# THE REFLEXIVITY OF CONTRACTIONS WITH NONREDUCTIVE *-RESIDUAL PARTS

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Let $T$ be a contraction on a separable Hilbert space, and suppose that there exist an isometry $V$ ($\neq 0$) and a (bounded linear) operator $Y$ with dense range such that $YT = VY$, which means that the contraction $T$ is not of class $C_{0\cdot}$, that is, $\lim_{n\to\infty}\|T^n x\| \neq 0$ for some $x$ (cf. [6, Proposition II.3.5]). If $V$ is non-unitary, then it is easily seen that every point in the open unit disc $D = \{\lambda : |\lambda| < 1\}$ is an eigenvalue of $T^*$, and so $T$ has many invariant subspaces. It was proved in [2] that $T$ is even reflexive in this case. But, in the case in which $V$ is unitary, it is not yet known whether such a $T$ always has a nontrivial invariant subspace (cf. [8]). In a recent paper [5], Kérchy has proved that if $V$ is a bilateral shift then $T$ has a nontrivial invariant subspace, and under the additional assumption that $T$ is of class $C_{11}$ (i.e., $\lim_{n\to\infty}\|T^n x\| \neq 0$ and $\lim_{n\to\infty}\|T^{*n} x\| \neq 0$ for every nonzero $x$), $T$ is reflexive. The purpose of the present note is to prove a reflexivity theorem which extends these results.

For an operator $T$, let $\operatorname{Alg} T$ denote the weakly closed algebra generated by $T$ and the identity $I$. Let $\operatorname{Lat} T$ and $\operatorname{Alg}\operatorname{Lat} T$ denote the lattice of all invariant subspaces for $T$ and the algebra of all operators $A$ such that $\operatorname{Lat} T \subseteq \operatorname{Lat} A$, respectively. Recall that $T$ is reflexive if $\operatorname{Alg} T = \operatorname{Alg}\operatorname{Lat} T$.

THEOREM. *If $T$ is a contraction on a separable Hilbert space and there exists an operator $Y$ with dense range such that $YT = WY$ for some bilateral shift $W$ ($\neq 0$), then $T$ is reflexive.*

The proof of [1, Theorem 5] shows that in the proof of our Theorem it suffices to consider the case where $T$ is completely non-unitary, that is, where $T$ has no nonzero invariant subspace on which it acts as a unitary operator.

Let $T$ be a completely non-unitary contraction. We use the functional model of Sz.-Nagy and Foiaş [6] for $T$. Let $\Theta$ be the characteristic function of $T$; thus $\Theta$ is an operator-valued $H^\infty$-function on the unit circle $\partial D$ whose values are contractions from $\mathfrak{D}$ to $\mathfrak{D}_*$, where $\mathfrak{D} = (\operatorname{ran}(I - T^*T))^-$ and $\mathfrak{D}_* = (\operatorname{ran}(I - TT^*))^-$. We set

$$\Delta(\zeta) = (I - \Theta(\zeta)^*\Theta(\zeta))^{1/2} \quad \text{and} \quad \Delta_*(\zeta) = (I - \Theta(\zeta)\Theta(\zeta)^*)^{1/2}$$

for $\zeta \in \partial D$ and consider $T$ being defined on the space

$$H(\Theta) = [H^2(\mathfrak{D}_*) \oplus (\Delta L^2(\mathfrak{D}))^-] \ominus \{\Theta h \oplus \Delta h : h \in H^2(\mathfrak{D})\}$$

by

$$T(f \oplus g) = P(\chi f \oplus \chi g), \tag{1}$$

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where for a separable Hilbert space $\mathcal{E}$, $L^2(\mathcal{E})$ and $H^2(\mathcal{E})$ denote (respectively) the spaces of $\mathcal{E}$-valued $L^2$- and $H^2$-functions on $\partial D$, $P$ denotes the orthogonal projection onto $H(\Theta)$, and $\chi(\zeta)=\zeta$, $\zeta\in\partial D$. If $T$ is not of class $C_{0\cdot}$ or, equivalently, if the function $\Delta_*$ is nonzero (cf. [6, Proposition VI.3.5]), then the unitary operator $R_*$ of multiplication by the function $\chi(\zeta)=\zeta$ on $(\Delta_* L^2(\mathfrak{D}_*))^-$ is called the *-residual part of $T$, and it follows from the relation $\Delta_*\Theta=\Theta\Delta$ that the operator $X\colon H(\Theta)\to(\Delta_* L^2(\mathfrak{D}_*))^-$ defined by

