ON THE MEAN VALUE THEOREM FOR ANALYTIC FUNCTIONS

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In all that follows $D(r,a)$ will denote the disk $|z-a|<r$ in the complex plane C; we abbreviate $D(r,0)$ by $D(r)$ and $D(1)$ by D. The plane including the point at infinity will be denoted by C_∞. For any region $R\subset C$, $B(R)$ will denote the family of all analytic functions f on D for which $f'(D)\subset R$. The mean value theorem of the differential calculus can be viewed as saying that if f is a differentiable real valued function on an open interval I for which $f'(I)\subset J$, then $f[b,a]=(f(b)-f(a))/(b-a)\in J$ for all pairs of distinct points a,b in I. The direct analogue of this is not true for analytic functions in D since, for example, $f[b,a]$ can be 0 even though $f'(D)$ does not contain 0. As a way of quantifying the degree to which the mean value theorem does hold for functions in $B(R)$, one is led to consider the supremum, denoted by $m(R)$, of all $r<1$ for which $f\in B(R)$ implies that $f[b,a]\in R$ for all $a,b\in D(r)$. (Here and in what follows, $f[b,a]$ is defined to be $f'(b)$ in the case that $a=b$.)

In the course of this discussion use is made of several theorems of classical complex analysis such as Bloch's theorem and the uniformization theorem; since these facts are well known and fully treated in many texts, no explicit references are given for them. The main facts established in what follows appear as indented statements, labeled with the letters (A)–(G).

1. Conditions for $m(R)<1$ and $m(R)>0$. The purpose of this section is to characterize those $R\subset C$ for which the study of $m(R)$ is interesting, that is, to determine when $0<m(R)<1$. We begin by pointing out that

(A) $m(R)=1$ if and only if R is convex.

The sufficiency of the convexity is well known and is easy to see since $f[b,a]=\int_0^1 f'(a+t(b-a))\,dt$. Conversely, let $R\subset C$ be nonconvex. Then ([2, Theorem 4.8]) there exists $c\in\partial R$ such that for some $r>0$ and real t, $e^{it}(R-c)$ contains the set $S=\{z: 0<|z|<r,\ \mathrm{Re}\{z\}\geq 0\}$. From this it follows that for some $\delta\in(0,r/2)$, $e^{it}(R-c)$ contains the set $T=S\cup D(\delta, ir/2)\cup D(\delta, -ir/2)$, which is symmetric with respect to the real axis. Let g' map D one-to-one onto T in such a way that g' is real on the real axis and $g'(-e^{i\theta})=-ir$ and $g'(-e^{-i\theta})=ir$. Then for $\theta>0$ sufficiently small we have $g[-e^{-i\theta}, -e^{i\theta}]<0$. But for s real, $g[-se^{-i\theta}, -se^{i\theta}]$ is real and tends to $g'(0)>0$ as s tends to 0. Thus there exist distinct points a,b in D such that $g(a)=g(b)$. But then $f=e^{-it}g+cz\in B(R)$ and $f[b,a]=c\notin R$, so that $m(R)<1$. It is to be noted that for any given nonconvex R a construction along these lines may be used to determine an upper bound (less than 1) for $m(R)$.

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It is also not difficult to determine when $m(R)>0$. For this purpose it is convenient to work with the radius of univalence $u(R)$ of $B(R)$, that is, with $u(R)=\sup\{r<1: f\in B(R)$ implies that f is univalent in $D(R)\}$. Clearly, $m(R)$ is the supremum of all $r<1$ such that $f[b,a]\notin \partial R$ for all $f\in B(R)$ and all $a,b\in D(r)$. Since for $a\neq b$, $f[b,a]=c$ if and only if $g(b)=g(a)$, where $g=f-cz$, we see that

$$m(R)=\inf\{u(R-c): c\in\partial R\}. \tag{1}$$

For $0<r<s$ let $A(r,s)=\{z: r<|z|<s\}$ and define

$$\mu(R)=\tfrac{1}{2}\sup\{\log(s/r): A(r,s)\subset R\};$$

we take $\mu(R)$ to be 0 if R contains no ring centered at 0; clearly, $\mu(R)=\infty$ if $0\in R$. We show that

$$\frac{3}{4}\beta(\mu^2(R)+\pi^2)^{-1/2}\leq u(R)\leq\frac{\pi}{\mu(R)}, \tag{2}$$

where β is the Bloch constant defined as $\inf\{r>0:$ all f analytic in D with $|f'(0)|\geq 1$ cover a disk of radius $r\}$.

In order to prove (2), we may assume that $0\notin R$, since otherwise $u(R)=0$ and $\mu(R)=\infty$. Next we define $W(S)=\sup\{r>0: D(r,c)\subset S$ for some $c\in C\}$ and $d(D,z)=\sup\{|f'(z)|: f(D)\subset S\}$.

