INVARIANT PSEUDODIFFERENTIAL OPERATORS ON TWO STEP NILPOTENT LIE GROUPS, II

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In [7] a method was given for constructing parametrices and inverses for invariant hypoelliptic pseudodifferential operators which are homogeneous with respect to the natural dilations on a step two nilpotent Lie group. The construction made use of a calculus for invariant pseudodifferential operators described in [6]. It will be shown here that a similar calculus is also valid in the case of arbitrary dilations on a step two group. The parametrix construction of [7] can then be easily extended to include operators homogeneous with respect to arbitrary dilations. As noted in [8], this construction can be "microlocalized".

In [4] Melin gave a somewhat different parametrix construction on the Heisenberg group and extended this procedure to arbitrary graded Lie groups with the natural dilations in [5]. Glowacki's construction of a commutative approximate identity, given in [2] for arbitrary dilations on the Heisenberg group, makes use of the parametrix construction in [4]. A pseudodifferential operator calculus such as that given below is a prerequisite for extending the results of [2] and [4] to all step two groups with arbitrary dilations.

The classes of pseudodifferential operators considered here differ from those considered in [6] in that here we require estimates for derivatives in all directions, not just the orbit directions. The asymptotic formula (17) for a composition product $p\#q$ is also valid for the classes considered in [6], since the estimates for derivatives of $p\#q$ in the orbit directions will be seen to depend only on estimates for derivatives of p and q in the orbit directions.

The point to be made in this paper is that the calculus in the orbit directions follows naturally from the Weyl calculus of Hörmander [3], while the estimates in non-orbit directions can then be obtained by making use of identities derived from the Lie algebra structure. We note that the development here is somewhat more natural than that in [6], since we have not needed to polarize the orbits.

DEFINITION. A family of dilations on a finite-dimensional Lie algebra $\mathcal{G}$ is a one-parameter family $\delta=\{\delta_r: r>0\}$ of automorphisms of $\mathcal{G}$ such that

$$\delta_r e_j = r^{\mu_j} e_j, \quad \mu_j > 0, \tag{1}$$

for some basis $\{e_1, \ldots, e_n\}$ for $\mathcal{G}$. A connected, simply connected nilpotent Lie group is said to be a homogeneous group if its Lie algebra is endowed with a family of dilations ([1]).

Without loss of generality we may assume that $\min \mu_j = 1$. It can be easily shown that there is a linearly independent set $S=\{e_1, \ldots, e_N\}$ which generates $\mathcal{G}$, satisfies (1), and such that $\mathcal{G}_1 = \operatorname{span} S$ intersects $\mathcal{G}_2 = [\mathcal{G}, \mathcal{G}]$ trivially. Assuming for the

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rest of the paper that $\mathcal{G}$ is step two, let $\{e_{N+1}, \ldots, e_n\}$ be a basis for $\mathcal{G}_2$ chosen so that each e_k, $k > N$, is a multiple of $[e_i, e_j]$ for some $i < j \leq N$. Since δ is a family of automorphisms, if the numbers γ_{ij}^k are defined by

$$[e_i, e_j] = \sum \gamma_{ij}^k e_k, \tag{2}$$

then

$$\gamma_{ij}^k \neq 0 \quad \text{implies} \quad \mu_i + \mu_j = \mu_k. \tag{3}$$

For $x \in \mathcal{G}$ let $|x| = (x_1^2 + \cdots + x_n^2)^{1/2}$, where $(x_1, \ldots, x_n)$ are the coordinates of x with respect to the basis $\{e_1, \ldots, e_n\}$. By replacing each e_k, $N < k \leq n$, by ce_k for sufficiently large c we may assume that

$$|[x, y]| \leq |x|\,|y|, \quad \text{for all } x \text{ and } y \text{ in } \mathcal{G}. \tag{4}$$

We fix a basis $\mathcal{B} = \{e_1, \ldots, e_n\}$ for $\mathcal{G}$ having the properties just described. Coordinates and norms on $\mathcal{G}$ and $\mathcal{G}^*$ will always be with respect to this basis or its dual $\{e_1^*, \ldots, e_n^*\}$.