$$X(f\oplus g)=-\Delta_* f+\Theta g \tag{2}$$

is nonzero and intertwines $T$ with $R_*$, that is,

$$XT=R_*X \tag{3}$$

(cf. [4]). The assumption of the Theorem implies that the *-residual part $R_*$ of $T$ has a bilateral shift summand, that is, $R_*$ is nonreductive (cf. [3, Proposition VII.5.2]). Indeed, if $T$ satisfies the assumption of the Theorem, then for the operators $Y$ and $W$ we have (by [4, Proposition 4]) an operator $Z$ such that $Y=ZX$ and $ZR_*=WZ$ (see also the proof of [2, Proposition 16]). Since $Y=ZX$ has dense range, $Z$ has dense range too, so it follows from the relation $ZR_*=WZ$ that the unitary operator $R_*$ has a bilateral shift summand (unitarily equivalent to $W$) (cf. [3, Proposition II.5.7]). It is also well known that $R_*$ has a bilateral shift summand exactly when $\Delta_*(\zeta)\neq 0$ a.e. on $\partial D$ (cf. [3, Proposition VII.5.2]).

The following lemma was proved in [5] for the case in which $\Delta(\zeta)=0$ and rank $\Delta_*(\zeta)=1$ a.e.

LEMMA. *Let $\Theta$ be an operator-valued $H^\infty$-function on $\partial D$ whose values are contractions from $\mathfrak{D}$ to $\mathfrak{D}_*$, and let*

$$\Delta(\zeta)=(I-\Theta(\zeta)^*\Theta(\zeta))^{1/2}\quad\textit{and}\quad \Delta_*(\zeta)=(I-\Theta(\zeta)\Theta(\zeta)^*)^{1/2}\quad \textit{for } \zeta\in\partial D.$$

*If $\Delta_*(\zeta)\neq 0$ a.e., then there exist functions $u\in H^2(\mathfrak{D}_*)$ and $v\in(\Delta L^2(\mathfrak{D}))^-$ such that*

$$\|-\Delta_*(\zeta)u(\zeta)+\Theta(\zeta)v(\zeta)\|>\delta\ \textit{a.e.} \tag{4}$$

*for some positive number $\delta$.*

*Proof.* Let us consider first the case in which $\Theta$ is *-outer. We set

$$\alpha=\{\zeta:\|\Delta_*(\zeta)\|>1/2\}$$

and choose a dense sequence $\{x_k\}$ in the unit sphere of $\mathfrak{D}_*$. For $k=1,2,\ldots$, let

$$\alpha_k^{(0)}=\{\zeta\in\alpha:\|\Delta_*(\zeta)x_k\|>1/2\},$$

and we define the sequence $\{\alpha_k\}$ by

$$\alpha_1=\alpha_1^{(0)}\quad\text{and}\quad \alpha_k=\alpha_k^{(0)}\Big\backslash\Big(\bigcup_{j=1}^{k-1}\alpha_j\Big)\quad\text{for } k\geq 2.$$

Obviously $\{\alpha_k\}$ consists of pairwise disjoint measurable sets, and since

$$\|\Delta_*(\zeta)\|=\sup_k\|\Delta_*(\zeta)x_k\|\quad\text{a.e. on } \partial D,$$

we have $\alpha=\bigcup_k \alpha_k$. Take a sequence $\{\rho_k\}$ of positive numbers such that $\sum_k \rho_k=\rho<1/2$. For $k=1,2,\ldots$, let $\hat{u}_k\in H^\infty$ be a function such that

$$|\hat{u}_k|=\chi_{\alpha_k}+\rho_k\chi_{\partial D\setminus\alpha_k} \text{ a.e.,} \tag{5}$$

where $\chi_\beta$ denotes the characteristic function of a measurable set $\beta$ (cf. [3, Corollary IV.6.4]), and set $u_k=\hat{u}_k x_k\in H^2(\mathfrak{D}_*)$. Then, as in the proof of [5, Lemma], we have

$$\sum_{k=1}^\infty \|u_k(\zeta)\|=\sum_{k=1}^\infty \rho_k=\rho \text{ a.e. on } \partial D\setminus\alpha$$

and

$$\sum_{k=1}^\infty \|u_k(\zeta)\|=1+\sum_{k\neq j}\rho_k<1+\rho \text{ a.e. on } \alpha_j$$

for $j=1,2,\ldots$. Thus, for almost every $\zeta\in\partial D$, the series $\sum_k u_k(\zeta)$ converges in $\mathfrak{D}_*$ and we can define the function $u\in H^2(\mathfrak{D}_*)$ by $u(\zeta)=\sum_k u_k(\zeta)$.