We have

$$W(S)\leq d(S,0)\leq W(S)/\beta, \tag{3}$$

where the first inequality follows since if $D(r,c)\subset S$ then $rz+c$ maps D into S, and the second from the fact that if $f(D)\subset S$, then S contains a disk of radius $\beta|f'(0)|$, so that $W(S)\geq\beta d(S,0)$. If $g(D)\subset S$ then $f(D)\subset S$ also, where $f(w)=g((w+z)/(1+\bar{z}w))$ for any $z\in D$. Since $f'(0)=g'(z)(1-|z|^2)$, we conclude that

$$d(S,z)=\frac{d(S,0)}{(1-|z|^2)}. \tag{4}$$

If $\log(R)$ denotes the set of all values if $\log z$ for $z\in R$, then

$$\mu(R)\leq W(\log(R))\leq(\mu^2(R)+\pi^2)^{1/2}. \tag{5}$$

The first of these inequalities follows since if R contains the ring $A(r,s)$ then $\log(R)$ contains a strip of width $\log(s/r)$ and therefore also a disk of radius $\frac{1}{2}\log(s/r)$. To see the second inequality of (5) note that if $\log(R)$ contains the disk $D(r,c)$, then $\log(R)$ contains the rectangle

$$\{z: |\operatorname{Im}\{z-c\}|\leq\pi, |\operatorname{Re}\{z-c\}|<(r^2-\pi^2)^{1/2}\},$$

so that R contains the ring $A(e^{\operatorname{Re}\{c\}-(r^2-\pi^2)^{1/2}}, e^{\operatorname{Re}\{c\}+(r^2-\pi^2)^{1/2}})$ and this means that $\mu(R)\geq(r^2-\pi^2)^{1/2}$. If $f\in B(R)$ one has by (3), (4), and (5) that

$$|f''(z)/f'(z)|\leq d(\log(R),z)\leq\frac{(\mu^2(R)+\pi^2)^{1/2}}{(1-|z|^2)\beta}\leq\frac{4}{3\beta}(\mu^2(R)+\pi^2)^{1/2}$$

for $|z|\leq 1/2$. The first inequality of (2) now follows since $|z(f''(z)/f'(z))|\leq 1$ in $D(r)$ and $f'(0)\neq 0$ imply that f is convex, and hence univalent, in $D(r)$. For the upper bound, let $\epsilon>0$ and r, s be such that $A(r^2, s^2)\subset R$ and $\log(s/r)>\mu(R)-\epsilon$. Then the function $g(z)=(rs/\log(s/r))e^{z\log(s/r)}$ belongs to $B(A(r^2,s^2))\subset B(R)$ and $g(i\pi/\log(s/r))=g(-i\pi/\log(s/r))$. Upon letting ϵ tend to 0 we conclude that $u(R)\leq\pi/\mu(R)$.

It follows from (1) and (2) that

(B) $\quad m(R)>0$ if and only if $\sup\{\mu(R-c): c\in\partial R\}<\infty$.

We close this section with an observation about the case of simply connected R which is relevant to what is to be done in the next section. Let us define m_{sc} to be $\inf\{m(R): R \text{ is simply connected}\}$. If R is simply connected and 0 is not in R, then $\mu(R)=0$, so that by (2) $u(R)\geq 3\beta/4\pi$ and therefore by (1) m_{sc} is bounded below by the same number. It is easy, however, to derive a better lower bound for m_{sc}. If $R\neq C$ is simply connected and $f\in B(R)$, then $f'(z)=g(h(z))$, where g maps D conformally onto R and $h(D)\subset D$ with $h(0)=0$. It is well known that $g(D(\rho))=R'\subset R$ is convex, where $\rho=2-\sqrt{3}$. Let $f_1(z)=f(\rho z)/\rho$. Then $f_1'(D)=f'(D(\rho))\subset R'$, so that if $a,b\in D(\rho)$, $f[b,a]=f_1[b/\rho, a/\rho]\in R'\subset R$ by (A). Thus we have that $m_{sc}\geq 2-\sqrt{3}$. The determination of the exact value of m_{sc} is an interesting problem and it would seem reasonable to conjecture that it is $m(C\backslash[1,\infty))$.

2. Extremal functions. In the next section we shall show how a simple variational argument can be used to obtain some information about $m(R)$ for a wide range of domains R. In order to do this one must work with a function in $B(R)$ which displays extremal behavior with regard to $m(R)$. We define an *extremal function* for R to be a function $f_0\in B(R)$ such that $f_0[b,a]\in\partial R$ for some pair a, b on $\partial D(m(R))$. Before discussing conditions on R that guarantee the existence of an extremal function, we point out that

(6) $\quad$ if $f\in B(R)$ and a, b are two points of $\overline{D(m(R))}$ for which $f[b,a]\notin R$, then f is an extremal function for R, $|a-b|\geq m(1-m)$, and $a,b\in\partial D(m)$, where $m=m(R)$.