For $\xi \in \mathcal{G}^* - \{0\}$, define $[\xi]$ by $[\xi] = r$ if $|\delta_r^{-1}\xi| = 1$. Note that, in terms of the chosen coordinate system,

$$[\xi] \approx \sum_{j=1}^{n} |\xi_j|^{1/\mu_j}.$$

Let $\chi: \mathcal{G}^* \to \mathbf{R}$ be a smooth function such that $\chi(\xi) \approx [\xi] + 1$. For ξ and η in $\mathcal{G}^*$ define

$$g_\xi(\eta) = |\delta_{\chi(\xi)}^{-1}\eta|^2.$$

We consider g as determining a Riemannian metric on each of the orbits of the coadjoint action of G in $\mathcal{G}^*$. Since $\mathcal{G}$ is step two nilpotent, if $\mathcal{O}_\xi$ is that orbit containing ξ then $\mathcal{O}_\xi$ is an affine space, $\mathcal{O}_\xi = \xi + T\mathcal{O}_\xi$, where $T\mathcal{O}_\xi = \{\operatorname{ad} x^* \xi : x \in \mathcal{G}\}$. $\mathcal{O}_\xi$ has a natural symplectic structure defined as follows: If η and ζ are in $T\mathcal{O}_\xi$, define $\sigma_\xi(\eta, \zeta) = \langle \eta, z \rangle$ for any z such that $(\operatorname{ad} z)^* \xi = \zeta$. As in Hörmander [3], for $\eta \in T\mathcal{O}_\xi$ define

$$g_\xi^\sigma(\eta) = \sup\{|\sigma_\xi(\eta, \zeta)|^2 / g_\xi(\zeta) : \zeta \in T\mathcal{O}_\xi\}.$$

PROPOSITION. *There exist N, C, and $c > 0$ such that*

$$[\xi - \eta] \leq c\chi(\xi) \quad \textit{implies} \quad c\chi(\eta) \leq \chi(\xi) \leq C\chi(\eta); \tag{5}$$

$$g_\xi(\eta) \leq c \quad \textit{implies} \quad g_\xi(\zeta) \approx g_{\xi+\eta}(\zeta) \quad \textit{for } \zeta \in \mathcal{G}^*; \tag{6}$$

$$g_\xi(\eta) \leq g_\xi^\sigma(\eta) \quad \textit{for all } \xi \in \mathcal{G}^*,\ \eta \in T\mathcal{O}_\xi; \tag{7}$$

$$\chi(\eta) \leq C\chi(\xi)(1 + g_\xi(\eta - \xi))^{1/2} \quad \textit{for all } \xi \in \mathcal{G}^*,\ \eta \in \mathcal{G}^*; \quad \textit{and} \tag{8}$$

$$\chi(\xi) \leq C\chi(\eta)(1 + g_\xi^\sigma(\eta - \xi))^N \quad \textit{for all } \xi \in \mathcal{G}^*,\ \eta \in \mathcal{O}_\xi. \tag{9}$$

Proof. Since $[\xi] \leq C([\eta] + [\xi - \eta])$, there exist $c_1 > 0$ and $C_1 \geq 1$ such that $[\xi - \eta] \leq c_1\chi(\xi)$ implies $\chi(\xi) \leq C_1\chi(\eta)$. (5) follows by letting $c = c_1 C_1^{-1}$, and (6) follows immediately from (5).

Let $\delta=\delta_{\chi(\xi)}$. Then

$$g_\xi^\sigma(\eta)^{1/2}=\sup\{|\langle\eta,z\rangle|/|\delta^{-1}\operatorname{ad}z^*\xi|: z\in\mathcal{G}\}$$
$$=\sup\{|\langle\delta^{-1}\eta,z\rangle|/|\operatorname{ad}z^*\delta^{-1}\xi|: |z|=1\},$$

since $\delta^{-1}\operatorname{ad}z^*\xi=(\operatorname{ad}\delta z)^*\delta^{-1}\xi$. Note that

$$|\operatorname{ad}z^*\delta^{-1}\xi|=\sup\{|\langle\delta^{-1}\xi,[z,y]\rangle|: |y|=1\}\le|\delta^{-1}\xi'|\,|z|$$

by (4), where $\xi'=\xi\mid\mathcal{G}_2$. Thus $g_\xi^\sigma(\eta)^{1/2}\ge|\delta^{-1}\xi'|^{-1}\sup\{|\langle\delta^{-1}\eta,z\rangle|: |z|=1\}$. Consequently,

$$g_\xi(\eta)^{1/2}g_\xi^\sigma(\eta)^{-1/2}\le|\delta_{\chi(\xi)}^{-1}\xi'|, \tag{10}$$

which proves (7), since $|\delta_{\chi(\xi)}^{-1}\xi'|\le|\delta_{[\xi]}^{-1}\xi|=1$.

