will use the theory of parallel surfaces developed in [4] to compute the Jacobian determinant. Recall that $\psi^t(p, X)$ is the geodesic with initial point p and velocity X. Define $i_t(x) = \psi^t(i(x), N_{i(x)})$. Then

$$G \circ i(x) = \lim_{t \to \infty} i_t(x).$$

The limit is taken in the Euclidean topology on $\mathbf{B}^3$.

Let X_i denote the coordinate vector field $di_t(\partial_{x_i})$. As observed in [4], X_i is a solution to the Jacobi equation

$$(1.1) \qquad \frac{D^2 X_i}{Dt^2} + R(N, X_i)N = 0.$$

We normalize so that the point $p \in M$ is $(0,0,0)$ and the unit normal N is $(0,0,1)$. Furthermore, we choose coordinates (x_1, x_2) so that $X_i(0)$ are unit principal directions. In this case we can solve (1.1) explicitly to obtain

$$X_i(t) = \tfrac{1}{2}[(1-k_i)e^t + (1+k_i)e^{-t}]\tilde{X}_i(t),$$

where $\tilde{X}_i(t)$ is the parallel translate of $X_i(0)$ along $\psi^t(p, N)$. We can express $\tilde{X}_i(t)$ in terms of the Euclidean parallel translate $\bar{X}_i(t) = X_i(0)$ by

$$\tilde{X}_i(t) = (\operatorname{ch} t/2)^{-2}\bar{X}_i(t).$$

The differential of $P_t = i_t \circ i^{-1}$ is expressed at p in terms of $X_i(t)$ by $dP_t(X_i(0)) = X_i(t)$. If M is C^2, then

$$\begin{aligned} dG(X_i) &= \lim_{t\to\infty} dP_t(X_i) \\ &= 2(1-k_i)\bar{X}_i(\infty). \end{aligned} \tag{1.2}$$

As X_1 and X_2 are orthogonal it follows that $J(p)$, the Jacobian determinant of G as p, is given by

$$J(p) = 16(1-k_1)^2(1-k_2)^2. \tag{1.3}$$

We derived (1.3) under the normalization described above. As this normalization is accomplished by applying a hyperbolic isometry, it follows that for any compact set $K \subset \Sigma$ there are positive constants C_1 and C_2 such that

$$C_1(1-k_1)^2(1-k_2)^2 \le J(p) \le C_2(1-k_1)^2(1-k_2)^2 \tag{1.3'}$$

To sum up, we have proven the following.

LEMMA 1.4. (a) *If M is a C^k-immersion then the Gauss map of M is C^{k-1}.*
(b) *If $k \ge 2$ and neither principal curvature of M (relative to the normal vector used to define G) is $+1$ then M is locally represented as an envelope of horospheres $\{H(\theta, \rho(\theta))\}$; $\rho(\theta)$ is a C^{k-1} function defined on a domain in S^2.*

A convex surface has nonpositive principal curvatures relative to the outer normal vector; thus Proposition 1.3 and Lemma 1.4 combine to show that a C^k-convex surface M is globally represented as an envelope of horospheres $\{H(\theta, \rho(\theta)) : \theta \in G(M)\}$. $\rho(\theta)$ is a C^{k-1} function on $G(M)$. Thus if M is at least C^5 the theory presented in [4] can be applied to study $ds_\infty^2 = e^{2\rho} d\sigma^2$, where $d\sigma^2$ is the round metric on S^2.

PROPOSITION 1.5. *If M is a complete convex C^5 imbedded surface then the metric ds_∞^2 is complete as well.*

REMARK. C^5 is very probably more than is required; C^2 should suffice.

Proof. Let (x, y) denote a stereographic coordinate system on S^2 centered at $\theta = G(p)$. Proposition 5.1 of [4] states that the metric tensor of M at p is given in these coordinates by

$$g_{ij}(p) = \begin{bmatrix} \frac{e^\rho}{2} + \left(\rho_{xx} + \frac{1}{2}(\rho_y^2 - \rho_x^2 - 1)\right)e^{-\rho} & (\rho_{xy} - \rho_x\rho_y)e^{-\rho} \\ (\rho_{xy} - \rho_x\rho_y)e^{-\rho} & \frac{e^\rho}{2} + \left(\rho_{yy} + \frac{1}{2}(\rho_x^2 - \rho_y^2 - 1)\right)e^{-\rho} \end{bmatrix}^2$$