To see this note that application of the definition of m to the function $f_1(z)=f((1-m)z+a)/(1-m)$ which also belongs to $B(R)$ shows that $f[c,a]\in R$ for $c\in D(m(1-m),a)$. This implies that $|a-b|\geq m(1-m)$. That $f[b,a]\in\partial R$ follows since the continuity of $f[z,w]$ in both arguments implies that $f[b,a]$ cannot lie in the interior of the complement of R. The hypotheses imply that f' is not constant, so that $f[z,a']$ is a nonconstant analytic function of z in D for each $a'\in D$. If $b\in D(m)$ then, for a' sufficiently near a, $f[D(m),a']$ contains $f[b,a]$. This means that there are two points $a',b'\in D(m)$ for which $f[b',a']\in\partial R$, which contradicts the definition of m. Thus $b\in\partial D(m)$, and by symmetry $a\in\partial D(m)$.

For the remainder of this section we shall assume that

(7) $\quad Q=C_\infty\backslash R$ has n components $Q_1,\ldots,Q_n$.

Since we are interested in R for which $m(R)>0$, in light of (B) no additional restriction is imposed by also assuming that

(8) $\quad$ each Q_i contains more than one point and one of the Q_i contains ∞.

It is slightly more convenient to work with the chordal metric

$$\chi(z,w)=\frac{|z-w|}{((1+|z|^2)(1+|w|^2))^{1/2}}$$

on C_∞ than with the Euclidean metric on C. We use $\operatorname{diam}(X)$ to denote the diameter of $X\subset C_\infty$ with respect to χ, $\delta(Q)$ to denote $\min\{\operatorname{diam}(Q_i):1\le 1\le n\}$, and $N(r,p)$ to denote the r-neighborhood of p with respect to χ. The following gives a condition which guarantees the existence of an extremal function.

(C) Let (7) and (8) hold for R. If $m=m(R)<m_{sc}$, then there exists an extremal function f_0 for R. Furthermore, for any such extremal function

$$\chi(f_0'(0),Q)\ge\frac{\delta^3}{16(1-m/m_{sc})^{-2}+2\delta^2},$$

where $\delta=\delta(Q)$.

The fact that one can get an explicit lower bound for $\chi(f_0'(0),Q)$ has some bearing on the discussion in the following section. It is also significant that this lower bound tends to 0 as m/m_{sc} tends to 1. The proof of (C) is not difficult and is based on the following elementary lemma:

(9) Let $S\subset C_\infty$ be a simply connected domain. If $\operatorname{diam}(C_\infty\backslash S)\ge\delta>0$, $g(D)\subset S$, and $d=\chi(g(0),C_\infty\backslash S)<\delta/2$, then

$$\chi(g(z),g(0))\le\frac{16d}{\delta(\delta-2d)(1-|z|)^2}.$$

To see that this is so, let $c\in C_\infty\backslash S$ satisfy $\chi(g(0),c)=d$. Since we are dealing with the chordal metric we may assume that $\infty\notin S$ and that $\chi(c,\infty)\ge\delta/2$. This means that $\chi(g(0),\infty)\ge\delta/2-d$, so that from the formula for the chordal metric we have $|g(0)-c|\le 4d/(\delta(\delta-2d))$. Let h map D one-to-one conformally onto S with $h(0)=g(0)$. The ¼-theorem implies that S contains a circle of radius $|h'(0)|/4$ about $h(0)$, so that $|h'(0)|\le 16d/(\delta(\delta-2d))$. An application of the distortion theorem for normalized univalent functions then gives $|h(z)-h(0)|\le 16d/(\delta(\delta-2d)(1-|z|)^2)$. The desired bound now follows since $g-g(0)$ is subordinate to $h-h(0)$ and $\chi(g(z),g(0))\le|g(z)-g(0)|$.

The next step in the proof of (C) is to deduce the following from (9).

(10) Let (7) and (8) hold for R and let $\delta=\delta(Q)$. If $f\in B(R)$ and $f[b,a]\notin R$ for some pair a,b of points in $D(rm_{sc})$, then

$$\chi(f'(0),Q)\ge\delta^3/(16(1-r)^{-2}+2\delta^2).$$

To see this, let $f\in B(R)$ and $d=\chi(f'(0),Q)=\chi(f'(0),Q_i)$. If $d<\delta/2$, we can apply (9) with $S=C\backslash Q_i$ to conclude that $f'(D(r))$ is contained in a component of $N(\rho,f'(0))\backslash Q$, where $\rho=16d/(\delta(\delta-2d)(1-r)^2)$. Such a set will be simply connected provided that $\rho<\delta$. Thus for a given $r<1$ there can exist $a,b\in D(rm_{sc})$ for which $f[b,a]\notin R$ only if $\rho\ge\delta$ or $d\ge\delta/2$, which yields the desired conclusion.