It follows from (5) that (8) holds if $[\xi-\eta]\le c\chi(\eta)$. Thus to prove (8) it suffices to show that $[\xi-\eta]\le C\chi(\xi)(1+g_\xi(\eta-\xi))^{1/2}$, which follows from $\chi(\xi)^{-1}[\zeta]=[\delta_{\chi(\xi)}^{-1}\zeta]\le C(1+|\delta_{\chi(\xi)}^{-1}\zeta|)$ for all $\zeta\in\mathcal{G}^*$. (8) implies that

$$g_\xi(\zeta)\le Cg_\eta(\zeta)(1+g_\xi(\eta-\xi))^{\bar\mu/2} \tag{11}$$

for all $\zeta\in\mathcal{G}^*$, where $\bar\mu=\max\mu_j$. Hence $g_\eta^\sigma(\zeta)\le Cg_\xi^\sigma(\zeta)(1+g_\xi^\sigma(\eta-\xi))^{\bar\mu/2}$ for $\zeta\in T\mathcal{O}_\xi$ and $\eta\in\mathcal{O}_\xi$. Taking $\zeta=\eta-\xi$, this implies that

$$1+g_\eta^\sigma(\eta-\xi)\le C(1+g_\xi^\sigma(\eta-\xi))^{1+\bar\mu/2},$$

which proves (9) since $\chi(\xi)\le C\chi(\eta)(1+g_\eta^\sigma(\eta-\xi))$ by (8). □

In the terminology of Hörmander [3], (6) and (11) imply that g is slowly varying and σ-temperate on each of the orbits, with constants c and C independent of the orbit. Inequalities (6), (8), and (9) imply that for any real m, χ^m is g continuous and σ, g temperate on each orbit with constants independent of the orbit.

DEFINITION. Let $\delta=\{\delta_r: r>0\}$ be a family of dilations on $\mathcal{G}$ and let $m\in\mathbf{R}$. $S^m(\mathcal{G}^*,\delta)$ is the set of $p\in C^\infty(\mathcal{G}^*)$ such that for every integer $j\ge0$, $\|p\|^j$ is finite, where

$$\|p\|^j=\sup|d^{(j)}p(\xi;\eta_1,\dots,\eta_j)|\chi(\xi)^{-m}\prod g_\xi(\eta_i)^{-1/2}, \tag{12}$$

with the supremum taken over all $\xi\in\mathcal{G}^*$, $(\eta_1,\dots,\eta_j)\in\mathcal{G}^*\times\cdots\times\mathcal{G}^*$. Here $d^{(j)}p$ denotes the jth total derivative of p.

Let $\mathcal{B}^*=\{e_1^*,\dots,e_n^*\}$ be the basis for $\mathcal{G}^*$ chosen earlier and let μ_j be defined by (1). If α is a multi-index, let $\mu\alpha=\sum\mu_j\alpha_j$ and let D^α denote the αth partial derivative with respect to the coordinate system determined by $\mathcal{B}^*$. Noting that $g_\xi(e_j^*)^{1/2}=\chi(\xi)^{-\mu_j}$ we obtain the following characterization of $S^m(\mathcal{G}^*,\delta)$: $p\in S^m(\mathcal{G}^*,\delta)$ if and only if, for every multi-index α,

$$|D^\alpha p(\xi)|\le C_\alpha\chi(\xi)^{m-\mu\alpha}\quad\text{for all }\xi\in\mathcal{G}^*. \tag{13}$$

Note that if $p\in C^\infty(\mathcal{G}^*)$ is homogeneous of degree m with respect to δ for large ξ, then $p\in S^m(\mathcal{G}^*,\delta)$.

For $\xi\in\mathcal{G}^*$, let $\xi'=\xi\mid\mathcal{G}_2$. Define $h(\xi)=|\delta_{\chi(\xi)}^{-1}\xi'|$. It follows from (5) and (8) that h is g-continuous and σ, g temperate on each orbit, with constants independent of the orbit. By (10), $\sup g_\xi(\eta)g_\xi^\sigma(\eta)^{-1}\le h(\xi)^2$, the supremum taken over $\eta\in T\mathcal{O}_\xi$.