Next let us take $h\in(\Delta L^2(\mathfrak{D}))^-$ such that $\|h(\zeta)\|=\chi_{\partial D\setminus\beta}(\zeta)$ a.e., where $\beta=\{\zeta:\Delta(\zeta)=0\}$, and define

$$v=\chi_{\partial D\setminus\alpha}h\in(\Delta L^2(\mathfrak{D}))^-.$$

For almost every $\zeta\in\beta$, $\Theta(\zeta)$ is isometric, and since $\Delta_*(\zeta)\neq 0$ by assumption, we have $\|\Delta_*(\zeta)\|=1$. Therefore it follows that $\beta\subseteq\alpha$, and so

$$\|v(\zeta)\|=\chi_{\partial D\setminus\alpha}(\zeta) \text{ a.e.} \tag{6}$$

We shall show that the functions $u$ and $v$ satisfy (4) with $\delta=1/2-\rho$. For almost every $\zeta\in\alpha_k$ $(k=1,2,\ldots)$, using the relations (5), (6),

$$\|\Delta_*(\zeta)x_k\|>1/2 \quad\text{and}\quad \|\Delta_*(\zeta)\|\leq 1,$$

we have

$$\begin{aligned}\|-\Delta_*(\zeta)u(\zeta)+\Theta(\zeta)v(\zeta)\|&=\|\Delta_*(\zeta)u(\zeta)\|\\&\geq|\hat{u}_k(\zeta)|\,\|\Delta_*(\zeta)x_k\|-\sum_{j\neq k}|\hat{u}_j(\zeta)|\,\|\Delta_*(\zeta)x_j\|\\&>\frac{1}{2}-\sum_{j\neq k}\rho_j>\frac{1}{2}-\rho=\delta.\end{aligned}$$

On the other hand, for almost every $\zeta\in\partial D\setminus\alpha$, since $\|\Delta_*(\zeta)\|\leq 1/2$, we have $I-\Theta(\zeta)\Theta(\zeta)^*\leq(1/4)I$, and so $\Theta(\zeta)\Theta(\zeta)^*\geq(3/4)I$, which shows that $\Theta(\zeta)^*$ is left-invertible. But, since $\Theta$ is *-outer by our assumption, $\Theta(\zeta)^*$ has dense range for almost every $\zeta\in\partial D$ (cf. [6, Proposition V.2.4]). It follows that $\Theta(\zeta)$ is invertible and $\Theta(\zeta)^*\Theta(\zeta)\geq(3/4)I$ a.e. on $\partial D\setminus\alpha$. Therefore, for almost every $\zeta\in\partial D\setminus\alpha$,

$$\begin{aligned}\|-\Delta_*(\zeta)u(\zeta)+\Theta(\zeta)v(\zeta)\|&\geq\|\Theta(\zeta)v(\zeta)\|-\|\Delta_*(\zeta)u(\zeta)\|\\&\geq\frac{\sqrt{3}}{2}\|v(\zeta)\|-\sum_k|\hat{u}_k(\zeta)|\,\|\Delta_*(\zeta)x_k\|\\&\geq\frac{\sqrt{3}}{2}-\sum_k\rho_k=\frac{\sqrt{3}}{2}-\rho>\delta\end{aligned}$$

because, by (5) and (6), $\|v(\zeta)\|=1$ and $|\hat{u}_k(\zeta)|=\rho_k$ $(k=1,2,\ldots)$ for almost every $\zeta\in\partial D\backslash\alpha$. This completes the proof of the case where $\Theta$ is *-outer.