It is now an easy matter to prove (C). By definition of m there exists a sequence $\{f_n\}$ of functions in $B(R)$ with $f_n(0)=0$ and two sequences $\{a_n\}$ and $\{b_n\}$ of points such that $a_n, b_n \in D(m+1/n)$ and $f_n[b_n, a_n] \notin R$. As in the proof of (6), $|a_n-b_n| \geq m(1-m-1/n)$ and, by (10),

$$\chi(f_n'(0), Q) \geq \frac{\delta^3}{16(1-(m+1/n)/m_{sc})^{-2}+2\delta^2}.$$

Since $f_n'(D) \subset R$, $\{f_n'\}$ contains a subsequence which converges locally uniformly in D either to a function mapping D into R or to a constant (possibly ∞) on ∂R. This latter possibility is, however, ruled out by the preceding lower bound for $\chi(f_n'(0), Q)$. Thus $\{f_n\}$ has a subsequence which converges locally uniformly in D to $f_0 \in B(R)$, and by choosing a subsequence of this subsequence we may assume that $a_n \to a$ and $b_n \to b$. Since $|a-b| \geq m(1-m)>0$, it follows that $f_0[b, a] \notin R$, so that by (6) f_0 is an extremal function. The validity of the second sentence of (C) follows from (10).

Without further assumptions about the nature of ∂R, it would appear that the hypothesis that $m<m_{sc}$ made in the statement of (C) is necessary; we shall discuss an example to this effect at the end of the next section. However, there necessarily exists an extremal function for any nonconvex region whose boundary consists of a finite number of *continuously differentiable* Jordan curves. To see this, let R be such a region and let $\{f_n\}$, $\{a_n\}$, and $\{b_n\}$ be as in the preceding paragraph. Since R is bounded, there exists a constant L such that $|p-q| \leq L\chi(p, q)$ for all p, q in the closure of R. Let $\delta=\delta(Q)$ and let $r=(m+1)/2$. Let $p_n \in \partial R$ be such that the absolute value of $q_n=f_n'(0)-p_n$ is equal to the (Euclidean) distance from $f_n'(0)$ to ∂R. If $|q_n|$ has a positive lower bound, then the existence of an extremal function follows as in the preceding paragraph. Thus, in order to prove the existence of an extremal function it suffices to obtain a contradiction from the assumption that $\chi(f_n'(0), Q)<\delta/4$ for all n and that $|q_n| \to 0$. If $z \in D(r)$, then by (9) and the fact that $\chi(z_1, z_2) \leq |z_1-z_2|$ we have

$$|f_n'(z)-p_n| \leq |f_n'(z)-f_n'(0)|+|f_n'(0)-p_n| \leq L\chi(f_n'(z), f_n'(0))+|q_n| \leq L'|q_n|,$$

where $L'=1+32L(\delta(1-r))^{-2}$. Thus $f_n'(D(r)) \subset D(p_n, L'|q_n|) \cap R=R_n$. By our asumption that the components of ∂R are smooth Jordan curves, $(R_n-p_n)/q_n$ tends to $D^+=\{z \in D(L'): \operatorname{Re}\{z\}>0\}$ with respect to the Blaschke metric on sets. By replacing $\{f_n\}$ by one of its subsequences we have that $w_n=(f_n-p_n z)/q_n$ tends locally uniformly on $D(r)$ to a function w for which $w'(D(r))$ is contained in the closure of D^+. But $(f_n'(0)-p_n)/q_n=1$, so that $w'(D(r)) \subset D^+$. Since D^+ is convex, we have by (A) that $w[b, a] \in D^+$ for $a, b \in D(r)$, so that for n sufficiently large $w_n[b_n, a_n] \in (R_n-p_n)/q_n$, or equivalently $f_n[b_n, a_n] \in R_n \subset R$. Since this contradicts the defining properties of $\{f_n\}$, $\{a_n\}$, and $\{b_n\}$, we have the existence of an extremal function.

For a given R with a finite number of continuously differentiable boundary components it is possible to give an a priori lower bound for the distance from $f'(0)$ to ∂R. We shall not, however, go into the details of how this can be done.

3. Variation of extremal functions. We now assume that an extremal function f_1 for R exists and apply a simple variational argument to draw some conclusions about the nature of such functions. Let $m=m(R)$, $a, b\in\partial D(m)$ satisfy $f_1[b,a]=c\in\partial R$, and let $f_0=f_1-cz$. Then f_0 is an extremal function for $R-c$ and $f_0(b)=f_0(a)$. Let $g_0=f_0'$, $b=me^{i\phi}$ and $g_0(b)=Ae^{i\tau}$, $A>0$. If $-\pi/2<\theta<\pi/2$ and $0<\rho<2m\cos\theta$, then $b-\delta\in D(m)$, where $\delta=\rho e^{i(\theta+\phi)}$. Assume that

(11) $g_\epsilon(z)=g_0(z)+\epsilon q(z)+O(\epsilon^2)$ uniformly on compact subsets of D and $g_\epsilon(D)\subset R-c$ for $0\le\epsilon\le\epsilon_0$.