DEFINITION. If $\alpha=(\alpha_1,\dots,\alpha_n)$ is a multi-index, let $\alpha'=(\alpha_{N+1},\dots,\alpha_n)$, $N=\dim \mathcal{G}_1$, and $n=\dim \mathcal{G}$. Given $m\in\mathbf{R}$ and $k\geq 0$, define $S^{m,k}(\mathcal{G}^*,\delta)$ to be the set of those functions $p\in S^m(\mathcal{G}^*,\delta)$ such that, for every α,

$$|D^\alpha p(\xi)|\leq C_\alpha h(\xi)^{\max\{k-|\alpha'|,0\}}\chi(\xi)^{m-\mu\alpha},\quad \xi\in\mathcal{G}^*.$$

Let $S_0^{m,k}(\mathcal{G}^*,\delta)$ be the set of symbols for which the corresponding estimates are required to hold only for derivatives parallel to the orbits (see [6]).

The symbol classes $S^{m,k}$ were introduced by Melin in [4] for the case of the natural dilations on the Heisenberg group.

Given $\zeta\in\mathcal{G}^*$, define B_ζ on $\mathcal{G}\times\mathcal{G}$ by $B_\zeta(x,y)=\langle\zeta,[x,y]\rangle$. B_ζ is the symbol of a second-order differential operator $B_\zeta(D)$ on $\mathcal{G}^*\times\mathcal{G}^*$ $(D=-i\partial)$. Given p and q in $C^\infty(\mathcal{G}^*)$ and an integer $j\geq 0$, define

$$\{p,q\}_j(\xi)=B_\xi(D)^j(p\otimes q)(\xi,\xi).$$

LEMMA. *If $p\in S^{m_1,k_1}(\mathcal{G}^*,\delta)$ and $q\in S^{m_2,k_2}(\mathcal{G}^*,\delta)$, then*

$$\{p,q\}_j\in S^{m_1+m_2,k_1+k_2+j}(\mathcal{G}^*,\delta).$$

Proof. Suppose that g is a slowly varying Riemannian metric on an affine space with corresponding vector space V, m is g-continuous, and $u\in S(m,g)$ in the notation of Hörmander [3]. Let $\|u\|_k=\sum_{j\leq k}\|u\|^j$, where $\|u\|^j$ is the norm in $S(m,g)$ analogous to (12). If B is a real bilinear form on $V^*\otimes V^*$ with corresponding linear map $B\colon V^*\to V$, $g_x^B(t)=\sup|\langle\xi,t\rangle|^2/g_x(B\xi)$, and $h(x)^2=\sup g_x(t)/g_x^B(t)$, then for each j there is a C such that

$$|B(D)^j u(x)|\leq Ch(x)^j m(x)\|u\|_{2j}. \tag{14}$$

The constant C depends only on j, not on g, m, or B.

If $\mathcal{O}$ is any orbit of the coadjoint action of G on $\mathcal{G}^*$, define $\bar{g}$ on $\mathcal{O}\times\mathcal{O}$ by $\bar{g}_{\xi_1\xi_2}(\eta_1,\eta_2)=g_{\xi_1}(\eta_1)+g_{\xi_2}(\eta_2)$, where ξ_1 and ξ_2 are in $\mathcal{O}$, η_1 and η_2 in $T\mathcal{O}$. Applying (14) with $B=B_\xi$ on $\mathcal{O}_\xi\times\mathcal{O}_\xi$ and $m=h^{k_1}\chi^{m_1}\otimes h^{k_2}\chi^{m_2}$ as in [3] yields

$$|\{p,q\}_j(\xi)|\leq C_j\, h(\xi)^{k_1+k_2+j}\chi(\xi)^{m_1+m_2}\|p\|_{2j}\|q\|_{2j}, \tag{15}$$

where $\|\ \|_{2j}$ now refers to a seminorm on $S_0^{m_i,k_i}(\mathcal{G}^*,\delta)$, $i=1,2$.