$$= h_{ij}^2. \tag{1.4}$$

It follows from Proposition 5.3 and equation (5.11) of [4] that

$$\begin{aligned} \operatorname{tr}(h_{ij}) &= e^\rho(1 - K_\infty) \\ &= \frac{e^\rho(2 + k_1 + k_2)}{(1 + k_1)(1 + k_2)}, \end{aligned} \tag{1.5}$$

and, by Proposition 5.5,

$$\begin{aligned} \det h_{ij} &= \frac{K_\infty}{K} e^{2\rho} \\ &= \frac{e^{2\rho}}{(1 + k_1)(1 + k_2)}; \end{aligned} \tag{1.6}$$

k_1 and k_2 are the principal curvatures of M at p with respect to the inner normal, and thus are nonnegative. By a rotation of the (x, y) coordinates we can diagonalize $g_{ij}(p)$ while retaining the conformal nature of ds_∞^2. Using (1.4) through (1.6) to calculate the eigenvalues of $g_{ij}(p)$ we obtain that, in the rotated coordinates,

$$g_{ij}(p) = e^{2\rho(\theta)} \begin{pmatrix} (1 + k_1)^{-1} & 0 \\ 0 & (1 + k_2)^{-1} \end{pmatrix}^2, \tag{1.7}$$

while

$$ds_\infty^2 |_\theta = e^{2\rho(\theta)}(dx^2 + dy^2). \tag{1.8}$$

As both k_1 and k_2 are positive, it is evident from (1.7) and (1.8) that ds_∞^2 dominates the metric on M. The Gauss map is proper and M is assumed to be complete; hence ds_∞^2 is complete as well. □

REMARKS. (1) The results in this section are true in any number of dimensions under the assumption that all principal curvatures relative to the inner normal of the imbedded hypersurface M are positive.

(2) Formulae (1.7) and (1.8) can be used to prove Proposition 5.4 in [4].

PROPOSITION 5.4. *The Gauss map of a surface Σ is conformal if and only if Σ is either totally umbilic or has mean curvature* 2.

2. Proof of the theorem. In this section we prove the main theorem; the principal curvatures are relative to the inner normal and are therefore nonnegative.

THEOREM. *If M is a C^5 imbedding of $\mathbf{R}^2$ into $\mathbf{H}^3$ as a complete surface with nonnegative Gauss curvature then $\partial_\infty M$ is exactly one point.*

Proof. If $\partial_\infty M$ is empty, then M is a compact surface and therefore not an imbedding of $\mathbf{R}^2$. The Gauss curvature of M is $K = k_1 k_2 - 1$. K is positive and therefore $k_1 k_2$ is never zero. The results of the preceding section apply to show that M bounds a convex region and therefore is represented as a smooth envelope of horospheres $\{H(\theta, \rho(\theta)) : \theta \in G(M)\}$; $ds_\infty^2 = e^{2\rho} d\sigma^2$ is a complete conformal metric, and the curvature of ds_∞^2 is given by

$$K_\infty = \frac{k_1 k_2 - 1}{(1+k_1)(1+k_2)} = \frac{K}{(1+k_1)(1+k_2)}.$$

Thus K_∞ is clearly nonnegative. $G(M)$ is a simply connected planar region contained in $S^2 \setminus \partial_\infty M$. We apply a theorem of Huber.

THEOREM 15 [5]. *If S is an open Riemann surface with a complete conformal metric ds^2 with curvature K such that $K^- = \min(K, 0)$ satisfies*

$$\left| \int K^- \, dA \right| < \infty,$$

then S is a parabolic surface.

We conclude that $G(M)$ is a parabolic surface. $G(M)$ is simply connected and thus the uniformization theorem implies that $G(M) = S^2 \setminus \{\theta\}$. From this the theorem follows immediately. □

REMARKS. The hypothesis that the Gauss curvature of M be nonnegative can be weakened to: $K > -1$ everywhere and

$$\left| \int_M K^- \, dA \right| < \infty;$$

for $K\,dA = K_\infty\, dA_\infty$ and G preserves orientation thus:

$$\int_M K^- \, dA = \int_{G(M)} K_\infty^- \, dA_\infty.$$

3. Examples. In this section we will use the representation of surfaces as envelopes of horospheres to construct complete imbedded surfaces of positive curvature. Let ℓ be a diameter of the unit ball and N the north pole with respect to ℓ; define θ to be the azimuthal angle measured with respect to N. Let

$$\rho_\alpha(\theta) = -(1-\alpha)\log(1-\cos\theta).$$

When $\alpha = 0$, $\Sigma(\rho_\alpha)$ is a horosphere tangent to S^2 at N. For $\alpha > 0$, $\Sigma(\rho_\alpha)$ is a surface of revolution with axis of symmetry ℓ. We will prove the following.