Since $f_0(b)=f_0(a)$, for any such ϵ, ρ, and θ, we have

$$\begin{aligned}\int_a^{b-\delta} g_\epsilon(z)\,dz &= \int_b^{b-\delta} g_0(z)\,dz+\epsilon\int_a^b q(z)\,dz+O(\epsilon\rho+\epsilon^2)\\ &= -g_0(b)\delta+\epsilon\int_a^b q(z)\,dz+O(\epsilon^2+\rho^2)\\ &= -A\rho e^{i(\theta+\tau+\phi)}+\epsilon\int_a^b q(z)\,dz+O(\epsilon^2+\rho^2)\\ &= e^{i(\phi+\tau)}\left(-A\rho e^{i\theta}+\epsilon e^{-i(\phi+\tau)}\int_a^b q(z)\,dz+O(\epsilon^2+\rho^2)\right).\end{aligned}$$

It is clear from the form of this last expression that if $e^{-i(\phi+\tau)}\int_a^b q(z)\,dz$ lies in the right half-plane, then for each sufficiently small ϵ there will exist θ,ρ (satisfying the stipulated conditions) for which $\int_a^{b-\delta} g_\epsilon(z)\,dz=0$. Since $b-\delta\in D(m)$ and $0\notin R-c$, this would imply that there exist $a', b'\in D(m)$ for which $g_\epsilon(a')=g_\epsilon(b')$, which contradicts the definition of m. Thus

$$\operatorname{Re}\left\{e^{-i(\phi+\tau)}\int_a^b q(z)\,dz\right\}\le 0. \tag{12}$$

Let U be the strip $\{z\colon -1<z<1\}$ and let K be any fixed conformal mapping of U onto the universal covering surface of R. We will not explicitly distinguish between K and its projection onto R; this should cause no confusion. Let $h(z)=K^{-1}(g_0(z)+c)$. Then $h(D)\subset U$, so that $v_0=\operatorname{Re}\{h\}$ satisfies $|v_0(z)|\le 1$ in D. This implies that the radial limit $v_0(\zeta)=\lim_{r\to 1} v_0(r\zeta)$ exists for almost all $\zeta\in\partial D$. Using the variational condition (12) we show that

(D) ∂D can be expressed as the union of an even number k of closed arcs with nonempty disjoint interiors, such that in the interior of each arc v_0 is constantly either 1 or -1 and the signs alternate as ∂D is traversed.

To see this we define, for each fixed $\xi\ge 0$,

$$N_-(\xi)=\{\zeta\in\partial D\colon v_0(\zeta)>\xi-1\}\quad\text{and}\quad N_+(\xi)=\{\zeta\in\partial D\colon v_0(\zeta)<1-\xi\}.$$

If $\xi>0$ and u is any nonnegative bounded function on ∂D whose support lies in $N_+(\xi)$, then for $0<\epsilon\le\epsilon_0$ we have that $-1\le v_0(\zeta)+\epsilon u(\zeta)\le 1$ for almost all $\zeta\in\partial D$, where ϵ_0 is some small positive number. By the Poisson representation of an analytic function in D in terms of the values of its real part on ∂D, we have that

$$g_\epsilon(z)=K\left(\int_0^{2\pi} P(\zeta,z)(v_0(\zeta)+\epsilon u(\zeta))\,\delta\theta+i\,\mathrm{Im}\{h(0)\}\right)-c \tag{14}$$

satisfies (11) with

$$q(z)=K'(h(z))\int_0^{2\pi} P(\zeta,z)u(\zeta)\,d\theta,$$

where $\zeta=e^{i\theta}$ and $P(\zeta,z)=(\zeta+z)/2\pi(\zeta-z)$. We therefore conclude from (12) that

$$\mathrm{Re}\left\{\int_0^{2\pi}\left(e^{-i(\phi+\tau)}\int_a^b K'(h(z))P(\zeta,z)\,dz\right)u(\zeta)\,d\theta\right\}\le 0.$$

Since $\xi>0$ is arbitrary and since $u(\zeta)$ is any nonnegative bounded function on ∂D with support in $N_+(\xi)$, we have that

$$V(\zeta)=e^{-i(\phi+\tau)}\int_a^b K'(h(z))P(\zeta,z)\,dz \tag{15}$$

satisfies $\mathrm{Re}\{V(\zeta)\}\le 0$ for almost all $\zeta\in N_+(0)$. In the same manner one sees that $\mathrm{Re}\{V(\zeta)\}\ge 0$ for almost all $\zeta\in N_-(0)$. Now, V is analytic in $C_\infty\backslash[a,b]$ and can easily be seen to have logarithmic singularities at a and b. Since V is therefore not a constant, its real part can vanish at only a finite number of points on ∂D. This means that to within sets of measure zero $N_+(0)\subset\{\zeta:\mathrm{Re}\{V(\zeta)\}<0\}$ and $N_-(0)\subset\{\zeta:\mathrm{Re}\{V(\zeta)\}>0\}$. But then, except for a set of measure zero, $\mathrm{Re}\{V(\zeta)\}>0$ implies that ζ is not in $N_+(0)$, which means that $v_0(\zeta)=1$. Similarly, $v_0(\zeta)=-1$ a.e. on $\{\zeta:\mathrm{Re}\{V(\zeta)\}<0\}$. Since a bounded harmonic function with constant radial limit a.e. on an open arc is actually harmonic on that arc, it follows that ∂D can be expressed as a union of arcs on the interior of each of which v_0 is constantly 1 or -1. It is also to be noted that if v_0 is constantly 1 (or -1) on the interiors of two contiguous arcs, then it follows that v_0 is harmonic in a neighborhood of the point separating these two arcs. After joining together all groups of consecutive arcs on which v_0 has the same value, v_0 will have the stated properties.