We need similar estimates for the derivatives of $\{p,q\}_j$. To that end, note that if p and q are in $\mathcal{S}(\mathcal{G}^*)$, then

$$\{p,q\}_s(\xi)=\iint e^{i\langle\xi,x+y\rangle}\langle\xi,[x,y]\rangle^s\hat{p}(x)\hat{q}(y)\,dx\,dy.$$

Define γ_{ij}^k by (2). It follows that

$$D_k\{p,q\}_s=\{D_kp,q\}_s+\{p,D_kq\}_s-s\sqrt{-1}\sum\gamma_{ij}^k\{D_ip,D_jq\}_{s-1} \tag{16}$$

for all p and q in $C^\infty(\mathcal{G}^*)$. Since $p\in S^{m,k}(\mathcal{G}^*,\delta)$ implies $D_jp\in S^{m-\mu_j,k}(\mathcal{G}^*,\delta)$ if $j\leq N$, or $D_jp\in S^{m-\mu_j,k-1}(\mathcal{G}^*,\delta)$ if $j>N$, and since $\gamma_{ij}^k=0$ unless $\mu_i+\mu_j=\mu_k$, applying (16) and estimate (15) inductively yields $\{p,q\}_j\in S^{m_1+m_2,k_1+k_2+j}(\mathcal{G}^*,\delta)$. □

DEFINITION. If $p \in \mathcal{S}^*(\mathcal{G}^*)$, define $F_1 p \in \mathcal{S}^*(G)$ by $F_1 p = Fp \circ \log$, where F is the Euclidean Fourier transform and $\log: G \to \mathcal{G}$ is the inverse of the exponential map. If p and q are in $\mathcal{S}(\mathcal{G}^*)$, define

$$p\#q = F_1^{-1}(F_1 p * F_1 q),$$

where $*$ is convolution on G.

Note that if the right invariant operator $Pu = F_1 p * u$ is associated to the symbol p (or if the left invariant operator $\tilde{P}u = u * F_1^{-1}p$ is associated to p), then $p\#q$ is the symbol of PQ (resp., of $\tilde{P}\tilde{Q}$).

Define the weak topology on symbol spaces as in [3].

THEOREM. *The map* $(p, q) \mapsto p\#q$ *from* $\mathcal{S}(\mathcal{G}^*) \times \mathcal{S}(\mathcal{G}^*)$ *to* $\mathcal{S}(\mathcal{G}^*)$ *extends to a weakly continuous map from* $S^{m_1, k_1}(\mathcal{G}^*, \delta) \times S^{m_2, k_2}(\mathcal{G}^*, \delta)$ *to* $S^{m_1+m_2, k_1+k_2}(\mathcal{G}^*, \delta)$. *For any integer* $J \geq 0$ *define*

$$r_J = p\#q - \sum_{j<J} (i/2)^j \{p, q\}_j / j!. \tag{17}$$

Then the map $(p, q) \mapsto r_J$ *is weakly continuous from* $S^{m_1, k_1}(\mathcal{G}^*, \delta) \times S^{m_2, k_2}(\mathcal{G}^*, \delta)$ *to* $S^{m_1+m_2, k_1+k_2+J}(\mathcal{G}^*, \delta)$.

Proof. Let p and q be in $\mathcal{S}(\mathcal{G}^*)$. Then

$$\begin{aligned} p\#q(\xi) &= \int_{\mathcal{G}} \int_{\mathcal{G}} e^{i\langle \xi, x+y+(1/2)[x,y]\rangle} \hat{p}(x)\hat{q}(y)\, dy\, dx \\ &= \int_{\mathcal{G}^*} \int_{\mathcal{G}} e^{i\langle \xi-\eta, x\rangle} p(\eta) q(\xi + \operatorname{ad} \tfrac{1}{2}x^*\xi)\, dx\, d\eta. \end{aligned} \tag{18}$$

Let R_ξ be the radical of the bilinear form B_ξ,

$$R_\xi = \{x : \langle \xi, [x, y]\rangle = 0 \quad \text{for all } y \in \mathcal{G}\}.$$

By applying the Fourier inversion theorem on R_ξ and noting that $T\mathcal{O}_\xi = R_\xi^\perp$, we obtain

$$p\#q(\xi) = \int_{\mathcal{O}_\xi} \int_{\mathcal{G}/R_\xi} e^{i\langle \xi-\eta, x\rangle} p(\eta) q(\xi + \operatorname{ad} \tfrac{1}{2}x^*\xi)\, dx\, d\eta. \tag{19}$$

In particular, $p\#q(\xi)$ is determined by the values of p and q on $\mathcal{O}_\xi$.