Let us next prove a general case. Let $\Theta=\Theta_2\Theta_1$ be the *-canonical factorization of $\Theta$ (cf. [6, Chapter V]), that is, $\Theta_1$ is a *-inner function whose values are contractions from $\mathfrak{D}$ to some Hilbert space $\mathcal{E}$ and $\Theta_2$ is a *-outer function whose values are contractions from $\mathcal{E}$ to $\mathfrak{D}_*$. Let $\Delta_i(\zeta)=(I-\Theta_i(\zeta)^*\Theta_i(\zeta))^{1/2}$ $(i=1,2)$ and $\Delta_{*2}(\zeta)=(I-\Theta_2(\zeta)\Theta_2(\zeta)^*)^{1/2}$ for $\zeta\in\partial D$. Since $\Theta_1$ is *-inner (i.e., $\Theta_1(\zeta)^*$ is isometric a.e.), $\Delta_*(\zeta)=\Delta_{*2}(\zeta)$ a.e. Thus we can apply the already proved case to the *-outer function $\Theta_2$, so we obtain $u\in H^2(\mathfrak{D}_*)$ and $v'\in(\Delta_2L^2(\mathcal{E}))^-$ such that

$$\|-\Delta_*(\zeta)u(\zeta)+\Theta_2(\zeta)v'(\zeta)\|>\delta \text{ a.e.} \tag{7}$$

for some $\delta>0$. Since the factorization $\Theta=\Theta_2\Theta_1$ is regular (cf. [6, Chapter VII]), we have the unitary operator $Z$ from $(\Delta L^2(\mathfrak{D}))^-$ onto $(\Delta_2L^2(\mathcal{E}))^-\oplus(\Delta_1L^2(\mathfrak{D}))^-$ such that $Z(\Delta g)=\Delta_2\Theta_1 g\oplus\Delta_1 g$ for $g\in L^2(\mathfrak{D})$. Let $v=Z^{-1}(v'\oplus 0)\in(\Delta L^2(\mathfrak{D}))^-$ and let us take a sequence $\{g_n\}$ in $L^2(\mathfrak{D})$ such that $\Delta g_n\to v$ in $L^2(\mathfrak{D})$. Then

$$v'\oplus 0=Zv=\lim_{n\to\infty}Z\Delta g_n=\lim_{n\to\infty}(\Delta_2\Theta_1 g_n\oplus\Delta_1 g_n),$$

so that $v'=\lim_{n\to\infty}\Delta_2\Theta_1 g_n$. But by the relations $\Theta_2\Delta_2=\Delta_{*2}\Theta_2$, $\Theta\Delta=\Delta_*\Theta$ and $\Delta_*=\Delta_{*2}$, we have $\Theta_2\Delta_2\Theta_1=\Theta\Delta$ and hence

$$\Theta_2 v'=\lim_{n\to\infty}\Theta_2\Delta_2\Theta_1 g_n=\lim_{n\to\infty}\Theta\Delta g_n=\Theta v.$$

Therefore it follows from (7) that $u$ and $v$ satisfy (4), and the proof is completed. □

This lemma is used to show that the contraction $T$ in the Theorem has many invariant subspaces $\mathfrak{M}$ for which there are nonzero operators $X$ such that

$$X(T\mid\mathfrak{M})=SX$$

for a unilateral shift $S$, so we can apply the reflexivity result in [2] stated above to the contraction $T\mid\mathfrak{M}$.

For a completely non-unitary contraction $T$, we also use the $H^\infty$-functional calculus of Sz.-Nagy and Foiaş (cf. [6, Chapter III]), which defines a weak*-weak continuous algebra homomorphism $\phi\to\phi(T)$ from $H^\infty$ to $\operatorname{Alg}T$. If $\phi(T)=0$ for some nonzero $\phi\in H^\infty$ then $T$ is said to be of class $C_0$.

*Proof of Theorem.* We consider that $T$ is defined by (1) on $H(\Theta)$ and use the operator $X\colon H(\Theta)\to(\Delta_*L^2(\mathfrak{D}_*))^-$ defined by (2) and the *-residual part $R_*$ of $T$. Since

$$\{\phi(T)\colon\phi\in H^\infty\}\subseteq\operatorname{Alg}T\subseteq\operatorname{Alg}\operatorname{Lat}T,$$

it suffices to show that, for each $A\in\operatorname{Alg}\operatorname{Lat}T$, there exists $\phi\in H^\infty$ such that $A=\phi(T)$.