PROPOSITION 3.1. *For α sufficiently close to zero, $\Sigma(\rho_\alpha)$ is a complete imbedded surface with positive curvature.*

As $\Sigma(\rho_\alpha)$ is a surface of revolution it suffices to study the generating curve. Let h be a plane containing ℓ; $h \cap \Sigma(\rho_\alpha)$ is a generating curve for $\Sigma(\rho_\alpha)$. We will call it γ_α; γ_α is the envelope of the family of horocycles in h defined by $\rho_\alpha(\theta)\,|_{\partial_\infty h}$, where θ is now taken as a coordinate for $S^1 = \partial_\infty h$. As $\rho_\alpha(\theta)$ is even there is no ambiguity arising from the fact that θ initially ran from 0 to π; henceforth θ will run from 0 to 2π. We can use the simpler two-dimensional theory of envelopes to study γ_α. The following formulae are easily derived from formulae in [4, §§3–6].

LEMMA 3.2. *If $\rho(\theta)$ is a smooth function on a domain Ω in S^1, then the envelope of the horocycles $H(\theta, \rho(\theta))$ is given by the formula*

$$R_\rho(\theta) = \frac{\rho'^2 + (e^{2\rho} - 1)}{\rho'^2 + (e^\rho + 1)^2}(\cos\theta, \sin\theta) + \frac{2\rho'}{\rho'^2 + (e^\rho + 1)^2}(-\sin\theta, \cos\theta). \tag{3.1}$$

If k is the geodesic curvature of $R_\rho(\theta)$, then

$$2\rho'' = \rho'^2 + 1 + (1+k)(1-k)^{-1}e^{2\rho}; \tag{3.2}$$

the induced line element of R_ρ is

$$ds_H^2 = e^{2\rho}(1-k)^{-2}d\theta^2. \tag{3.3}$$

One can also consider $R_\rho(\theta)$ as a curve in the Euclidean plane. The Euclidean line element is related to the hyperbolic line element by

$$ds_E^2 = \frac{(1-R^2)^2}{4}ds_H^2.$$

Using (3.1) and (3.3), we easily obtain

$$ds_E^2 = \left(\frac{e^{2\rho}}{\rho'^2 + (e^\rho + 1)^2}\right)^2 \frac{d\theta^2}{(1-k)^2}. \tag{3.4}$$

Putting ρ_α into (3.1) and simplifying somewhat, we obtain

$$\begin{pmatrix} x_\alpha(\theta) \\ y_\alpha(\theta) \end{pmatrix} = A^{-1}\begin{pmatrix} \cos\theta + (1-\cos\theta)^{1-2\alpha}[\alpha\cos\theta(\alpha\cos\theta + \alpha - 2) + 2 - 2\alpha] \\ \sin\theta[1 + \alpha(1-\cos\theta)^{1-2\alpha}(\alpha\cos\theta + \alpha - 2)] \end{pmatrix},$$
$$A = (1-\alpha)^2(1+\cos\theta)(1-\cos\theta)^{1-2\alpha} + (1 + (1-\cos\theta)^{1-\alpha})^2. \tag{3.5}$$

Putting ρ_α into (3.2), we obtain

$$(1+k)(1-k)^{-1} = -\alpha(1-\cos\theta)^{1-2\alpha}[\alpha + (\alpha-2)\cos\theta]. \tag{3.6}$$

From (3.6) it follows that for $\alpha < \frac{1}{2}$ there are constants a_α and b_α such that

$$-\infty < a_\alpha \leq k(\theta) \leq b_\alpha < 1. \tag{3.7}$$

Using (3.7) and (3.4), one easily sees that when $\alpha < \frac{1}{2}$, ds_E^2 is bounded above and below by constant multiples of $d\theta^2$. Finally, using (3.5) we see that $R_\alpha(\theta) = (x_\alpha(\theta), y_\alpha(\theta))$, considered as a curve in the Euclidean plane, has continuous nonzero tangent vector so long as $\alpha < \frac{1}{4}$. In fact,

$$\sup_{S^1}(|R_\alpha - R_\beta| + |\dot{R}_\alpha - \dot{R}_\beta|) = O(|\beta - \alpha|) \tag{3.8}$$

if α, β are both smaller than $\frac{1}{4}$. Here $|\cdot|$ denotes the Euclidean metric and $\dot{R}_\alpha$ is differentiation with respect to θ. Altogether we have proven the following.