Likewise the Fourier inversion theorem implies that if $\zeta \in \mathcal{O}_\xi$, then

$$\begin{aligned} (i/2)B_\xi(D)(p \otimes q)(\xi, \zeta) &= \int_{\mathcal{G}} \int_{\mathcal{G}} e^{i(\langle \xi, x\rangle + \langle \zeta, y\rangle)} i\langle \xi', [x/2, y]\rangle \hat{p}(x)\hat{q}(y)\, dy\, dx \\ &= \int_{\mathcal{O}_\xi} \int_{\mathcal{G}/R_\xi} e^{i\langle \xi-\eta, x\rangle} p(\eta)\, dq(\zeta; \operatorname{ad} \tfrac{1}{2}x^*\xi)\, dx\, d\eta. \end{aligned}$$

Since the exponential of the directional derivative $q \mapsto dq(\cdot; \operatorname{ad} \frac{1}{2}x^*\xi)$ is the operator of translation by $\operatorname{ad} \frac{1}{2}x^*\xi$, we find that

$$p\#q(\xi) = \exp(\tfrac{1}{2}iB_\xi(D))(p \otimes q)(\xi, \xi).$$

By Theorem 3.6 of [3] and the Proposition above, $(p,q)\mapsto r_j$ is weakly continuous from $S_0^{m_1,k_1}(\mathcal{G}^*,\delta)\times S_0^{m_2,k_2}(\mathcal{G}^*,\delta)$ to $S_0^{m_1+m_2,k_1+k_2+J}(\mathcal{G}^*,\delta)$. In particular,

$$|r_J(\xi)|\le C\chi(\xi)^{m_1+m_2}h(\xi)^{k_1+k_2+J}\|p\|\,\|q\| \tag{20}$$

for some seminorm $\|\ \|$ on $S_0^{m_i,k_i}(\mathcal{G}^*\delta)$, $i=1,2$.

We still need to prove the appropriate estimates for derivatives of r_j. If p and q are in $\mathcal{S}(\mathcal{G}^*)$, consider the derivatives dp and dq as elements of $\mathcal{S}(\mathcal{G}^*)\otimes\mathcal{G}$. Given $\zeta\in\mathcal{G}_2^*$, $B_\zeta\colon\mathcal{G}\times\mathcal{G}\to\mathcal{G}$ and $\#\colon\mathcal{S}(\mathcal{G}^*)\times\mathcal{S}(\mathcal{G}^*)\to\mathcal{S}(\mathcal{G}^*)$ are bilinear. We define $B_\zeta^\#(p,q)=(\#\otimes B_\zeta)(dp,dq)$. If $\zeta=\sum\zeta_k e_k^*$ and γ_{ij}^k are defined by (2), then

$$B_\zeta^\#(p,q)=\sum\gamma_{ij}^k\zeta_k\partial_i p\#\partial_j q. \tag{21}$$

Let $B_j^\#=B_{e_j}^\#$ if $j>N$, $B_j=0$ if $j\le N$. It follows from (18) that

$$D_j(p\#q)=D_jp\#q+p\#D_jq-\tfrac{1}{2}B_j^\#(p,q). \tag{22}$$

Applying (20) to D_ip and D_lq when $\gamma_{il}^j\ne 0$, (21) implies that

$$\Big|-\tfrac{1}{2}B_j^\#(p,q)(\xi)-\tfrac{1}{2}\sum_{s<J}\sum_{i,l}(i/2)^{s-1}\gamma_{il}^j\{D_ip,D_lq\}_{s-1}(\xi)\Big|$$
$$\le C\chi(\xi)^{m_1+m_2-\mu_j}h(\xi)^{k_1+k_2+J-1}\|p\|\,\|q\|.$$

It follows from (22) and (16) that

$$|D_jr_J(\xi)|\le\begin{cases}C\chi(\xi)^{m_1+m_2-\mu_j}h(\xi)^{k_1+k_2+J} & \text{if } j\le N;\\ C\chi(\xi)^{m_1+m_2-\mu_j}h(\xi)^{k_1+k_2+J-1} & \text{if } j>N.\end{cases}$$

Estimates for higher derivatives follow by induction. The theorem now follows from the fact that every $p\in S^m(\mathcal{G}^*,\delta)$ is the weak limit of a sequence in $\mathcal{S}(\mathcal{G}^*)$. □

Concerning other matters that are usually part of a pseudodifferential operator calculus, we note that if $Pu=F_1p*u$, then $P^*u=F_1\bar{p}*u$. Also, by the same proof as in [6], if $p\in S^0(\mathcal{G}^*,\delta)$, then $P\colon L^2(G)\to L^2(G)$ is bounded.

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