As noted above, the characteristic function $\Theta$ of $T$ satisfies the assumption of the Lemma, so by the Lemma there exist functions $u\in H^2(\mathfrak{D}_*)$ and $v\in(\Delta L^2(\mathfrak{D}))^-$ satisfying (4). We set

$$x = P(u \oplus v) \in H(\Theta).$$

Then it follows from the relation $\Delta_* \Theta = \Theta \Delta$ that $Xx = -\Delta_* u + \Theta v$, and since (4) implies $\log \|-\Delta_* u + \Theta v\| \in L^1$ we have $Xx = gE$, where $g$ is a scalar outer function in $H^2$ and $E$ is a function in $L^2(\mathfrak{D}_*)$ such that $\|E(\zeta)\| = 1$ a.e. (cf. [3, Corollary IV.6.4 and Theorem IV.6.5]). Therefore,

$$(X\mathfrak{M}_x)^- = \bigvee_{n \geq 0} R_*^n Xx = \{fE : f \in H^2\},$$

where $\mathfrak{M}_y = \bigvee_{n \geq 0} T^n y$ for $y \in H(\Theta)$, so that the isometry $R_* \mid (X\mathfrak{M}_x)^-$ is a unilateral shift and by (3) the operator $X \mid \mathfrak{M}_x$ $(\neq 0)$ intertwines $T \mid \mathfrak{M}_x$ with $R_* \mid (X\mathfrak{M}_x)^-$:

$$(X \mid \mathfrak{M}_x)(T \mid \mathfrak{M}_x) = (R_* \mid (X\mathfrak{M}_x)^-)(X \mid \mathfrak{M}_x).$$

By [2, Theorem 4] it follows that

$$\text{(8)} \qquad \operatorname{Alg\,Lat}(T \mid \mathfrak{M}_x) = \{\phi(T) \mid \mathfrak{M}_x : \phi \in H^\infty\}.$$

Now let us take $A \in \operatorname{Alg\,Lat} T$. Then $\mathfrak{M}_x \in \operatorname{Lat} A$ and $A \mid \mathfrak{M}_x \in \operatorname{Alg\,Lat}(T \mid \mathfrak{M}_x)$; hence by (8) there exists $\phi \in H^\infty$ such that $A \mid \mathfrak{M}_x = \phi(T) \mid \mathfrak{M}_x$. Let us show $A = \phi(T)$. Consider the set

$$\mathcal{C} = \{y \in H(\Theta) : \operatorname*{ess\,sup}_{\zeta \in \partial D} \|(Xy)(\zeta)\| < \delta\}.$$

(Here $\delta$ is a positive number such that $\|(Xx)(\zeta)\| > \delta$ a.e.) Since $\mathcal{C}$ clearly contains the set

$$\{P(f \oplus g) : f \in H^2(\mathfrak{D}_*),\ g \in (\Delta L^2(\mathfrak{D}))^- \text{ and } \operatorname*{ess\,sup}_{\zeta \in \partial D} \|f(\zeta) \oplus g(\zeta)\| < \delta/\sqrt{2}\},$$

whose closed linear span is the whole space $H(\Theta)$, in order to show $A = \phi(T)$ it suffices to prove that $Ay = \phi(T)y$ for every $y \in \mathcal{C}$. Let $y \in \mathcal{C}$ and $z = x + y$. Then, for almost every $\zeta \in \partial D$, we have

$$\|(Xz)(\zeta)\| \geq \|(Xx)(\zeta)\| - \|(Xy)(\zeta)\| > \delta - \operatorname{ess\,sup} \|(Xy)(\zeta)\| > 0;$$

hence, by the argument given for $x$ above, $R_* \mid (X\mathfrak{M}_z)^-$ is a unilateral shift and $(X \mid \mathfrak{M}_z)(T \mid \mathfrak{M}_z) = (R_* \mid (X\mathfrak{M}_z)^-)(X \mid \mathfrak{M}_z)$, so there is $\psi \in H^\infty$ such that $A \mid \mathfrak{M}_z = \psi(T) \mid \mathfrak{M}_z$. Similarly for $w := x + z = 2x + y$, we obtain $\eta \in H^\infty$ such that $A \mid \mathfrak{M}_w = \eta(T) \mid \mathfrak{M}_w$. Then we have

$$\begin{aligned} \eta(T)x + \eta(T)z &= \eta(T)(x+z) = A(x+z) \\ &= Ax + Az = \phi(T)x + \psi(T)z, \end{aligned}$$

so that

$$(\eta - \phi)(T)x = (\psi - \eta)(T)z \in \mathfrak{M}_x \cap \mathfrak{M}_z.$$