LEMMA 3.3. *For $\alpha \in [0, \frac{1}{4})$, $R_\alpha(\theta)$ is a continuous family of C^1-curves immersed in $\mathbf{R}^2$.*

From this fact it follows easily that $R_\alpha(\theta)$ is imbedded for small α.

LEMMA 3.4. *Let $c_\alpha(\theta)\colon [0, \epsilon) \times S^1 \to \mathbf{R}^2$ be a family of C^1 immersions of S^1 such that*

$$\sup_{S^1}(|c_\alpha(\theta) - c_\beta(\theta)| + |\dot{c}_\alpha(\theta) - \dot{c}_\beta(\theta)|) = \omega(|\beta - \alpha|); \tag{3.9}$$

$\omega(\cdot)$ is a continuous monotone function with $\omega(0) = 0$. Suppose that $c_0(\theta)$ is imbedded; then $c_\alpha(\theta)$ is imbedded for small enough α.

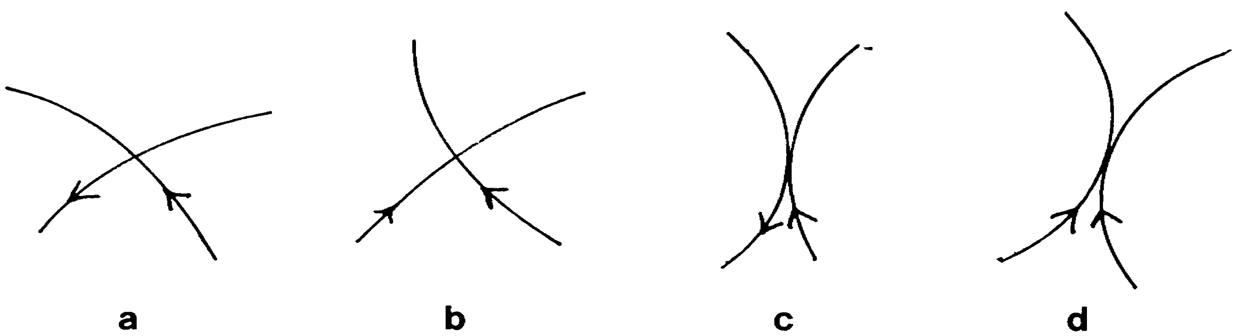
Proof of Lemma 3.4. Suppose not; then we can find $\alpha_n \to 0$ and $\theta_n^1 \neq \theta_n^2$ such that

$$c_{\alpha_n}(\theta_n^1) = c_{\alpha_n}(\theta_n^2).$$

Without loss of generality we can suppose that $|\dot{c}_\alpha(\theta)| \geq c > 0$ for all (α, θ). As $c_0(\theta)$ is imbedded it follows easily that, as $n \to \infty$, $|\theta_n^1 - \theta_n^2|$ tends to zero. From (3.9) it follows that

$$\lim_{n \to \infty} \dot{c}_{\alpha_n}(\theta_n^1) = \lim \dot{c}_{\alpha_n}(\theta_n^2). \tag{3.10}$$

Suppose that θ_n^1 precedes θ_n^2 in some fixed orientation of S^1. At the point of intersection there are four possible configurations.



transverse intersection or tangent contact

One of them must occur infinitely often. We will show this leads to a contradiction. In case a, (3.10) implies that the angle between the two branches must tend to π; from this it follows that the variation in the direction of the tangent vector between $c_{\alpha_n}(\theta_n^1)$ and $c_{\alpha_n}(\theta_n^2)$ must tend to at least 2π, an obvious contradiction. In case b, (3.10) implies that the variation in the direction of the tangent vector must tend to at least 2π. (3.10) and case c are mutually exclusive; case d requires the variation in the tangent vector to tend to at least 2π. Thus the lemma is proved. □