It is clear from the foregoing proof that

(E) the integer k appearing in (D) is bounded above the number of zeros of $\mathrm{Re}\{V\}$ on ∂D.

It is also worth pointing out that (D) simply says that $h(D)$ covers U $k/2$ times, so that as an immediate consequence of (D) we have that

(F) if f_1 is an extremal function for R, then $f_1'(z)=p(\sigma(z))$, where p is the projection onto R of a one-to-one conformal maping of D onto the universal covering surface of R and σ is a Blaschke product with $k/2$ factors.

In many instances it is possible, in theory at least, to obtain an a priori estimate for the number k of (D) based entirely on simple geometric properties of R, and we indicate briefly how this can be done. Let f_1 be an extremal function for R and let a,b be as above. Using (1) and (2) one can get a lower bound $\gamma_1>0$ for $m=m(R)$, and, as indicated at the end of the first paragraph of §1, a construction along the lines of that described there may be used to obtain an upper bound $\gamma_2<1$ for $m(R)$. By (6) we then have that $|b-a|\geq\min\{\gamma_i(1-\gamma_i): i=1,2\}=\gamma_3$. By replacing K by the composition of K with an appropriate one-to-one conformal mapping of U onto itself we may assume that $h(0)=0$. Assume, for simplicity, that R is contained in a disk of radius d_1. Assume further that we have a lower bound $d_0>0$ for the distance from $f_1'(0)=K(0)$ to ∂R, such as the one given in (C). Since $(4/\pi)\arctan z$ maps D one-to-one onto U, $d_0z+K(0)$ is subordinate to $K((4/\pi)\arctan z)$, so that $|K'(0)|\geq\pi d_0/4$. It is not difficult to find explicit functions $\gamma_4(r)=\gamma_4(d_0,d_1,r)>0$ and $\gamma_5(r)=\gamma_5(d_0,d_1,r)$ such that

$$\gamma_4(r)\leq|w'((4/\pi)\arctan z)|\leq\gamma_5(r)\quad\text{for } z\in D(r)$$

for all functions w on U which satisfy

$$w(U)\subset D(d_1,p),\quad |w'(0)|\geq\pi d_0/4,\quad\text{and}\quad 0\notin w'(U),$$

for some p. Since $w=K$ satisfies these conditions, and $h(z)$ is subordinate to $(4/\pi)\arctan z$, we have that $\gamma_4(|z|)\leq|K'(h(z))|\leq\gamma_5(|z|)$. Now, the function $X(\zeta)=(V(\zeta)+\overline{V(1/\bar{\zeta})})/2$ is analytic for $m<|\zeta|<1/m$ and $X(\zeta)=\mathrm{Re}\{V(\zeta)\}$ for $\zeta\in\partial D$. From the bounds for $|K'(h(z))|$, the bound $|b-a|\geq\gamma_3$, and the definition (15) of V, one can derive bounds of the form

$$\gamma_6-\gamma_7\log|\zeta-a|\leq|X(\zeta)|\leq\gamma_8-\gamma_9\log((|\zeta|-m)(1/m-|\zeta|))\quad\text{for } m<|\zeta|<1/m,$$

where γ_6, γ_7, γ_8, γ_9 depend only on d_0, d_1, γ_1, γ_2, and where $\gamma_7>0$. From this it follows that one can find $\rho>m$ and γ_{10} which depend only on d_0, d_1, γ_1, γ_2 such that $|X(\zeta_0)|\geq 1$ and $|X(\zeta)|\leq\gamma_{10}$, for some ζ_0 in $\partial D(\rho)$ and all ζ for which $(\rho+m)/2\leq|\zeta|\leq 2/(\rho+m)$. Finally, using this one can derive an upper bound $N(d_0,d_1,\gamma_1,\gamma_2)$ for the number of zeros of X on ∂D, which, by (E), is an upper bound for k.

Although it is significant that in many cases one could find an a priori upper bound for k, to carry out the program described above would be tedious and would yield a very large bound. I believe that in a large number of cases $k=2$, and in the following section it will be shown that this is the case when $m(R)$ is small. It should be pointed out that the considerations outlined in the preceding paragraph are of an analytical nature and do not take into account the geometrical significance of f_1.