Since $\phi(T) = A = \psi(T)$ on $\mathfrak{M}_x \cap \mathfrak{M}_z$, it follows that

$$\text{(9)} \qquad ((\phi - \psi)(\eta - \phi))(T)x = ((\phi - \psi)(\psi - \eta))(T)z = 0.$$

Now, note that it follows from the relation $XT = R_* X$ $(Xx \neq 0)$ that $T \mid \mathfrak{M}_x$ is not of class $C_0$ (cf. [6, Proposition III.4.1]). Similarly, $T \mid \mathfrak{M}_z$ is not of class $C_0$.

Thus (9) implies $(\phi-\psi)(\eta-\phi)=(\phi-\psi)(\psi-\eta)=0$, so that $\phi=\psi$. This shows $A=\phi(T)$ on $\mathfrak{M}_x \vee \mathfrak{M}_z$; in particular, $Ay=\phi(T)y$. This completes the proof. □

It was proved in [7] that a contraction whose restriction to some invariant subspace is a unilateral shift is reflexive. The following proposition shows that our Theorem also extends this result of [7] as well as the one of [2] used above.

PROPOSITION. *Let $T$ be a completely non-unitary contraction and let $\Theta$ be its characteristic function. Then the following conditions are equivalent.*

(i) *$\Theta(\zeta)^*$ is non-isometric for almost every $\zeta \in \partial D$.*
(ii) *There is an operator $Y$ with dense range such that $YT=WY$ for some bilateral shift $W$.*
(iii) *There are $\mathfrak{M} \in \operatorname{Lat} T$ ($\mathfrak{M} \neq \{0\}$) and an operator $Y$ with dense range such that $Y(T\,|\,\mathfrak{M})=WY$ for some bilateral shift $W$.*
(iv) *There are $\mathfrak{M} \in \operatorname{Lat} T$ and a nonzero operator $Y$ such that $Y(T\,|\,\mathfrak{M})=SY$ for some unilateral shift $S$.*

*Proof.* (i)$\Rightarrow$(ii): Let $X$ be the operator used above which intertwines $T$ with its $*$-residual part $R_*$. If (i) holds, then (by the proof of the Theorem) $\operatorname{ran} X$ contains the function of the form $gE$, where $g$ is a scalar outer function and $E$ is a function in $L^2(\mathfrak{D}_*)$ such that $\|E(\zeta)\|=1$ a.e. We define the operator $Y_1$: $(\Delta_* L^2(\mathfrak{D}_*))^- \to L^2$ by $(Y_1 f)(\zeta)=k(\zeta)\langle f(\zeta), E(\zeta)\rangle$, $\zeta \in \partial D$, where $k$ is a function in $L^\infty$ such that $k(\zeta)\neq 0$ a.e. and $\log|k| \notin L^1$ (and $\langle\cdot,\cdot\rangle$ denotes the inner product on $\mathfrak{D}_*$). Clearly the operator $Y=Y_1X$ intertwines $T$ with the bilateral shift $W$ on $L^2$; $YT=WY$, which implies $(\operatorname{ran} Y)^- \in \operatorname{Lat} W$. Since $kg \in \operatorname{ran} Y$ and $g$ is outer, it follows that $kH^2 \subseteq (\operatorname{ran} Y)^-$. But, by Szegö's Theorem (cf. [3, Theorem IV.5.13]), the conditions on $k$ imply $(kH^2)^-=L^2$; hence it follows that $Y$ has dense range. This shows (ii).

The implication (ii)$\Rightarrow$(i) was remarked above (and used in the proof of the Theorem). (i)$\Rightarrow$(iv) was shown in the proof of the Theorem and (iv)$\Rightarrow$(iii) follows from the facts that $S\,|\,(\operatorname{ran} Y)^-$ is a unilateral shift and the unilateral shift $S\,|\,(\operatorname{ran} Y)^-$ is a quasiaffine transform of a bilateral shift $W$ of the same multiplicity; that is, there is an injection $Z$ with dense range such that $Z(S\,|\,(\operatorname{ran} Y)^-)=WZ$.