As $\gamma_0(\theta)$ is circle, Lemma 3.4 implies that $\gamma_\alpha(\theta)$ is imbedded for α sufficiently small. Thus $\Sigma(\rho_\alpha)$ is a smoothly imbedded surface for small enough α. (3.7) and (3.3) imply that ds_H^2 is complete on γ_α and thus $\Sigma(\rho_\alpha)$ is complete as well. The curvature of $ds_\infty^2 = e^{2\rho_\alpha} d\sigma^2$ is always positive:

$$\begin{aligned} K_\infty(\alpha) &= (1 - \Delta_{S^2}\rho_\alpha)e^{-2\rho_\alpha} \\ &= 4\alpha(1-\cos\theta)^{2-2\alpha}. \end{aligned}$$

From (3.7) and the fact that $\Sigma(\rho_\alpha)$ is a surface of revolution it follows that the Gauss map is orientation preserving; hence $K(\alpha)$ has the same sign as $K_\infty(\alpha)$. This completes the proof of Proposition 3.1. □

4. Convexity and imbeddedness. For surfaces in Euclidean space, various local convexity assumptions along with completeness have been shown to imply that a surface is actually imbedded and the boundary of a convex region. The names usually associated with this fact are Hadamard, Bouligand, Stoker, Van Heijenoort and Sacksteder. The theorem took a definitive form in [8], where noncompact, nonsmooth surfaces are considered. A generalization of these results to spaces of constant curvature is proved in [3]. The surface is required to be compact. In hyperbolic space some hypothesis about the behavior of the surface near infinity is required. If one appends Hypothesis F below, then Van Heijenoort's theorem and his proof extend to surfaces in hyperbolic space.

HYPOTHESIS F. *Suppose there exists a smooth foliation of* $\mathbf{H}^3$ *by planes* $\{H_t : t \in \mathbf{R}\}$ *such that*

(a) *For all* t, $M \cap H_t$ *is a compact set.*

(b) H_0 *is a local support plane at a point where* M *is locally strictly convex.*

We have the following extension of Van Heijenoort's theorem.

THEOREM 4.1 [8]. *Let* M *be a connected topological surface and* ψ *an immersion of* M *into* $\mathbf{H}^3$ *such that:*

(1) ψ *is locally one-to-one;*

(2) *every point* p *in* M *has a neighborhood* N *such that* $\psi(N)$ *is part of the boundary of a compact convex set;*

(3) $\psi(M)$ *is locally strictly convex at some point* (*as in Hypothesis* F);

(4) *the metric on* M *defined by pulling back the hyperbolic metric via* ψ *is complete; and*

(5) *Hypothesis* F *holds.*

Then $\psi(M)$ *is the boundary of a convex set in* $\mathbf{H}^3$.

Van Heijenoort's proof works essentially without modification, so we will not reproduce it. The foliation H_t serves as the family of parallel planes used in his proof. The corollary of his theorem is as follows.

COROLLARY 4.3. *M is either homeomorphic to a sphere or to a plane. If M is homeomorphic to a plane then* $\partial_\infty\psi(M)$ *is a single point.*

REMARK. The compact case is mentioned in [3].

Proof. The compact case is obvious. If M is noncompact but Hypothesis F holds, then $\psi(M)$ lies in the half-space $\{H_t : t \geq 0\}$. $\psi(M) \cap H_0$ is a point, and $\psi(M) \cap H_t$ (for each positive t) is a compact convex set and therefore a disk compactly contained in H_t. The topological part of the corollary follows from this. Since the planes H_t foliate $\mathbf{H}^3$, it is clear that H_t tends to a point on S^2 as t tends to infinity. As $\psi(M) \cap H_t$ is compactly contained in H_t it also tends to a point on S^2. □

This corollary has the following partial converse.

COROLLARY 4.4. *If $\psi(M)$ is everywhere strictly locally convex, complete as in Theorem* 4.1(4) *and if $\partial_\infty \psi(M)$ is a single point, then $\psi(M)$ is imbedded.*

Proof. Let θ be the asymptotic boundary of $\psi(M)$. The hypotheses of the corollary imply that some geodesic γ with endpoint θ meets $\psi(M)$ transversally. Let $p \in \psi(M)$ be a point of transverse intersection; let H_0 be a local support plane to $\psi(M)$ at p which is transverse to γ. We define the foliation of $\mathbf{H}^3$ by parallel translating H_0 along γ. Let H_t be the parallel plane such that the distance from $H_0 \cap \gamma$ to $H_t \cap \gamma$ is t. We orient time so that H_t tends to θ as t tends to infinity. This foliation clearly satisfies the conditions in Hypothesis F; thus the corollary follows from Theorem 4.1. □

If $\partial_\infty \psi(M)$ has two points, then $\psi(M)$ need not be convex even if it is locally strictly convex. We construct a counterexample in the Klein model. This is a model of $\mathbf{H}^3$ on $\mathbf{B}^3$ in which a surface is locally (strictly) convex in the hyperbolic sense if and only if it is locally (strictly) convex in the Euclidean sense. For details see [7, pp. 2.7, 8.10].