Before closing this section we show that for a simply connected region there does not necessarily exist an extremal function, even if the region is bounded by a piecewise smooth Jordan curve. Specifically we will show that $R=\{re^{i\theta}: 0<r<1, |\theta|<\lambda\}$, where $\pi/2<\lambda<\pi$, does not have an extremal function. Assume to the contrary that f_1 is in an extremal function for R and let m, a, b, c, g_0, and f_0 be as at the beginning of this section. Since c cannot belong to the boundary of the convex hull of R, $|\arg c|=\lambda$ or $c=0$. Let K map U onto R (which is its own universal covering surface) in such as way that the right boundary line of U corresponds to the circular portion of ∂R, and $K(1)=1$. Since g_0 is not constant, in this case the number k appearing in (D) is at least 2. Let α denote one of the $k/2$ arcs on which v_0 is 1. Since $v_0=\mathrm{Re}\{h\}$, $h(\zeta)$ traverses the right boundary line of U as ζ traverses α. Thus there is a point $\zeta_0 \in \alpha$ such that $h(\zeta_0)=1$. By the reflection principle, K is analytic on the right boundary line of U, so that there exists $r>0$ such that $K(D(r,1)) \subset D(1,1)$. Let $t>0$ be such that $|\zeta-\zeta_0|<t$ implies that $|h(\zeta)-h(\zeta_0)|<r/2$ $(\zeta \in \partial D)$. Let u be a nonnegative smooth real valued function on ∂D which is not identically zero and whose support lies in $D(t,\zeta_0) \cap \partial D$. Then the harmonic conjugate of u is bounded, so that for ϵ sufficiently small the function g_ϵ defined in (14) satisfies $g_\epsilon(D) \subset (R \cup D(1,1))-c$. (Note that $g_\epsilon(\zeta) \in \partial R$ for all ζ not in the support of u.) It follows from the proof of (D) that $\operatorname{Re}\{V(\zeta)\}>0$ for all but at most a finite number of points on α, and therefore the variational argument given in that proof tells us that for any sufficiently small $\epsilon>0$ there exist $a',b' \in D(m)$ for which $\int_{a'}^{b'} g_\epsilon(z)\,dz=0$. However, for ϵ sufficiently close to 0, $g_\epsilon(D)+c \subset R \cup D(1,1) \subset 2R$. If f is any antiderivative of $(g_\epsilon+c)/2$, then $f \in B(R)$ and $f[b',a']=c/2 \notin R$, since $|\arg c|=\lambda$ or $c=0$. Since this contradicts the definition of $m=m(R)$ we are done.

4. The case of small $m(R)$. This section is devoted to showing that

(G) there exists an $\epsilon_0>0$ such that if $0<m(R) \le \epsilon_0$ and R has an extremal function f_1, then f_1' maps D one-to-one onto the universal covering surface of R.

In what follows we let f_1 be an extremal function for R, $f_1[b,a]=c \in \partial R$ where $a,b \in \partial D(m)$, and we freely use the notation introduced in §3. We shall prove that f_1' has the desired mapping property by showing that the integer k of (D) is 2. This will be done by showing that if m is sufficiently small the function $W(\zeta)=V(1/\zeta)$ maps D one-to-one onto a convex domain which by (E) means that k is at most 2. The desired conclusion will then follow immediately from (F).

Let $d=d(\log(R-c),0)$. Since $u(R-c)=m$, from (2), (3), and (5) we have that

$$\left(\left(\frac{3\beta}{4m}\right)^2-\pi^2\right)^{1/2} \le d \le \frac{\pi}{\beta}\left(\frac{1}{m^2}+1\right)^{1/2},$$

so that there exist positive constants C_1, C_2 independent of R such that $C_1/m \le d \le C_2/m$ whenever m is sufficiently small. For notational convenience we denote by Λ any function of d which tends to 1 as d tends to ∞ (or equivalently, as m tends to 0). In addition, we write $\alpha=O(\beta)$ to mean that there exists a universal constant M such that $|\alpha| \le M|\beta|$ whenever d is sufficiently large. An analogous meaning is ascribed to the statement $\alpha=o(\beta)$. We will show the following.

(16) Let $\gamma=f_0''(0)/f_0'(0)$. Then $|\gamma|=d\Lambda$ and $\{a,b\}=\{i\pi\Lambda/\gamma, -i\pi\Lambda/\gamma\}$. In addition, we can choose the mapping K of U onto the universal covering surface of R so that $K'(h(z))=L(e^{\gamma z}+O(1/d))$ in $D(m)$, where $L \ne 0$ is a constant (which depends on R).

Application of this and the definition of V in (15) shows that

$$W(\zeta)=\frac{e^{-i(\phi+\tau)}L}{2\pi}\int_a^b (e^{\gamma z}+O(1/d))\frac{1+\zeta z}{1-\zeta z}\,dz.$$

We then have that

$$W'(\zeta)=\frac{e^{-i(\phi+\tau)}L}{\pi}\left(\int_a^b e^{\gamma z}z\,dz+O(1/d^3)\right)=\pm 2ie^{-i(\phi+\tau)}L\Lambda(1/\gamma^2+o(1/d^2)),$$

and $W''(\zeta)=O(L/d^3)$, from which it follows that $W'(0)\neq 0$ for m sufficiently small and $W''(\zeta)/W'(\zeta)=O(1/d)$ for ζ in D. From this it follows that W indeed maps D one-to-one onto a convex domain for sufficiently small m.