(iii)$\Rightarrow$(i): Let $\Theta=\Theta_2\Theta_1$ be the regular factorization of $\Theta$ which induces the invariant subspace $\mathfrak{M}$. Since the purely contractive part of $\Theta_1$ coincides with the characteristic function of $T\,|\,\mathfrak{M}$ (cf. [6, Proposition VII.2.1]), applying the implication (ii)$\Rightarrow$(i) to $T\,|\,\mathfrak{M}$ we see that $\Theta_1(\zeta)^*$ is nonisometric a.e. on $\partial D$. Then, since the factorization $\tilde{\Theta}=\tilde{\Theta}_1\tilde{\Theta}_2$ is regular (where for an operator-valued analytic function $A$ on $D$, $\tilde{A}(\lambda)=A(\bar{\lambda})^*$, $\lambda \in D$), it follows that $\Theta(\zeta)^*$ is nonisometric a.e. (cf. [6, Proposition VII.3.3]). The proof is completed. □

Finally, we show that our Theorem can be extended from the case where the space on which $T$ acts is separable to the nonseparable case. Thus, suppose that $T$ is a contraction acting on a nonseparable Hilbert space $\mathcal{H}$ for which there is an operator $Y$ with dense range such that $YT=WY$ for some bilateral shift $W$, and let us show that $T$ is reflexive.

By the proof of [1, Theorem 5], we may assume that $T$ is completely non-unitary. We may also assume that the multiplicity of the bilateral shift $W$ is one, so that $W$ acts on a separable space. (Indeed, consider $W_0$ and $P_0 Y$ instead of $W$ and $Y$, respectively, where $W_0$ is a bilateral shift summand of $W$ of multiplicity one and $P_0$ is the projection onto the subspace on which $W_0$ acts.) Thus the space $\mathcal{H} \ominus \ker Y = (\operatorname{ran} Y^*)^-$ is separable. Let $\mathcal{H}_1$ be the $T$-reducing subspace generated by $\mathcal{H} \ominus \ker Y$, which is separable. We have $T = T_1 \oplus T_2$ on $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and note that $T_1$ satisfies the conditions of the Theorem. Take $A \in \operatorname{Alg Lat} T$. Clearly $A = A_1 \oplus A_2$ on $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and $A_i \in \operatorname{Alg Lat} T_i$ $(i = 1, 2)$. Since $T_1$ satisfies the conditions of the Theorem, the proof of the Theorem shows that

$$A_1 = \phi(T_1) \tag{10}$$

for some $\phi \in H^\infty$, and there is a vector $x \in \mathcal{H}_1$ and an operator $X \neq 0$ such that

$$X(T_1 \mid \mathfrak{M}_x) = SX, \tag{11}$$

where $S$ is a unilateral shift on $\mathcal{K}$ and $\mathfrak{M}_x = \bigvee_{n \geq 0} T^n x = \bigvee_{n \geq 0} T_1^n x$. Let us show $A_2 = \phi(T_2)$, so that $A = \phi(T) \in \operatorname{Alg} T$, which proves the reflexivity of $T$. The relation (11) implies that the (nonzero) operator $\tilde{X} = (X, 0) \colon \mathfrak{M}_x \oplus \mathcal{H}_2 \to \mathcal{K}$ satisfies $\tilde{X}(T \mid (\mathfrak{M}_x \oplus \mathcal{H}_2)) = S\tilde{X}$, and since $A \mid (\mathfrak{M}_x \oplus \mathcal{H}_2) \in \operatorname{Alg Lat}(T \mid (\mathfrak{M}_x \oplus \mathcal{H}_2))$, it follows from [2, Theorem 4] that

$$A \mid (\mathfrak{M}_x \oplus \mathcal{H}_2) = \psi(T) \mid (\mathfrak{M}_x \oplus \mathcal{H}_2)$$

for some $\psi \in H^\infty$, which means $A_1 \mid \mathfrak{M}_x = \psi(T_1) \mid \mathfrak{M}_x$ and $A_2 = \psi(T_2)$. Then by (10) we have $\phi(T_1) \mid \mathfrak{M}_x = \psi(T_1) \mid \mathfrak{M}_x$, and since the relation (11) implies that $T_1 \mid \mathfrak{M}_x$ is not of class $C_0$, we have $\phi = \psi$ and so $A_2 = \phi(T_2)$.

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