Let N and S be antipodal points on the unit sphere, ℓ the diameter of $\mathbf{B}^3$ connecting them, and P the equatorial plane perpendicular to ℓ. Our example is constructed as the double cone of a curve C lying in P. C is shown in Figure 2.

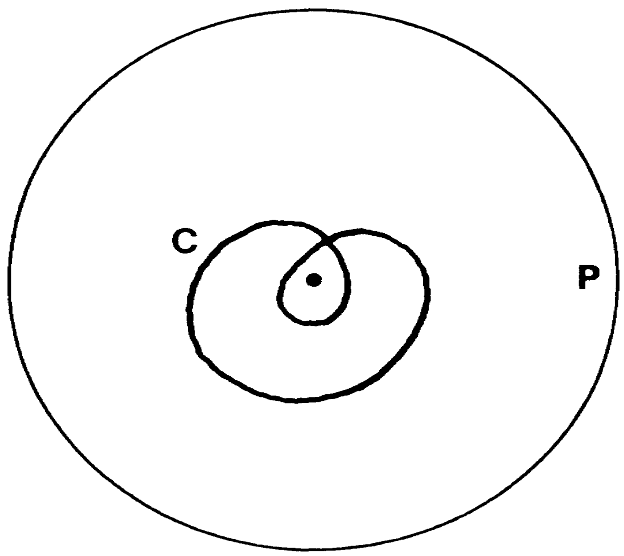


Figure 2

ℓ passes through P at 0. We form the double cone with respect to N and S to obtain $\tilde{M}$ (see Figure 3).

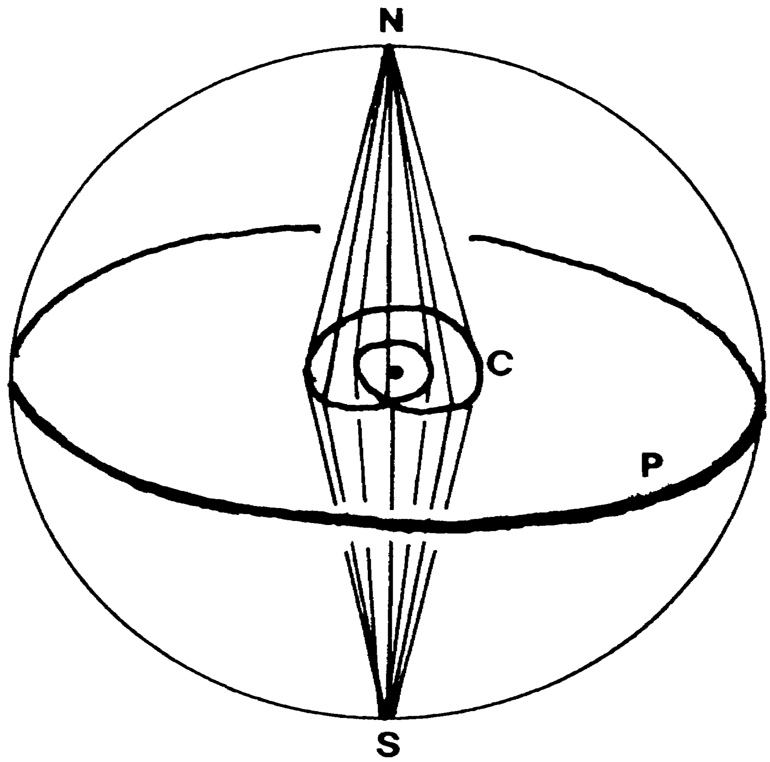


Figure 3

Inflating $\tilde{M}$ a little, we obtain a smooth locally strictly convex surface with self intersections. From simple topological considerations it is clear that this surface cannot have everywhere nonnegative curvature.

This example was suggested by one in [6, p. 172] and a conversation with Bill Thurston.

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