We now prove (16). Let $X, Y\in D(2\pi/d)$, $Z=(Y+X)/2$ and $\xi=(Y-X)/2$, so that $Y=Z+\xi$ and $X=Z-\xi$. Let $f\in B(R-c)$. Then

$$\log f'(z)=\alpha+\eta(z-Z)+O(d(z-Z)^2)\quad \text{for } z\in D(1/2), \tag{17}$$

where $\eta=O(d)$ by (4). We have

$$f(Y)-f(X)=e^{\alpha}\int_{-\xi}^{\xi} e^{\eta z}(1+O(dz^2))\,dz=e^{\alpha}(g(\eta\xi)+O(\xi))\xi, \tag{18}$$

where $g(z)=(e^z-e^{-z})/z$. Here we have taken into account that $O(d\xi^3)=O(\xi^2)$ since $|\xi|<\pi/d$. There holds

$$|g(z)|\geq M\min\{|z-i\pi|, |z+i\pi|\}\quad \text{for } |z|\leq 3\pi/2, \tag{19}$$

where M is a positive constant. Let ϵ be any positive number. We now apply the above with $Z=0$ to any function $f\in B(R-c)$ for which $|\eta|\geq d-\epsilon$. Let $|\xi|=(\pi\pm\epsilon)/|\eta|$. Then $\min\{|\eta\xi-i\pi|, |\eta\xi+i\pi|\}=\epsilon$, so that by (19) $|g(\eta\xi)|>O(\xi)$ for sufficiently large d. Since $g(\pm i\pi)=0$, we conclude from Rouché's theorem and (18) that $m\leq(\pi+\epsilon)/(d-\epsilon)$ for d sufficiently large. This shows that $m\leq\pi\Lambda/d$.

Now assume that a and b are as indicated just after the statement of (G) and apply the foregoing with $f=f_0$, $X=a$, and $Y=b$. Since $f_0(b)-f_0(a)=0$, (18) and (19) imply that $\min\{|\eta\xi-i\pi|, |\eta\xi+i\pi|\}=O(\xi)=O(m)=O(1/d)$. Thus $\eta\xi=\pm i\pi+O(1/d)$, and therefore $|\eta|=(\pi+O(1/d))/|\xi|\geq(\pi+O(1/d))/m\geq d\Lambda$, where we have used the fact that $m\leq\pi\Lambda/d$ established at the end of the preceding paragraph. Since it follows from (4) that $|\eta|\leq d\Lambda$, we have that $|\eta|=d\Lambda$ and $|\xi|=\pi\Lambda/d$. Thus $|b-a|=2\pi\Lambda/d$, and since $a, b\in D(m)$ we conclude that $Z=o(1/d)$. Since $\gamma=f_0''(0)/f_0'(0)$, we have $\gamma=\eta+O(dZ)=\eta+o(1)$, so that $|\gamma|=d\Lambda$. Also $\{a,b\}=\{Z\pm\xi\}=\{\pm\xi+o(1/d)\}=\{\pm\xi\Lambda\}=\{\pm i\pi\Lambda/\eta\}=\{\pm i\pi\Lambda/\gamma\}$, as desired.

Finally, we must establish the expression for $K'(h(z))$ in (16). Since K can be any mapping of the strip U onto the universal covering surface of R, we may assume that $h(0)=0$. For any mapping j of D into U, $\log(K(j(z))-c)$ maps D into $\log(R-c)$. Thus if $M_1=\sup\{|j'(0)|: j(D)\subset U, j(0)=0\}$ ($M_1=4/\pi$, but the exact value is not important), then

$$\begin{aligned} M_1|K'(0)/(K(0)-c)|\Lambda\leq d\Lambda=|\gamma|&=|f_0''(0)/f_0'(0)|\\ &=|K'(0)h'(0)/(K(0)-c)|\leq M_1|K'(0)/(K(0)-c)|, \end{aligned}$$

so that $h'(0)=M_1\Lambda$. From this and the fact that

$$h'(z)=h'(0)+O(m)=h'(0)+O(1/d)\ \text{ in } D(m),$$

we see that $h'(z) = h'(0)(1 + O(1/d))$ in that disk. Since from (17) it follows that $f_0''(z)/f_0'(z) = \gamma + O(1)$ in $D(m)$, we see that

$$K'(h(z))/(K(h(z)) - c) = f_0''(z)/(f_0'(z)h(z)) = \gamma(1 + O(1/d))/h'(0) \text{ in } D(m).$$

Thus, in $D(m)$,

$$\begin{aligned} K'(h(z)) &= \frac{\gamma}{h'(0)}(1 + O(1/d))(K(h(z)) - c) = \frac{\gamma}{h'(0)}(1 + O(1/d))f_0'(z) \\ &= \frac{\gamma}{h'(0)}(1 + O(1/d))e^{\alpha}e^{\gamma z}e^{O(1/d)} \end{aligned}$$

by the expansion (17). From this the desired expression for $K'(h(z))$ follows with $L = \gamma e^{\alpha}/h'(0) = f_0''(0)/h'(0)$ (since $e^{\alpha} = f_0'(0)$).

The interested reader might care to compare the contents of this section with those of §1 of [1].

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